

The Finite Element Method of Smoothing Quadratic Splines

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1. Introduction

The purpose of this paper is to present a kind of finite element method which is using a B-spline bases of a bivariate spline space. The method shown in this paper can be used to solve the general partial differential equations, such that the solution has the property of C^1 -continuity.

We have used our method to solve the 2-dimensional linear stable electromagnetic field, and the continuity of flux density has been guaranteed.

2. Basic theory

Let D be a polygonal domain in R^2 , and T an arbitrary triangulation of D with M vertices. Each triangle of the triangulation T , says ABC will be subdivided into six smaller triangles shown in Figure 1, where o is any given interior point in the triangle ABC , and the point P , Q , and R are obtained by joining O to those given

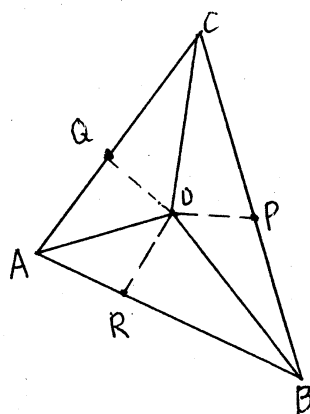


Figure 1.

interior points in triangles of adjacent to the triangle ABC respectively. If some edge of the triangle ABC is a boundary edge of the triangulation T , says bc , then p may be any interior point in the edge BC . The subdivided triangulation of T is denoted by $\tilde{W}$. Denote by $S_2^1(W)$ The bivariate spline space

$$S_2^1(W) = \{S \in C^1(W) \mid \text{the restriction of } S \text{ to each triangle of } W \text{ is a quadratic polynomial}\}$$

Powell and Sabin^([1]) have shown the dimension of bivariate spline

space $S_2^1(W)$:

$$\dim S_2^1(W) = 3M. \quad (1)$$

Furthermore, Wang and others ([2]) have constructed B-spline bases of the space $S_2^1(W)$.

For any given interior vertex, says V_i , of the original triangulation T , we can construct 3 B-splines $[B_i^{(1)}(x,y), B_i^{(2)}(x,y), B_i^{(3)}(x,y)]$ which are supported on the polygonal domain $V_{i1} \dots V_{in_i}$ (see Fig.2), where each of V_{ij} ($j=1,2,\dots,n_i$) is neighbour vertex around V_i in T . It notes that each i_j ($1,\dots,M$). All B-splines $[B_i^{(1)}(x,y), B_i^{(2)}(x,y), B_i^{(3)}(x,y)]_{i=1}^M$ satisfy the following properties ($i,j=1,2,\dots,M$).

$$\begin{aligned} B_i^{(1)}(V_j) &= \delta_{ij}, & B_i^{(2)}(V_j) &= 0, & B_i^{(3)}(V_j) &= 0, \\ \frac{\partial}{\partial X} B_i^{(1)}(V_j) &= 0, & \frac{\partial}{\partial X} B_i^{(2)}(V_j) &= \delta_{ij}, & \frac{\partial}{\partial X} B_i^{(3)}(V_j) &= 0, \\ \frac{\partial}{\partial Y} B_i^{(1)}(V_j) &= 0, & \frac{\partial}{\partial Y} B_i^{(2)}(V_j) &= 0, & \frac{\partial}{\partial Y} B_i^{(3)}(V_j) &= \delta_{ij}. \end{aligned} \quad (2)$$

It is clear that $[B_i^{(1)}, B_i^{(2)}, B_i^{(3)}]_{i=1}^M$ are linear independent, therefore they are the B-spline bases of the space $S_2^1(W)$ ([2]).

According to the property of the B-spline, the values of any function in $S_2^1(W)$ on a certain triangle $V_i V_j V_m$ will only depend on B-splines $[B_t^{(1)}, B_t^{(2)}, B_t^{(3)}]_{t=i,j,m}$. Therefore we only need to show how to represent the B-spline on a triangle $V_i V_j V_m$.

