

The varieties of intersections of lines and hypersurfaces in projective spaces

By

Atsushi IKEDA*

§ 1. Introduction

We denote by X_F the hypersurface in $\mathbf{P}^n$ defined by a homogeneous polynomial $F \in \mathbf{C}[x_0, \dots, x_n]$ of degree d , and denote by $\mathbf{G}$ the set of all lines in $\mathbf{P}^n$. Let $1 \leq m \leq d + 1$. Then the set

$$Y_{F,m} = \{(p, L) \in \mathbf{P}^n \times \mathbf{G} \mid L \text{ and } X_F \text{ intersect at } p \text{ with the multiplicity } \geq m\}$$

form a projective variety, whose defining equations are given by using the higher derivative of F (Theorem 2.1). For a general hypersurface X_F , the projective variety $Y_{F,m}$ is smooth of dimension $2n - m - 1$ (Theorem 3.2). The purpose of this research is to characterize some geometric properties of X_F by using the Hodge structure of $Y_{F,m}$. In this paper, we give a method to describe the Hodge cohomologies of $Y_{F,m}$ by the Jacobian rings, which is a generalization of the theory of Jacobian ring for a hypersurface in $\mathbf{P}^n$ by Griffiths [2]. Using this method, we study the injectivity of the infinitesimal period map for $Y_{F,m}$ (Theorem 6.2). In the case $n = d = 3$, we also yield a Torelli type theorem for the map $X_F \mapsto Y_{F,3}$ (Proposition 6.5). The full-detailed version with all proofs of this article will be appeared somewhere.

§ 2. Varieties of intersections

Let $\mathbf{P}^n$ be a complex projective space of dimension n , and let $V = H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. We denote by $\mathbf{P} = \text{Grass}(n, V)$ the Grassmannian variety of all n -dimensional subspaces

Received September 26, 2007. Revised March 18, 2008.

2000 Mathematics Subject Classification(s): Primary 14C05; Secondary; 14C30

Partially supported by the 21st century COE program towards a new basic science; depth and synthesis, Osaka university.

*Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan.

email: atsushi@math.sci.osaka-u.ac.jp

© 2008 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

in V , and denote by $\mathcal{S}_{\mathbf{P}}$ (resp. $\mathcal{Q}_{\mathbf{P}}$) the universal sub (resp. quotient) bundle on $\mathbf{P}$. We have an exact sequence

$$0 \longrightarrow \mathcal{S}_{\mathbf{P}} \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes V \longrightarrow \mathcal{Q}_{\mathbf{P}} \longrightarrow 0.$$

Then $\mathbf{P}$ is naturally identified with $\mathbf{P}^n$, and $\mathcal{Q}_{\mathbf{P}}$ is identified with the tautological line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$. We denote by $\mathbf{G} = \text{Grass}(n-1, V)$ the Grassmannian variety of all $(n-1)$ -dimensional subspaces in V , and denote by $\mathcal{S}_{\mathbf{G}}$ (resp. $\mathcal{Q}_{\mathbf{G}}$) the universal sub (resp. quotient) bundle on $\mathbf{G}$. We have an exact sequence

$$0 \longrightarrow \mathcal{S}_{\mathbf{G}} \longrightarrow \mathcal{O}_{\mathbf{G}} \otimes V \longrightarrow \mathcal{Q}_{\mathbf{G}} \longrightarrow 0.$$

We remark that a point in $\mathbf{G}$ corresponds to a line in $\mathbf{P}^n$. Let $p_1 : \mathbf{P} \times \mathbf{G} \rightarrow \mathbf{P}$ and $p_2 : \mathbf{P} \times \mathbf{G} \rightarrow \mathbf{G}$ be the projections. We denote by Γ the subvariety of $\mathbf{P} \times \mathbf{G}$ defined as the zeros of the composition

$$p_2^* \mathcal{S}_{\mathbf{G}} \longrightarrow \mathcal{O}_{\mathbf{P} \times \mathbf{G}} \otimes V \longrightarrow p_1^* \mathcal{Q}_{\mathbf{P}}.$$

