

An introduction to the piecewise algebraic curve

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Abstract

In this paper, we review the recent development of our research on piecewise algebraic curves.

Keywords piecewise algebraic curve, spline function, conformality condition.

1 Introduction

Let us recall the formulation of splines at first. Let D be a bounded polygonal domain of R^2 and we partition D with irreducible algebraic curves into cells $\Delta_i, i = 1, \dots, N$. The partition is denoted by Δ . A function $f(x)$ defined on D is a spline function if $f(x) \in C^r(D)$ and $f(x)|_{\Delta_i} = p_i \in P_k$, which is expressed for short as follows:

$$f(x) \in S_n^r(D, \Delta).$$

In [1] R. H. Wang got the following basic results:

Let Δ_i and Δ_j be two adjacent cells with partitioning curve $l_{ij} = 0$. $f(x) \in c^r(\Delta_i \cup \Delta_j)$ if and only if

$$p_i - p_j = l_{ij}^{r+1} * q_{ij}.$$

where $q_{ij} \in P_{k-(\mu+1)d_{ij}}$ is called a smooth cofactor of the partitioning curve l_{ij} and d_{ij} is the degree of l_{ij} .

Further, $f(x) \in S_n^r(D, \Delta)$, if and only if there exists a smooth cofactor on each interior partitioning curve and

$$\sum_{l_{ij} \in L_k} l_{ij}^{r+1} * q_{ij} \equiv 0.$$

where L_k is the set of partitioning curves sharing the same interior vertex.

Algebraic curve Γ is defined as follows

$$(*) \quad \Gamma = \{(x, y) | p(x, y) = 0, p \in P\}.$$

The so-called piecewise algebraic curve is defined by using the piecewise polynomial or polynomial spline function $s(x, y)$ to replace the polynomial $p(x, y)$ in $(*)$, we have

$$\Gamma = \{(x, y) | s(x, y) = 0\}.$$

Let $\Gamma : s(x, y) = 0$ and $\gamma : t(x, y) = 0$ be two piecewise algebraic curves. γ is called a local branch of Γ , if there exists a union of cells in Δ

$$\Omega = \bigcup \delta_i$$

such that γ is a branch of Γ on Ω .

Why do we have to study piecewise algebraic curves? Let us consider the following interpolation problem: Let $d = \dim S_k^\mu(\Delta)$. How can we choose a set of knots $K = \{(x_i, y_i)\}_{i=1}^d$ such that for any given values $z_1, \dots, z_d$, there exists a unique $s \in S_k^\mu(\Delta)$ satisfying

$$s(x_i, y_i) = z_i, i = 1, \dots, d$$

According to the theory on bivariate spline mentioned above, the interpolation problem is a linear algebraic problem. Therefore there is a unique solution if and only if the linear homogeneous equations

$$s(x_i, y_i) = 0, i = 1, \dots, d$$

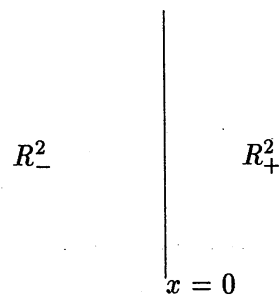
has only a trivial solution, that is, if and only if K does not lie on any piecewise algebraic curve $\Gamma : s(x, y) = 0, s \in S_k^\mu(\Delta)$. Denote by $p_i(x, y) \in P_k$ the polynomial defined by $s(x, y) \in S_k^\mu(\Delta)$ on Δ_i . Because there is the possibility that

$$\{(x, y) | p_i(x, y) = s|_{\Delta_i} = 0\} \cap \overline{\Delta_i} = \emptyset$$

it is difficult to derive the piecewise algebraic curve.