By using the "smoothing cofactor - conformality condition" method proposed by wang ([3]) and a coordinate transformatoin, we can get the representation of the B-spline:

$$\begin{aligned} \bar{B}_{i1}^{(1)}(x,y) &= (\sqrt{2}+1) l_{mj}^2(x,y) \\ \bar{B}_{i1}^{(2)}(x,y) &= K_{i1} l_{mj}^2(x,y) \\ \bar{B}_{i1}^{(3)}(x,y) &= K_{i2} l_{mj}^2(x,y) \\ \bar{B}_{i2}^{(1)}(x,y) &= (\sqrt{2}+1) l_{mj}^2(x,y) + K_{i3} l_{oj}^2(x,y) \\ \bar{B}_{i2}^{(2)}(x,y) &= K_{i1} l_{mj}^2(x,y) + K_{i4} l_{oj}^2(x,y) \\ \bar{B}_{i2}^{(3)}(x,y) &= K_{i1} l_{mj}^2(x,y) + K_{i5} l_{oj}^2(x,y) \end{aligned}$$

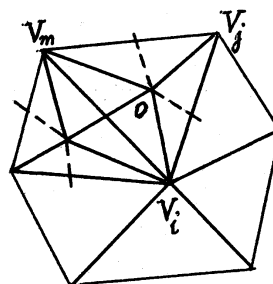


Figure 2.

$$\begin{aligned}
\bar{B}_{i3}^{(1)}(x,y) &= \bar{B}_{i2}^{(1)}(x,y) + K_{i6} l_{op}^2(x,y) \\
\bar{B}_{i3}^{(2)}(x,y) &= \bar{B}_{i2}^{(2)}(x,y) + K_{i7} l_{op}^2(x,y) \\
\bar{B}_{i3}^{(3)}(x,y) &= \bar{B}_{i2}^{(3)}(x,y) + K_{i8} l_{op}^2(x,y) \\
\bar{B}_{i4}^{(1)}(x,y) &= (\sqrt{2}+1) l_{mj}^2(x,y) + K_{i9} l_{mo}^2(x,y) \\
\bar{B}_{i4}^{(2)}(x,y) &= K_{i11} l_{mj}^2(x,y) + K_{i,10} l_{mo}^2(x,y) \\
\bar{B}_{i4}^{(3)}(x,y) &= K_{i12} l_{mj}^2(x,y) + K_{i,11} l_{mo}^2(x,y) \\
\bar{B}_{i5}^{(1)}(x,y) &= \bar{B}_{i4}^{(1)}(x,y) + K_{i6} l_{oq}^2(x,y) \\
\bar{B}_{i5}^{(2)}(x,y) &= \bar{B}_{i4}^{(2)}(x,y) + K_{i7} l_{oq}^2(x,y) \\
\bar{B}_{i5}^{(3)}(x,y) &= \bar{B}_{i4}^{(3)}(x,y) + K_{i,12} l_{oq}^2(x,y)
\end{aligned} \tag{3}$$

where the coefficients $K_{i,t}$ depend on the coordinates of the vertices.

3. Linear stable electromagnet

The problem on linear stable electromagnet is to solve the equation

$$\left\{ \begin{array}{l} D: \frac{\partial}{\partial x}(\beta \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\beta \frac{\partial u}{\partial y}) = -J, \\ S_1: u = u_0, \\ S_2: \beta \frac{\partial u}{\partial n} = -H_t. \end{array} \right. \tag{4}$$

The above problem is equivalent to finding a function u which satisfies the following variational problem

$$\left\{ \begin{array}{l} W(u) = \iint_D \frac{\beta}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy - \iint_D J u dx dy = \text{Min.}, \\ S_1: u = u_0. \end{array} \right. \tag{5}$$

By using the B-spline bases to construct the shape function defined on a triangle element e , we obtain a piecewise quadratic polynomial of satisfying interpolation conditions:

$$\begin{aligned}
u_e(x,y) &= u_{i1} B_i^{(1)}(x,y) + u_{i1} B_i^{(2)}(x,y) + u_{i2} B_i^{(3)}(x,y) \\
&\quad + u_{j1} B_j^{(1)}(x,y) + u_{j1} B_j^{(2)}(x,y) + u_{j2} B_j^{(3)}(x,y) \\
&\quad + u_{m1} B_m^{(1)}(x,y) + u_{m1} B_m^{(2)}(x,y) + u_{m2} B_m^{(3)}(x,y) \\
&= [u] \cdot [B(x,y)]^T,
\end{aligned} \tag{6}$$

where $[u] \cdot [B(x,y)]^T$ denotes the inner product of the vectors u and the transpose of $B(x,y)$.