Then Γ is the flag variety of all pairs (p, L) of a point $p \in \mathbf{P}^n$ and a line $L \subset \mathbf{P}^n$ containing the point p . By the first projection $\phi = p_1|_{\Gamma}$, the subvariety Γ is considered as the $\mathbf{P}^{n-1}$ -bundle

$$\phi : \Gamma = \text{Grass}(n-1, \mathcal{S}_{\mathbf{P}}) = \mathbf{P}(\mathcal{S}_{\mathbf{P}}) \longrightarrow \mathbf{P}.$$

By the second projection $\pi = p_2|_{\Gamma}$, the subvariety Γ is considered as the $\mathbf{P}^1$ -bundle

$$\pi : \Gamma = \text{Grass}(1, \mathcal{Q}_{\mathbf{G}}) = \mathbf{P}(\mathcal{Q}_{\mathbf{G}}) \longrightarrow \mathbf{G}.$$

We denote by $\mathcal{Q}_{\phi}$ the universal quotient bundle of the Grassmannian bundle $\phi : \Gamma \rightarrow \mathbf{P}$. We have exact sequences

$$0 \longrightarrow \pi^* \mathcal{S}_{\mathbf{G}} \longrightarrow \phi^* \mathcal{S}_{\mathbf{P}} \longrightarrow \mathcal{Q}_{\phi} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{Q}_{\phi} \longrightarrow \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow \phi^* \mathcal{Q}_{\mathbf{P}} \longrightarrow 0.$$

Note that $\mathcal{Q}_{\phi}$ is an invertible sheaf. We define a decreasing filtration

$$\text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} = \text{Fil}^0 \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} \supset \cdots \supset \text{Fil}^{d+1} \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} = 0$$

on the d -th symmetric product of $\pi^* \mathcal{Q}_{\mathbf{G}}$, as $\text{Fil}^m \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}$ being the image of the natural homomorphism

$$\text{Sym}^m \mathcal{Q}_{\phi} \otimes \text{Sym}^{d-m} \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}.$$

Let $F \in \text{Sym}^d V$. We denote by X_F the hypersurface in $\mathbf{P}$ defined as the zeros of the section $[F]_{\mathbf{P}} \in H^0(\mathbf{P}, \text{Sym}^d \mathcal{Q}_{\mathbf{P}})$ which is the image of F by the natural isomorphism

$$\text{Sym}^d V \simeq H^0(\mathbf{P}, \text{Sym}^d \mathcal{Q}_{\mathbf{P}}).$$

We denote by $Y_{F,m}$ the subvariety in Γ defined as the zeros of the section $[F]_{\Gamma,m} \in H^0(\Gamma, \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \text{Fil}^m \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}})$ which is the image of F by the natural homomorphism

$$\text{Sym}^d V \simeq H^0(\Gamma, \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}) \longrightarrow H^0(\Gamma, \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \text{Fil}^m \text{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}).$$

We denote by Z_F the subvariety in $\mathbf{G}$ defined as the zeros of the section $[F]_{\mathbf{G}} \in H^0(\mathbf{G}, \text{Sym}^d \mathcal{Q}_{\mathbf{G}})$ which is the image of F by the natural isomorphism

$$\text{Sym}^d V \simeq H^0(\mathbf{G}, \text{Sym}^d \mathcal{Q}_{\mathbf{G}}).$$

Then a point in Z_F corresponds to a line which is contained in X_F . Let L be a line in $\mathbf{P}^n$, and let p be a point on L . The fiber of the line bundle $\mathcal{Q}_{\phi}$ at the point $(p, L) \in \Gamma$ is naturally identified with the kernel of the restriction

$$H^0(L, \mathcal{O}_{\mathbf{P}^n}(1)|_L) \longrightarrow H^0(p, \mathcal{O}_{\mathbf{P}^n}(1)|_p).$$

Hence, L and X_F intersect at p with the multiplicity $\geq m$ if and only if the pair (p, L) represents a point in $Y_{F,m}$. We have a diagram

$$\begin{array}{ccccc} \mathbf{P} & \xleftarrow{\phi} & \Gamma & \xrightarrow{\pi} & \mathbf{G} \\ \cup & & \cup & & \cup \\ X_F = \phi(Y_{F,1}) & \longleftarrow & Y_{F,1} & \longrightarrow & \pi(Y_{F,1}) \\ \cup & & \cup & & \cup \\ \vdots & & \vdots & & \vdots \\ \cup & & \cup & & \cup \\ \phi(Y_{F,d}) & \longleftarrow & Y_{F,d} & \longrightarrow & \pi(Y_{F,d}) \\ \cup & & \cup & & \cup \\ \phi(Y_{F,d+1}) & \longleftarrow & Y_{F,d+1} & \longrightarrow & \pi(Y_{F,d+1}) = Z_F. \end{array}$$