2 Some Examples

Example 1 $D = R^2, \Delta : x = 0, 2$ cells



$$R_-^2 = \{(x, y) \in R^2 : x < 0\}$$

$$R_+^2 = \{(x, y) \in R^2 : x \geq 0\}$$

Define $s \in S_1^0(\Delta)$ as follows

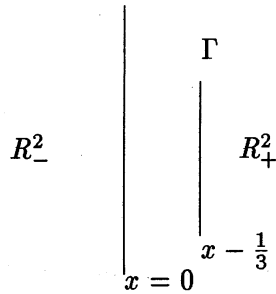
$$s(x, y) = \begin{cases} x - 1 & (x, y) \in R_-^2, \\ -x - 1 & (x, y) \in R_+^2. \end{cases}$$

The piecewise algebraic curve $\Gamma : s(x, y) = 0$ is empty.

Example 2 $s \in S_1^0(\Delta)$ is defined by

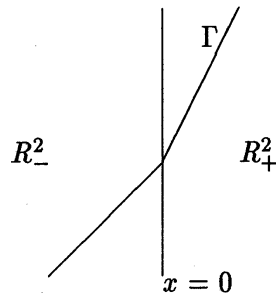
$$s(x, y) = \begin{cases} x - 1 & (x, y) \in R_-^2, \\ 3x - 1 & (x, y) \in R_+^2. \end{cases}$$

The piecewise algebraic curve $\Gamma : s(x, y) = 0$ is $s - \frac{1}{3} = 0$.



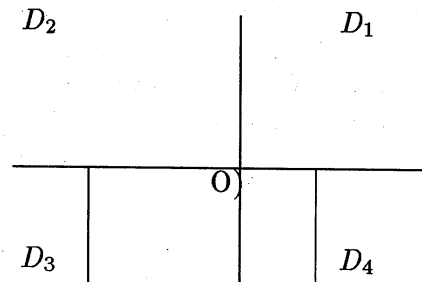
Example 3 $s \in S_1^0(\Delta)$ is defined as follows

$$s(x, y) = \begin{cases} x - y & (x, y) \in R_-^2, \\ 2x - y & (x, y) \in R_+^2. \end{cases}$$



Example 4 $D = R^2, \Delta : x = 0, y = 0, s \in S_2^1(\Delta)$ is defined as follows

$$s(x, y) = \begin{cases} 3x^2 + 3y^2 - 1 & (x, y) \in D_1, \\ x^2 + 3y^2 - 1 & (x, y) \in D_2, \\ x^2 - 1 & (x, y) \in D_3, \\ 3x^2 - 1 & (x, y) \in D_4. \end{cases}$$

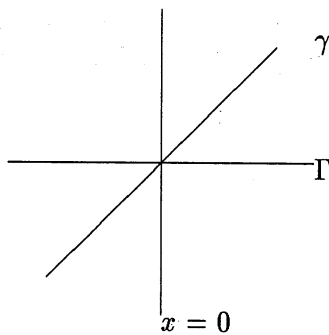


Example 5 $D = R^2, \Delta : x = 0$

$$\Gamma : s(x, y) = xy - y^2 - yx_+ = 0, s \in S_2^0(\Delta)$$

$$\gamma : t(x, y) = x - y = 0, t \in S_1(\Delta)$$

γ is a local branch of Γ on R^2 .



3 Intersection of piecewise algebraic curves

Denote by $Inter(\Gamma_1, \Gamma_2)$ the intersection set of the two piecewise algebraic curves $\Gamma_1 : s_1(x, y) = 0$ and $\Gamma_2 : s_2(x, y) = 0$. The number

$$BN(m_1, r_1; m_2, r_2)$$

$$:= \max\{Card Inter(\Gamma_1, \Gamma_2) < \infty; \Gamma_i : s_i(x, y) = 0, s_i \in S_{m_i}^{r_i}(\Delta), i = 1, 2\}$$

