

Pasting reproducing kernel Hilbert spaces

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Abstract: The aim of this article is to find the necessary and sufficient condition for the mapping $H_K(E) \ni f \mapsto (f|_{E_1}, f|_{E_2}) \in H_{K|_{E_1 \times E_2}}(E_1) \oplus H_{K|_{E_2 \times E_2}}(E_2)$ to be isomorphic, where K is a positive definite function on $E = E_1 + E_2$.

1 Introduction

Let E be a set and $K : E \times E \rightarrow \mathbb{C}$ be a positive definite function. For $f \in H_K(E)$, we can easily check that $f|_{E_0} \in H_{K|_{E_0 \times E_0}}(E_0)$ since

$$f|_{E_0} \otimes f|_{E_0} \ll \|f\|_{H_K(E)}^2 K|_{E_0 \times E_0} \otimes K|_{E_0 \times E_0}$$

in the sense that

$$\begin{aligned} & \sum_{j,k=1,2,\dots,n} (\|f\|_{H_K(E)}^2 K|_{E_0 \times E_0}(p_j, p_k) - f|_{E_0} \otimes f|_{E_0}(p_j, p_k)) z_j \overline{z_k} \\ &= \sum_{j,k=1,2,\dots,n} (\|f\|_{H_K(E)}^2 K(p_j, p_k) - f(p_j, p_k)) z_j \overline{z_k} \geq 0 \end{aligned}$$

for any finite set $\{p_1, p_2, \dots, p_k\} \subset E_0$ and $\{z_1, z_2, \dots, z_k\} \subset \mathbb{C}$. Therefore, when E is partitioned into the sum $E = E_1 + E_2$, the operation the mapping $R : H_K(E) \ni f \mapsto (f|_{E_1}, f|_{E_2}) \in H_{K|_{E_1 \times E_2}}(E_1) \oplus H_{K|_{E_2 \times E_2}}(E_2)$ makes sense. Note that R is injection, since $f|_{E_1} = 0$ and $f|_{E_2} = 0$ imply $f = 0$.

2 Main result

We show the necessary and sufficient condition for R to be isomorphic.

Theorem. *The mapping R is isomorphic if and only if $K|_{E_1 \times E_2} = 0$.*

Proof. Assume first that $K|_{E_1 \times E_2} = 0$. Let us first show that R is surjection. To this end, given $g_1 \in H_{K|_{E_1 \times E_1}}(E_1)$ and $g_2 \in H_{K|_{E_2 \times E_2}}(E_2)$, we define a function f on E by $f(p) = g_1(p)$ on E_1 and $f(p) = g_2(p)$ on E_2 . Let us check that $f \in H_K(E)$. To this end, we set $f_1 = \chi_{E_1} f$ and $f_2 = \chi_{E_2} f$. Then for $l = 1, 2$, we have

$$\begin{aligned} & \sum_{j,k=1,2,\dots,n} (\|f_l\|_{H_K(E)}^2 K(p_j, p_k) - f_l \otimes f_l(p_j, p_k)) z_j \overline{z_k} \\ & \geq \sum_{j,k=1,2,\dots,n, p_j, p_k \in E_l} (\|f_l\|_{H_K(E)}^2 K(p_j, p_k) - f_l \otimes f_l(p_j, p_k)) z_j \overline{z_k} \end{aligned}$$

by assumption. Since $\|f_l\|_{H_K(E)} \geq \|g_l\|_{H_{K|_{E_l \times E_l}}(E_l)}$ from a general result on the reproducing kernel Hilbert spaces, we have

$$\begin{aligned} & \sum_{j,k=1,2,\dots,n} (\|f_l\|_{H_K(E)}^2 K(p_j, p_k) - f_l \otimes f_l(p_j, p_k)) z_j \overline{z_k} \\ & \geq \sum_{j,k=1,2,\dots,n, p_j, p_k \in E_l} (\|g_l\|_{H_{K|_{E_l \times E_l}}(E_l)}^2 K(p_j, p_k) - g_l \otimes g_l(p_j, p_k)) z_j \overline{z_k} \geq 0. \end{aligned}$$

Thus, $f_l \in H_{K_l}(E_l)$ as was to be shown.

It remains to show that R is an isomorphism. In fact, $\{K(\cdot, p)\}_{p \in E_l}$ is a dense subspace in $H_{K|_{E_l \times E_l}}(E_l)$, we have only to show that

$$\begin{aligned} & \left(\left\| \sum_{m=1}^L (z_1^m K(\cdot, p_1^m) + z_2^m K(\cdot, p_2^m)) \right\|_{H_K(E)} \right)^2 \\ & = \left(\left\| \sum_{m=1}^L z_1^m K(\cdot, p_1^m) \right\|_{H_{K|_{E_1 \times E_1}}(E_1)} \right)^2 + \left(\left\| \sum_{m=1}^L z_2^m K(\cdot, p_2^m) \right\|_{H_{K|_{E_2 \times E_2}}(E_2)} \right)^2 \end{aligned}$$

for any $p_1^m \in E_1, p_2^m \in E_2, z_1^m, z_2^m \in \mathbb{C}$ with $m = 1, 2, \dots, L$.

Conversely, if R is an isomorphism, then

$$\begin{aligned} & \left(\|K(\cdot, p_1) + zK(\cdot, p_2)\|_{H_K(E)} \right)^2 \\ &= \left(\|K(\cdot, p_1)\|_{H_{K|E_1 \times E_1}(E_1)} \right)^2 + |z|^2 \left(\|K(\cdot, p_2)\|_{H_{K|E_2 \times E_2}(E_2)} \right)^2 \end{aligned}$$

for any $p_1 \in E_1$ and $p_2 \in E_2$ and $z \in \mathbb{C}$. Thus $K|E_1 \times E_2 = 0$. $\square$

This result is essentially based on [1, 2], however, the representation is arranged in the polished version.

References

- [1] Y. Sawano, *Pasting reproducing kernel Hilbert spaces*, Jaen Journal on Approximations, **3** (2011), no. 1, 135–141.
- [2] S. Saitoh and Y. Sawano, *Theory of Reproducing Kernels and Applications*, to appear in Springer.

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