The morphism $\phi|_{Y_{F,1}} : Y_{F,1} \rightarrow X_F$ is the $\mathbf{P}^{n-1}$ -bundle

$$\mathbf{P}(\mathcal{S}_{\mathbf{P}}|_{X_F}) = \mathbf{P}(\Omega_{\mathbf{P}}^1 \otimes \mathcal{Q}_{\mathbf{P}}|_{X_F}) \longrightarrow X_F.$$

If X_F is a smooth hypersurface, then $\phi|_{Y_{F,2}} : Y_{F,2} \rightarrow X_F$ is the $\mathbf{P}^{n-2}$ -bundle

$$\mathbf{P}(\Omega_{X_F}^1 \otimes \mathcal{Q}_{\mathbf{P}}|_{X_F}) \longrightarrow X_F.$$

The morphism $\pi|_{Y_{F,m}} : Y_{F,m} \rightarrow \pi(Y_{F,m})$ is generically finite for $1 \leq m \leq d$, and the morphism $\pi|_{Y_{F,d+1}} : Y_{F,d+1} \rightarrow Z_F$ is the $\mathbf{P}^1$ -bundle

$$\mathbf{P}(\mathcal{Q}_{\mathbf{G}}|_{Z_F}) \longrightarrow Z_F.$$

We remark that the isomorphism

$$\sigma : \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \mathrm{Fil}^m \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} \xrightarrow{\sim} \mathrm{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \mathrm{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}$$

is induced by the homomorphism

$$\begin{aligned} \mathrm{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} &\longrightarrow \mathrm{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \mathrm{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}; \\ A_1 \cdots A_d &\longmapsto \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} [A_{\sigma(1)} \cdots A_{\sigma(d-m+1)}]_{\mathbf{P}} \otimes A_{\sigma(d-m+2)} \cdots A_{\sigma(d)}, \end{aligned}$$

where $[A]_{\mathbf{P}} \in \mathrm{Sym}^j \phi^* \mathcal{Q}_{\mathbf{P}}$ denotes the image of a local section $A \in \mathrm{Sym}^j \pi^* \mathcal{Q}_{\mathbf{G}}$, and $\mathfrak{S}_d$ denotes the permutation group of the index set $\{1, \dots, d\}$.

For $F \in \mathrm{Sym}^d V$, we define the tensor $F_k \in \mathrm{Sym}^{d-k} V \otimes \mathrm{Sym}^k V$ by

$$F_k = \frac{(d-k+1)!}{d!} \sum_{0 \leq i_1, \dots, i_{k-1} \leq n} \frac{\partial^{k-1} F}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}} \otimes x_{i_1} \cdots x_{i_{k-1}},$$

which does not depend on the choice of the basis $(x_0, \dots, x_n)$ of V .

Theorem 2.1. *The subvariety $Y_{F,m}$ in Γ is defined as the zeros of the section*

$$F_m \in \mathrm{Sym}^{d-m+1} V \otimes \mathrm{Sym}^{m-1} V \simeq H^0(\Gamma, \mathrm{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \mathrm{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}).$$

§ 3. Smoothness and connectedness

Since the variety $Y_{F,d+1}$ is a $\mathbf{P}^1$ -bundle over Z_F , the following theorem is directly induced from the results in [1, Theorem 8] and [5, Chapter V. 4].

Theorem 3.1. *Assume $d \geq 1$.*

- (1) *If $d \geq 2n - 2$, then $Y_{F,d+1}$ is empty for general $F \in \mathrm{Sym}^d V \setminus \{0\}$.*
- (2) *If $d \leq 2n - 3$, then $Y_{F,d+1}$ is non-empty for any $F \in \mathrm{Sym}^d V \setminus \{0\}$.*
- (3) *If $d \leq 2n - 3$, then $Y_{F,d+1}$ is smooth of dimension $2n - d - 2$ for general $F \in \mathrm{Sym}^d V \setminus \{0\}$.*
- (4) *If $d \leq 2n - 4$ and $(d, n) \neq (2, 3)$, then $Y_{F,d+1}$ is connected for any $F \in \mathrm{Sym}^d V \setminus \{0\}$.*

For $1 \leq m \leq d$, we have the following theorem.