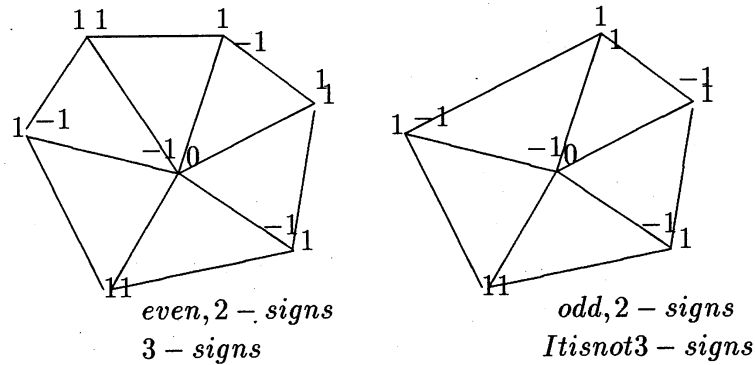
is called the Bezout number of $S_{m_1}^{r_1}$ and $S_{m_2}^{r_2}$. It is obvious that

$$BN(m_1, r_1; m_2, r_2) \leq Nm_1m_2,$$

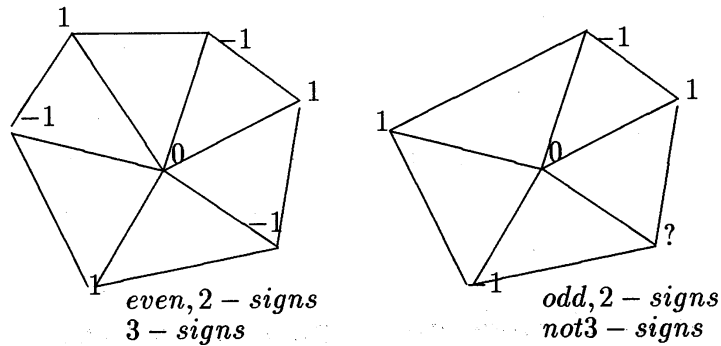
where N is the number of cells in Δ .

X.Q.Shi and R.H.Wang^[3] discussed the Bezout number of $S_m^0(\Delta)$ and $S_n^0(\Delta)$. We find that the Bezout number $BN(m, 0; n, 0)$ depends on some property of the triangulation Δ .

A triangulation Δ is called to be 2-signs, if one can mark -1 or 1 on each vertex of Δ such that the numbers marked on 3 vertices of any cell in Δ are not the same one. A triangulation Δ is called to be 3-signs, if one can mark $-1, 0$ or 1 on each vertex of Δ such that the numbers marked on 3 vertices of any cell in Δ are totally different.



Let v be an interior vertex of Δ . Denote by $d(v)$ the number of boundary vertices of the star $st(v)$. $d(v)$ is called the degree of v . An interior vertex is called to be even(odd) if $d(v)$ is even(odd). A triangulation Δ is called to be even, if all of its interior vertices are even.



Proposition The even triangulation of a simple connected domain is of 3-signs.
X.Q.Shi and R.H.Wang^[3] proved

Theorem 1 If Δ is a triangulation of a simple connected domain, then

- 1° $BN(1, 0; 1, 0) = t$, if Δ is even;
- 2° $BN(1, 0; 1, 0) \leq T - [(V_{\text{odd}} + 2)/3]$, otherwise,

where T is the number of cells in Δ , V_{odd} is the number of odd vertices of Δ , and $[x]$ denotes the maximum integer less than or equal to x .

Denote by $\delta = [v_1, v_2, v_3]$ the triangle with vertices v_1, v_2 and v_3 . Let $f, g \in S_1^0(\Delta)$, and

$$f_i = f(v_i), g_i = g(v_i), i = 1, 2, 3.$$

Then the piecewise algebraic curves $f = 0$ and $g = 0$ can be represented on δ as follows

$$f_1 u_1 + f_2 u_2 + f_3 u_3 = 0, \quad (1)$$

and

$$g_1 u_1 + g_2 u_2 + g_3 u_3 = 0 \quad (2)$$

respectively, where (u_1, u_2, u_3) are the barycentric coordinates of a point v with respect to the triangle δ . Suppose that (u_1^*, u_2^*, u_3^*) are the barycentric coordinates of the intersection point of $f_1 u_1 + f_2 u_2 + f_3 u_3 = 0$ and $g_1 u_1 + g_2 u_2 + g_3 u_3 = 0$, then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = u_1^* \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} + u_2^* \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} + u_3^* \begin{pmatrix} f_3 \\ g_3 \end{pmatrix} \quad (3)$$