$$\begin{aligned}
\frac{\partial u_e}{\partial x} &= u_{i1} \frac{\partial}{\partial x} B_i^{(1)}(x,y) + u_{i1} \frac{\partial}{\partial x} B_i^{(2)}(x,y) + u_{i2} \frac{\partial}{\partial x} B_i^{(3)}(x,y) \\
&\quad + u_{j1} \frac{\partial}{\partial x} B_j^{(1)}(x,y) + u_{j1} \frac{\partial}{\partial x} B_j^{(2)}(x,y) + u_{j2} \frac{\partial}{\partial x} B_j^{(3)}(x,y) \\
&\quad + u_{m1} \frac{\partial}{\partial x} B_m^{(1)}(x,y) + u_{m1} \frac{\partial}{\partial x} B_m^{(2)}(x,y) + u_{m2} \frac{\partial}{\partial x} B_m^{(3)}(x,y) \\
&= [u] \cdot \left[\frac{\partial}{\partial x} B(x,y) \right]^T,
\end{aligned} \tag{7}$$

similarly,

$$\frac{\partial u_e}{\partial y} = [u] \cdot \left[\frac{\partial}{\partial y} B(x,y) \right]^T. \tag{8}$$

Hence, the energy functional on the element e will be

$$\begin{aligned}
W_e(u) &= \iint_D \left\{ \frac{\beta}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] - Ju \right\} dx dy \\
&= \iint_D \left\{ \frac{\beta}{2} \cdot [u] \cdot \left(\left[\frac{\partial}{\partial x} B(x,y) \right] \cdot \left[\frac{\partial}{\partial x} B(x,y) \right]^T \right. \right. \\
&\quad \left. \left. + \left[\frac{\partial}{\partial y} B(x,y) \right] \cdot \left[\frac{\partial}{\partial y} B(x,y) \right]^T \right) \cdot [u]^T - J \cdot [u] \cdot [B(x,y)]^T \right\} dx dy.
\end{aligned} \tag{9}$$

According to the variation principle, In order to obtain the solution of (9), we need only to solve

$$\begin{aligned}
[K(B_i^{(1)}, B_j^{(1)}) u_i + K(B_i^{(2)}, B_j^{(1)}) u_{ix} + K(B_i^{(3)}, B_j^{(1)}) u_{iy}] &= p(B_j^{(1)}) \\
j=1,2,3, \quad l=1,2,3
\end{aligned} \tag{10}$$

It is denoted simply by

$$[K]^e [u]^e = [p]^e, \tag{11}$$

where

$$K_{k,l}^{(r,n)} = \iint_D \left[\frac{\partial B_k^{(r)}}{\partial x} \cdot \frac{\partial B_l^{(n)}}{\partial x} + \frac{\partial B_k^{(r)}}{\partial y} \cdot \frac{\partial B_l^{(n)}}{\partial y} \right] dx dy, \quad (12)$$

$$k, l = i, j, m; \quad r, n = 1, 2, 3.$$

$$P_{h,l} = \iint_D J B_h^{(l)}(x, y) dx dy, \quad (13)$$

$$h = i, j, m; \quad l = 1, 2, 3.$$

As a whole, the equations on the B-splines finite element method will be

$$\sum_{l=1}^m [K(B_i^{(1)}, B_j^{(1)}) u_i + K(B_i^{(2)}, B_j^{(1)}) u_{ix} + K(B_i^{(3)}, B_j^{(1)}) u_{iy}] = P(B_j^{(1)}), \quad (14)$$

$$l = 1, 2, 3; \quad j = 1, 2, \dots, n.$$

The matrix of coefficients is

$$[K] = \sum_{e=1}^{n_e} [K]_e \quad (15)$$

To deal with the coercive boundary conditions, we can get a modified equations. By means of the numerical method on linear equations we can find the potential function $u(x, y)$ at each discrete point (x, y) .

The flux density B on the element e is

$$B = \frac{\partial u_e}{\partial y} i + \frac{\partial u_e}{\partial x} j. \quad (16)$$

Substituting (7) and (8) into (16), we have the relation

$$B = [u] \cdot \left[\frac{\partial}{\partial x} B(x, y) \right]^T \cdot j + [u] \cdot \left[\frac{\partial}{\partial y} B(x, y) \right]^T \cdot i \quad (17)$$

Because of $B(x, y)$ is a piecewise quadratic polynomial, it is clear that the flux density B will be continuous.

References:

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