Theorem 3.2. *Assume $1 \leq m \leq d$.*

- (1) *If $m \geq 2n$, then $Y_{F,m}$ is empty for general $F \in \text{Sym}^d V \setminus \{0\}$.*
- (2) *If $m \leq 2n - 1$, then $Y_{F,m}$ is non-empty for any $F \in \text{Sym}^d V \setminus \{0\}$.*
- (3) *If $m \leq 2n - 1$, then $Y_{F,m}$ is smooth of dimension $2n - m - 1$ for general $F \in \text{Sym}^d V \setminus \{0\}$.*
- (4) *If $m \leq 2n - 2$, then $Y_{F,m}$ is connected for any $F \in \text{Sym}^d V \setminus \{0\}$.*

In the case when X_F is a cubic hypersurface in $\mathbf{P}^n$, the variety $Y_{F,m}$ is smooth of dimension $2n - m - 1$ if and only if X_F is smooth.

If $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then we can compute some topological invariants of $Y_{F,m}$. For example, if $m = d = 2n - 1$, then $\dim Y_{F,m} = 0$, and we can compute the number of the points of $Y_{F,m}$ by Schubert calculus;

$$\begin{cases} n = 1, m = d = 1 & \implies \#Y_{F,m} = 1, \\ n = 2, m = d = 3 & \implies \#Y_{F,m} = 9, \\ n = 3, m = d = 5 & \implies \#Y_{F,m} = 575, \\ n = 4, m = d = 7 & \implies \#Y_{F,m} = 99715, \\ & \dots \end{cases}$$

for general F . Similarly, if $d = 2n - 3$, then $\dim Z_F = 0$ and we have

$$\begin{cases} n = 2, d = 1 & \implies \#Z_F = 1, \\ n = 3, d = 3 & \implies \#Z_F = 9 \times 3 = 27, \\ n = 4, d = 5 & \implies \#Z_F = 575 \times 5 = 2785, \\ n = 5, d = 7 & \implies \#Z_F = 99715 \times 7 = 698005, \\ & \dots \end{cases}$$

for general F , that is known in [3]. When $\dim Y_{F,m} = 1$, we can compute the genus $g(Y_{F,m})$ of $Y_{F,m}$. For example, if $m = d = 2n - 2$, then $\dim Y_{F,m} = 1$ and we have

$$\begin{cases} n = 2, m = d = 2 & \implies g(Y_{F,m}) = 0, \\ n = 3, m = d = 4 & \implies g(Y_{F,m}) = 201, \\ n = 4, m = d = 6 & \implies g(Y_{F,m}) = 75601, \\ n = 5, m = d = 8 & \implies g(Y_{F,m}) = 39001985, \\ & \dots \end{cases}$$

for general F .

§ 4. Jacobian rings

We denote by

$$S = \mathbf{C}[x_0, \dots, x_n, z_0, \dots, z_n] = \bigoplus_{a,b \in \mathbf{Z}} S^{a,b}$$

the polynomial ring bi-graded by $\deg x_i = (1, 0)$ and $\deg z_j = (0, 1)$. We define homomorphisms δ and ε by

$$\delta : S^{a,b} \longrightarrow S^{a-1,b+1}; \quad A \mapsto \frac{1}{a} \sum_{i=0}^n \frac{\partial A}{\partial x_i} \cdot x_i$$

and

$$\varepsilon : S^{a,b} \longrightarrow S^{a+1,b-1}; \quad A \mapsto \frac{1}{b} \sum_{i=0}^n \frac{\partial A}{\partial z_i} \cdot z_i.$$