$$u_i^* \geq 0, u_1^* + u_2^* + u_3^* = 1$$

Lemma 1 Suppose (1) and (2) have only one intersection point p . Then the point p is an interior point of the triangle $\delta = [v_1, v_2, v_3]$, if and only if the origin $(0, 0)$ is an interior point of triangle $\delta^* = [\omega_1, \omega_2, \omega_3]$, where $\omega_i, i = 1, 2, 3$ are defined by

$$\omega_i = (f_i, g_i),$$

Note: $u_i^* > 0, i = 1, 2, 3$.

Lemma 2 Let v be an interior vertex of the triangulation Δ , and $f = 0, g = 0$ have only finite intersection points on $st(v)$, where $f, g \in S_1^0(\Delta)$. Then $f = 0$ and $g = 0$ have at most N intersection points:

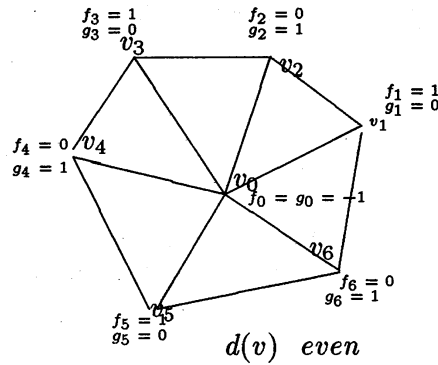
$$N = \begin{cases} d(v), & \text{if } d(v) \text{ is even,} \\ d(v) - 1, & \text{if } d(v) \text{ is odd.} \end{cases}$$

Proof. Assume $d(v) = 2m$, and $v_0, v_1, \dots, v_{2m}$ are the vertices of $st(v)$, where $v_0 = v$. Let ω_0, ω_1 and ω_2 be some points on R^2 such that the origin is an interior point of the triangle $[\omega_0, \omega_1, \omega_2]$, for example,

$$\omega_0 = (-1, -1), \omega_1 = (1, 0), \omega_2 = (0, 1).$$

Now we define two piecewise linear curves $f = 0, g = 0$ on $st(v)$ by using the following values

$$\begin{aligned} (f(v_0), g(v_0)) &= \omega_0 = (-1, -1), \\ (f(v_{2i-1}), g(v_{2i-1})) &= \omega_1 = (1, 0), (i = 1, \dots, m) \\ (f(v_{2i}), g(v_{2i})) &= \omega_2 = (0, 1). \end{aligned}$$



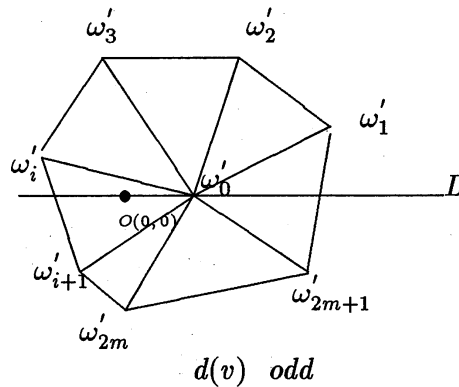
Because the origin is inside the triangle $[\omega_0, \omega_1, \omega_2]$, By Lemma 1, $N = d(v)$.

Now assume $d(v) = 2m + 1$. For two piecewise linear curves $f = 0$ and $g = 0$, suppose

$$\omega'_0 = (f(v_0), g(v_0)), \omega'_i = (f(v_i), g(v_i)),$$

$$i = 1, \dots, 2m + 1, v_0 = v$$

Lemma 1 shows that $f = 0$ and $g = 0$ have an intersection point inside the triangle $[v_0, v_i, v_{i+1}]$ if and only if the origin is inside the triangle $[\omega'_0, \omega'_i, \omega'_{i+1}]$ ($i = 1, \dots, 2m + 1, \omega'_1 = \omega'_{2m+2}$).