Let V be the $(n+1)$ -dimensional vector space as in Section 2. For $F \in \text{Sym}^d V$, we have a bi-homogeneous polynomial $F_1 \in S^{d,0}$ by considering $x_0, \dots, x_n$ as a basis of V . We set the bi-homogeneous polynomial F_k by

$$F_k = \delta^{k-1}(F_1) \in S^{d-k+1,k-1}$$

for $k \geq 1$. We define the bi-graded ring $S_{F,m}$ by

$$S_{F,m} = S / (F_k; \ 1 \leq k \leq m),$$

and we define the *Jacobian ring* $R_{F,m}$ as the bi-graded ring by

$$R_{F,m} = S_{F,m-1} / \left(\frac{\partial F_m}{\partial x_i} \cdot x_j + \frac{\partial F_m}{\partial z_i} \cdot z_j; \ 0 \leq i \leq n, \ 0 \leq j \leq n \right)$$

for $m \geq 1$, where we set $S_{F,0} = S$. Since

$$\frac{1}{d} \sum_{i=0}^n \left(\frac{\partial F_m}{\partial x_i} \cdot x_i + \frac{\partial F_m}{\partial z_i} \cdot z_i \right) = F_m,$$

$S_{F,m-1} \rightarrow R_{F,m}$ factors through $S_{F,m}$. We set

$$S_{F,m}^{a,b,c} = \text{Ker}(\varepsilon^c : S_{F,m}^{a,b} \longrightarrow S_{F,m}^{a+c,b-c}).$$

In the following, we describe the relation between these rings and the variety $Y_{F,m}$. We denote by T_Γ (resp. $T_{Y_{F,m}}$) the tangent bundle of Γ (resp. $Y_{F,m}$). Then we have the exact sequences

$$0 \longrightarrow \mathcal{O}_\Gamma \longrightarrow \phi^* \mathcal{S}_{\mathbf{P}}^\vee \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow T_\Gamma \longrightarrow 0,$$

where $\mathcal{S}_{\mathbf{P}}^{\vee}$ denotes the $\mathcal{O}_{\Gamma}$ -dual of $\mathcal{S}_{\mathbf{P}}$. If $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then we define the coherent sheaf $\mathcal{N}_m$ of $\mathcal{O}_{Y_{F,m-1}}$ -modules by

$$\mathcal{N}_m = \text{Coker}(T_{Y_{F,m-1}}(-\log Y_{F,m}) \longrightarrow T_{\Gamma}|_{Y_{F,m-1}}).$$

Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_{F,m-1}} \longrightarrow (\text{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \text{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}})|_{Y_{F,m-1}} \longrightarrow \mathcal{N}_m \longrightarrow 0.$$

Using Lemma 5.5 in the next section, we have the following proposition.

Proposition 4.1. *If $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then*

$$H^0(Y_{F,m-1}, \mathcal{N}_m) \simeq S_{F,m}^{d-m+1, m-1}$$

for $1 \leq m \leq n - 1$.

We remark that the composition

$$V^{\vee} \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow \phi^* \mathcal{S}_{\mathbf{P}}^{\vee} \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow T_{\Gamma} \longrightarrow \mathcal{N}_m$$

induces the homomorphism

$$\begin{aligned} V^{\vee} \otimes V &\simeq V^{\vee} \otimes H^0(\Gamma, \pi^* \mathcal{Q}_{\mathbf{G}}) \longrightarrow H^0(Y_{F,m-1}, \mathcal{N}_m) \simeq S_{F,m}^{d-m+1, m-1}; \\ x_i^{\vee} \otimes x_j &\longmapsto \frac{\partial F_m}{\partial x_i} \cdot x_j + \frac{\partial F_m}{\partial z_i} \cdot z_j, \end{aligned}$$

where $x_0^{\vee}, \dots, x_n^{\vee}$ denotes the dual basis of $x_0, \dots, x_n$. Using Lemma 5.6 in the next section, we have the following theorem.

Theorem 4.2. *If $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then there is a natural injective homomorphism*

$$\rho : R_{F,m}^{d-m+1, m-1} \longrightarrow H^1(Y_{F,m-1}, T_{Y_{F,m-1}}(-\log Y_{F,m})),$$

and it is an isomorphism for $m \leq n - 2$.

We set the integers $\alpha(n, m, d, q)$ and $\beta(n, m, q)$ by

$$\begin{cases} \alpha(n, m, d, q) = md - \frac{m(m-1)}{2} - n - 2 + q(d - m + 1), \\ \beta(n, m, q) = \frac{m(m-1)}{2} - n + q(m - 1). \end{cases}$$

Since $\Omega_{Y_{F,m-1}}^{2n-m}(Y_{F,m})$ is isomorphic to

$$(\text{Sym}^{\alpha(n, m, d, 0)} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \text{Sym}^{\beta(n, m, 0)} \mathcal{Q}_{\phi})|_{Y_{F,m-1}},$$

using Lemma 5.5 in the next section, we have the following theorem.