By joining the origin and ω'_0 , we obtain a straight line L . According to Lemma 1, two piecewise algebraic curves have a unique intersection point inside the triangle $[\omega'_0, \omega'_i, \omega'_{i+1}]$, if and only if the vertices ω'_i and ω'_{i+1} are located at two different sides of the straight line L . So it is obvious that

$$N \leq d(v) - 1,$$

where $d(v)$ is odd. Moreover, if we take

$$\omega'_0 = (-1, -1), \omega'_{2i} = (0, 1), \omega'_{2i+1} = (1, 0),$$

$i = 1, \dots, m$, then $f = 0$ and $g = 0$ have $2m = d(v) - 1$ intersection points.

The proof of Theorem 1:

Let $f, g \in S_1^0(\Delta)$ be defined by

$$(f(v), g(v)) = \omega_i, v \in \Delta,$$

where v is marked by $i, i = -1, 0, 1$,

$$\omega_{-1} = (-1, -1), \omega_0 = (1, 0) \text{ and } \omega_1 = (0, 1).$$

According to Lemma 1, the piecewise linear curves $f = 0$ and $g = 0$ have just an intersection point in each triangle of Δ , i.e. if Δ is even, then

$$BN(1, 0; 1, 0) = T.$$

Similarly, we can prove 2° in Theorem 1.

Note: One can find some triangulations satisfying

$$BN(1, 0; 1, 0) = T - [(V_{\text{odd}} + 2)/3].$$

Lemma 3 If the triangulation δ is of 2-signs, then

$$BN(1, 0; 2, 0) = 2T.$$

where T is the number of triangles in Δ .

Let $f \in S_1^0(\Delta)$ be defined as follow

$$f(v) = \begin{cases} 1 & \text{if } v \in \Delta \text{ is marked by } 1 \\ -1 & \text{if } v \in \Delta \text{ is marked by } -1 \end{cases} \quad (4)$$

Assuming that $\delta = [v_1, v_2, v_3] \in \Delta$ is a triangle, and $f(u_1, u_2, u_3) = f|_\delta = u_1 + u_2 - u_3$, where (u_1, u_2, u_3) are the barycentric coordinates of $(x, y) \in \delta$ with respect to δ .

Define $g(x, y) \in S_2^0(\Delta)$ by using the following way

$$\begin{aligned} g(x, y)|_\delta &= g(u_1, u_2, u_3) \\ &= u_1^2 + u_2^2 + u_3^2 - \frac{3}{2}(u_1 u_2 + u_2 u_3 + u_3 u_1) \end{aligned} \quad (5)$$

for any $\delta \in \Delta$.

It is no difficult to check that the piecewise algebraic curves $f(u_1, u_2, u_3) = 0$ and $g(u_1, u_2, u_3) = 0$ have two intersection points in δ . So

$$BN(1, 0; 2, 0) = 2T.$$

Lemma 4 If the triangulation Δ is of 2-signs, then

$$BN(1, 0; 3, 0) = 3T,$$

where T is the number of triangles in Δ .

Proof. Let $f \in S_1^0(\Delta)$ be defined as in Lemma 3, and let

$$\begin{aligned} g(u_1, u_2, u_3) &= u_1^3 + u_2^3 + u_3^3 + au_1^2 u_2 + u_2^2 u_3 + u_3^2 u_1 \\ &\quad + bu_1 u_2^2 + u_2 u_3^2 + u_3 u_1^2 + u_1 u_2 u_3. \end{aligned} \quad (6)$$