Theorem 4.3. *If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\gamma_0 : S_{F,m-1}^{\alpha(n,m,d,0),\beta(n,m,0),1} \longrightarrow H^0(Y_{F,m-1}, \Omega_{Y_{F,m-1}}^{2n-m}(Y_{F,m})),$$

and it is an isomorphism for $m \leq n-1$.

Here we remark that $S_{F,m-1}^{\alpha(n,m,d,0),\beta(n,m,0)} = S_{F,m-1}^{\alpha(n,m,d,0),\beta(n,m,0),1}$ for $\frac{m(m-1)}{2} \leq n$.

The following theorem is proved by the similar way as Theorem 4.2, by using the exact sequence

$$0 \rightarrow \Omega_{Y_{F,m-1}}^{2n-m-1}(\log Y_{F,m}) \rightarrow T_\Gamma|_{Y_{F,m-1}} \otimes \Omega_{Y_{F,m-1}}^{2n-m}(Y_{F,m}) \rightarrow \mathcal{N}_m \otimes \Omega_{Y_{F,m-1}}^{2n-m}(Y_{F,m}) \rightarrow 0.$$

Theorem 4.4. *If $\frac{m(m-1)}{2} = n$ and $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\gamma_1 : R_{F,m}^{\alpha(n,m,d,1),\beta(n,m,1)} \longrightarrow H^1(Y_{F,m-1}, \Omega_{Y_{F,m-1}}^{2n-m-1}(\log Y_{F,m})),$$

and it is an isomorphism for $m \leq n-2$.

§ 5. Computation of cohomology

In this section, we enumerate several lemmas, which is used in the proof of theorems in Section 4. For simplicity of notations, we set the invertible sheaf $\mathcal{O}_\Gamma(p, q)$ on Γ by

$$\mathcal{O}_\Gamma(p, q) = \begin{cases} \text{Sym}^p \phi^* \mathcal{Q}_\mathbf{P} \otimes \text{Sym}^q \mathcal{Q}_\phi & (p \geq 0, q \geq 0), \\ \text{Sym}^p \phi^* \mathcal{Q}_\mathbf{P} \otimes \text{Sym}^{-q} \mathcal{Q}_\phi^\vee & (p \geq 0, q < 0), \\ \text{Sym}^{-p} \phi^* \mathcal{Q}_\mathbf{P}^\vee \otimes \text{Sym}^q \mathcal{Q}_\phi & (p < 0, q \geq 0), \\ \text{Sym}^{-p} \phi^* \mathcal{Q}_\mathbf{P}^\vee \otimes \text{Sym}^{-q} \mathcal{Q}_\phi^\vee & (p < 0, q < 0), \end{cases}$$

and we set $Q_\mathbf{G}^r = \text{Sym}^r \pi^* \mathcal{Q}_\mathbf{G}$ for $r \geq 0$. For a sheaf $\mathcal{E}$ of $\mathcal{O}_\Gamma$ -modules, we set $\mathcal{E}(p, q) = \mathcal{E} \otimes \mathcal{O}_\Gamma(p, q)$.

Lemma 5.1. *Assume $r \geq 0$.*

$$H^0(\Gamma, Q_\mathbf{G}^r(p, q)) = \text{Ker}(\varepsilon^{r+1} : S^{p,q+r} \rightarrow S^{p+r+1,q-1}).$$

Lemma 5.2. *Assume $q \leq 0$ and $r \geq 0$.*

- (1) $H^j(\Gamma, Q_\mathbf{G}^r(p, q)) = 0$ for $1 \leq j \leq n-2$.
- (2) When $n \geq 2$, if $q \geq -n+1$ or $p+r \leq -2$, then $H^{n-1}(\Gamma, Q_\mathbf{G}^r(p, q)) = 0$.

Lemma 5.3. Assume $q \leq 0$.

- (1) $H^j(\Gamma, T_\Gamma(p, q)) = 0$ for $1 \leq j \leq n - 3$.
- (2) When $n \geq 3$, if $q \geq -n + 1$ or $p \leq -2$, then $H^{n-2}(\Gamma, T_\Gamma(p, q)) = 0$.

Lemma 5.4. Assume $q \leq 0$ and $r \geq 0$.

- (1) $H^1(Y_{F,m}, Q_{\mathbf{G}}^r(p, q)|_{Y_{F,m}}) = 0$ for $1 \leq m \leq n - 3$.
- (2) If $q \geq \frac{n^2-7n+8}{2}$ or $p + r \leq (n - 2)d - \frac{n^2-5n+10}{2}$, then

$$H^1(Y_{F,n-2}, Q_{\mathbf{G}}^r(p, q)|_{Y_{F,n-2}}) = 0.$$

Lemma 5.5. Assume $r \geq 0$.

- (1) $H^0(Y_{F,m}, Q_{\mathbf{G}}^r(p, q)|_{Y_{F,m}}) \simeq \text{Ker}(\varepsilon^{r+1} : S_{F,m}^{p,q+r} \rightarrow S_{F,m}^{p+r+1,q-1})$ for $1 \leq m \leq n - 2$.
- (2) If $\min\{q, 0\} \geq \frac{n^2-5n+4}{2}$ or $p + r + \max\{q, 0\} \leq (n - 1)d - \frac{n^2-3n+6}{2}$, then

$$H^0(Y_{F,n-1}, Q_{\mathbf{G}}^r(p, q)|_{Y_{F,n-1}}) \simeq \text{Ker}(\varepsilon^{r+1} : S_{F,n-1}^{p,q+r} \rightarrow S_{F,n-1}^{p+r+1,q-1}).$$

Lemma 5.6. Assume $q \leq 0$.

- (1) $H^1(Y_{F,m}, T_\Gamma(p, q)|_{Y_{F,m}}) = 0$ for $1 \leq m \leq n - 4$.
- (2) If $q \geq \frac{n^2-9n+14}{2}$ or $p \leq (n - 3)d - \frac{n^2-7n+16}{2}$, then

$$H^1(Y_{F,n-3}, T_\Gamma(p, q)|_{Y_{F,n-3}}) = 0.$$

§ 6. The case $n = 3$

In this section, we consider a hypersurface X_F in $\mathbf{P}^3$. Then $Y_{F,1}$ is a $\mathbf{P}^2$ -bundle over X_F . If X_F is a smooth hypersurface, then $Y_{F,2}$ is a $\mathbf{P}^1$ -bundle over X_F . If $d \geq 4$, then $Y_{F,4}$ is a smooth algebraic curve of genus $31d^3 - 158d^2 + 186d + 1$ for general F . If $d \geq 5$, then $\dim Y_{F,5} = 0$ and $\sharp Y_{F,5} = 5d(d - 4)(7d - 12)$ for general F . In the following, we study the variety $Y_{F,3}$. If $Y_{F,3}$ is smooth, then $Y_{F,3}$ is an algebraic surface of the square of the first chern class $c_1^2 = 2d(3d - 8)^2$ and the second chern class $c_2 = 2d(11d^2 - 48d + 54)$.

Proposition 6.1. If the variety $Y_{F,3}$ is smooth of dimension 2, then the morphism $\phi|_{Y_{F,3}} : Y_{F,3} \rightarrow X_F$ is the double covering branched along B_F , where B_F is the divisor on X_F defined by the equation

$$\det \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{0 \leq i, j \leq 3} = 0.$$

By the results in Section 4, we have natural injective homomorphisms

$$\rho : R_{F,3}^{d-2,2} \longrightarrow H^1(Y_{F,2}, T_{Y_{F,2}}(-\log Y_{F,3})),$$

$$\gamma_0 : S_{F,2}^{3d-8,0} \longrightarrow H^0(Y_{F,2}, \Omega_{Y_{F,2}}^3(Y_{F,3}))$$

and

$$\gamma_1 : R_{F,3}^{4d-10,2} \longrightarrow H^1(Y_{F,2}, \Omega_{Y_{F,2}}^2(\log Y_{F,3})).$$

By the similar way, we have a natural surjective homomorphism

$$R_{F,3}^{7d-18,2} \simeq H^1(Y_{F,2}, \Omega_{Y_{F,2}}^3(Y_{F,3}) \otimes \Omega_{Y_{F,2}}^2(\log Y_{F,3})) \longrightarrow H^1(Y_{F,2}, T_{Y_{F,2}}(-\log Y_{F,3}))^\vee.$$

Since the multiplication map

$$S_{F,2}^{3d-8,0} \otimes R_{F,3}^{4d-10,2} \longrightarrow R_{F,3}^{7d-18,2}$$

is surjective, we have the following theorem.