To find the conditions such that $f(u_1, u_2, u_3) = 0$ and $g(u_1, u_2, u_3) = 0$ have 3 intersection points in the triangle δ , take $u_1 + u_2 = u_3 = \frac{1}{2}$ and consider

$$g(u_1, u_2, \frac{1}{2}) = 0, u_1 + u_2 = \frac{1}{2}.$$

If there are 3 real constants α_1, α_2 and α_3 such that

$$\begin{aligned} g(u_1, u_2, \frac{1}{2}) &= g(u_1, u_2, u_1 + u_2) \\ &= u_1^3 + u_2^3 + 2(u_1 + u_2)^3 + au_1^2u_2 \\ &\quad + (u_1^2 + u_2^2)(u_1 + u_2) + bu_1u_2^2 + u_1u_2(u_1 + u_2) \\ &= 4u_1^3 + 4u_2^3 + (8+a)u_1^2u_2 + (8+b)u_1u_2^2 \\ &= 4(u_1 + \alpha_1u_2)(u_1 + \alpha_2u_2)(u_1 + \alpha_3u_2). \end{aligned}$$

then

$$\begin{cases} a &= 4(\alpha_1 + \alpha_2 + \alpha_3) - 8, \\ b &= 4(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) - 8, \\ \alpha_1\alpha_2\alpha_3 &= 1. \end{cases} \quad (7)$$

Choose $\alpha_1, \alpha_2, \alpha_3 > 0$ satisfying (7). one can obtain a special $g(u_1, u_2, u_3)$ by (7) such that $g(u_1, u_2, u_3) = 0$ and $f(u_1, u_2, u_3) = 0$ have 3 intersection points in the interval $u_1 \in (0, \frac{1}{2}), u_1 + u_2 = \frac{1}{2}$.

This shows that $f(x, y) = 0$ and $\bar{g}(x, y) = 0$ have $3T$ intersection points, where $\bar{g}(x, y)$ is defined by

$$\bar{g}(x, y)|_\delta = g(u_1, u_2, u_3), \forall \delta \in \Delta. \quad (8)$$

Theorem 2 If Δ is a 2-signs triangulation, and $\max\{m, n\} \geq 2$, then the Bezout number of the spaces $S_m^0(\Delta)$ and $S_n^0(\Delta)$ is mnT , i.e.

$$BN(m, 0; n, 0) = nmT,$$

where T is the number of triangles in Δ .

Proof. Let $f(x, y)$ and $g(x, y)$ be defined by (4) and (5), $\bar{g}(x, y)$ be defined as (8), and $m \geq n$.

If $m \geq 2$ is even, we define

$$f_1 = f^n, g_1 = g^{m/2},$$

then

$$f_1(x, y) \in S_n^0(\Delta), g_1(x, y) \in S_m^0(\Delta),$$

moreover, the piecewise algebraic curves $f_1(x, y) = 0, g_1(x, y) = 0$ have mnT intersection points in Δ . This means that

$$BN(m, 0; n, 0) = nmT.$$

If $m \geq 3$ is odd, we define

$$f_1 = f^n, g_1 = g^{\frac{m-3}{2}} \bar{g},$$

then theorem 2 can be also proved.

It seems that the Bezout number depends on whether the triangulation is of 2-signs or not. Based on many examples, however, the following conjecture may be right.

Conjecture Any triangulation is of 2-signs.

For the c^1 - *smoothness* cases

Let $s_i \in S_{m_i}^1(\Delta)$, $i = 1, 2$ be two bivariate splines. We are going to consider the problem on intersection of two piecewise curves $s_1(x, y) = 0$ and $s_2(x, y) = 0$.

A partition Δ of D is called a proper partition, if all angles of the intersection determined by any two adjacent edges of Δ are less than $\pi/2$.

By using the resultant of two bivariate splines s_1 and s_2 with respect to ρ in the polar coordinate (ρ, θ) , R.H.Wang and G.H.Zhao^[4] proved

Theorem 3 Let $\Gamma_i : s_i(x, y) = 0$, $i = 1, 2$ be two piecewise algebraic curves, where $s_i \in S_{m_i}^1(\Delta)$, $i = 1, 2$. For any given interior vertex of Δ , the cardinality Λ of the intersection set δ of Γ_1 and Γ_2 on $st(v)$ is upper bounded by

$$n_i(m_1 m_2 - 1) + 1$$

except that the cardinality of δ is infinite, where n_i is the number of edges passing through v .

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