Theorem 6.2. *If $d \geq 3$ and $Y_{F,3}$ is smooth of dimension 2, then the homomorphism*

$$H^1(Y_{F,2}, T_{Y_{F,2}}(-\log Y_{F,3})) \longrightarrow \text{Hom}_{\mathbf{C}}(H^0(Y_{F,2}, \Omega_{Y_{F,2}}^3(Y_{F,3})), H^1(Y_{F,2}, \Omega_{Y_{F,2}}^2(\log Y_{F,3})))$$

is injective.

We consider the period map

$$\psi : M \longrightarrow W; [X_F] \longmapsto [H^3(Y_{F,2} \setminus Y_{F,3})],$$

where M denotes the set of isomorphism classes of hypersurfaces X_F in $\mathbf{P}^3$ such that $Y_{F,3}$ is smooth, and W denotes the set of isomorphism classes of Hodge structures of weight 2. By Theorem 6.2, the differential $d\psi$ of the period map ψ at a general point in M is injective, where we remark that the sets M and W have geometric structure. Now we have a natural question of Torelli type.

Question 6.3. For smooth surfaces X_{F_1} and X_{F_2} in $\mathbf{P}^3$, if there is an isomorphism $H^3(Y_{F_1,2} \setminus Y_{F_1,3}) \simeq H^3(Y_{F_2,2} \setminus Y_{F_2,3})$ as Hodge structures, then is there an isomorphism $X_{F_1} \simeq X_{F_2}$ as algebraic varieties?

§ 6.1. The case $d = 3$

We assume that $d = 3$. If $Y_{F,3}$ is smooth, then $Y_{F,3}$ is a minimal algebraic surface with the geometric genus $p_g = 4$, the irregularity $q = 0$ and the square of the first chern class $c_1^2 = 6$. Such algebraic surfaces are classified by Horikawa, and $Y_{F,3}$ is called of

type Ib in [4]. For $F \in S^{3,0}$, the cubic surface X_F is smooth if and only if $Y_{F,3}$ is a smooth surface. If X_F is a smooth cubic surface, then X_F contains 27 lines, which means that $\sharp Z_F = 27$. Hence $Y_{F,4}$ is a disjoint union of 27 rational curves, which are (-3) -curves in $Y_{F,3}$.

Proposition 6.4. *If X_F is a smooth cubic surface, then B_F has at most nodes as its singularities. A point $p \in X_F$ is a node of B_F if and only if there are three lines in X_F which contains the point p .*

Since the morphism $\phi|_{Y_{F,3}} : Y_{F,3} \rightarrow \mathbf{P}^3$ is the canonical map for $d = 3$, we have the following proposition.

Proposition 6.5. *For smooth cubic surfaces X_{F_1} and X_{F_2} , there is an isomorphism $X_{F_1} \simeq X_{F_2}$ if and only if there is an isomorphism $Y_{F_1,3} \simeq Y_{F_2,3}$.*

In the case when $d = 3$, the Hodge structure $H^2(X_F)$ is trivial, but the Hodge structure $H^3(Y_{F,2} \setminus Y_{F,3})$ is not trivial. Hence the Question 6.3 is particularly interesting in this case.

References

- [1] W. Barth and A. Van de Ven, *Fano-Varieties of lines on hypersurfaces*, Arch. Math. (Basel) **31** (1978), 96–104.
- [2] P. Griffiths, *On the periods of certain rational integrals. I, II*, Ann. of Math. (2) **90** (1969), 460–495, 496–541.
- [3] J. Harris, *Galois groups of enumerative problems*, Duke Math. J. **46** (1979), 685–724.
- [4] E. Horikawa, *Algebraic surface of general type with small c_1^2 . III*, Invent. Math. **47** (1978), 209–248.
- [5] J. Kollár, *Rational curves on algebraic varieties*, Springer-Verlag Berlin Heidelberg (1996).