

On Terwilliger algebras with respect to subsets in Hamming graphs and Johnson graphs

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In this talk, we determine irreducible modules of the Terwilliger algebra of a Q -polynomial distance-regular graph Γ with respect to a subset with a special condition. Here we focus on the case where Γ is the Johnson graph. We construct irreducible modules of the Terwilliger algebra of Γ from those of binary Hamming graphs. This is a joint work with Hajime Tanaka.

1 Width and dual width

Let Γ be a Q -polynomial distance-regular graph of diameter D with vertex set X . We refer the reader to [1], [2] for terminology and background materials on Q -polynomial distance-regular graphs. Let C be a nonempty subset of X . Let $\chi \in \mathbb{C}^X$ be the *characteristic vector* of C , i.e.,

$$(\chi)_x = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_0, \dots, A_D$ be distance matrices of Γ . We write $A = A_1$. Let $E_0, \dots, E_D$ be primitive idempotents of Γ . Brouwer, Godsil, Koolen and Martin [3] introduced two parameters of C . The *width* w of C is defined as

$$w = \max\{i \mid \chi^T A_i \chi \neq 0\}.$$

Dually, the *dual width* w^* of C is defined as

$$w^* = \max\{i \mid \chi^T E_i \chi \neq 0\}.$$

We can verify that $w = \max\{\partial(x, y) \mid x, y \in C\}$, i.e., the maximum distance between two vertices in C . Obviously, $w = 0$ if and only if $C = \{x\}$ ($x \in X$). The following fundamental bound holds.

Theorem 1 [3]

$$w + w^* \geq D.$$

When the above bound is attained, Brouwer et.al. showed that some good properties hold:

Theorem 2 [3] *Suppose $w + w^* = D$. Then*

- (i) C is completely regular.
- (ii) C induces a Q -polynomial distance-regular graph whenever C is connected.

Recently, Tanaka proved the following:

Theorem 3 [8] *Suppose $w + w^* = D$. Then*

- (i) C induces a Q -polynomial distance-regular graph whenever $q \neq -1$.
- (ii) C is convex if and only if Γ has classical parameters.

The subsets with $w + w^* = D$ were classified for some Q -polynomial distance-regular graphs (see [3], [8]). Our current goal is to characterize Q -polynomial distance-regular graphs having subsets with $w + w^* = D$ in terms of Terwilliger algebras. We will see the definitions and basic terminology on Terwilliger algebras in the next section.

2 Terwilliger algebras and modules

Let $C \subset X$. Let $\Gamma_i(C) = \{x \in X \mid \partial(x, C) = i\}$, i.e., the i th subconstituent of Γ with respect to C . We define the diagonal matrix $E_i^* \in \text{Mat}_X(C)$ so that

$$(E_i^*)_{xx} = \begin{cases} 1 & \text{if } x \in \Gamma_i(C), \\ 0 & \text{otherwise.} \end{cases}$$

The *Terwilliger algebra* $\mathcal{T}(C)$ of Γ with respect to C is defined as follows:

$$\mathcal{T}(C) = \langle A, E_0^*, \dots, E_D^* \rangle \subset \text{Mat}_X(C).$$

It is known that $\mathcal{T}(C)$ is semisimple, and non-commutative in general. If we set $C = \{x\}$ ($x \in X$), then $\mathcal{T}(C)$ is identical to the ordinary Terwilliger algebra $\mathcal{T}(x)$ or the *subconstituent algebra* introduced by Terwilliger [10]. Suzuki generalized the theory of subconstituent algebras to the case associated with subsets [6].

Let $W \subset C^X$ be an irreducible $\mathcal{T}(C)$ -module. There are two types of decompositions of W into subspaces which are invariant under the action of E_i^* and E_i respectively:

$$W = E_0^*W + \dots + E_D^*W \quad (\text{direct sum}),$$

$$W = E_0W + \dots + E_DW \quad (\text{direct sum}).$$

We define parameters for W to describe isomorphism classes of irreducible modules; The *endpoint* ν of W is defined as $\nu = \min\{i \mid E_i^*W \neq 0\}$, and the *dual endpoint* μ of W is $\mu = \min\{i \mid E_iW \neq 0\}$. The *diameter* of W is defined as $d = |\{i \mid E_i^*W \neq 0\}| - 1$. W is called *thin* if $\dim E_i^*W \leq 1$ for all i .

Suppose C satisfies $w + w^* = D$. We have a preceding result on irreducible modules of endpoint 0:

Theorem 4 [5] *Suppose C satisfies $w + w^* = D$. Let W be an irreducible $T(C)$ -module of endpoint $\nu = 0$. Then W is thin with $d = w^*$.*

Our primary goal is to determine irreducible $T(C)$ -modules of arbitrary endpoint ν . In this article, we discuss the case of Johnson graphs.

3 Johnson graphs

Definition 3.1 *The binary Hamming graph $\tilde{\Gamma} = H(N, 2)$ ($N \geq 2D$) has vertex set*

$$\tilde{X} = \{(\overbrace{x_1 \cdots x_N}^N) \mid x_i \in \{0, 1\}\},$$

i.e., the set of binary words of length N , and two vertices $x, y \in \tilde{X}$ are adjacent if x and y differ in exactly 1 coordinate.

Definition 3.2 *The Johnson graph $\Gamma = J(N, D)$ has vertex set*

$$X = \tilde{\Gamma}_D(\mathbf{0}) = \{(x_1 \cdots x_N) \in \tilde{X} \mid (\# \text{ of } 1\text{s}) = D\},$$

i.e., the set of binary words of length N and weight D , and two vertices $x, y \in X$ are adjacent if x and y differ in exactly 2 coordinates.

Theorem 5 [3] *Let $\Gamma = J(N, D)$ and $C \subset X$. Suppose C satisfies $w + w^* = D$. Then*

$$C \cong \{(\overbrace{1 \cdots 1}^{w^*} \overbrace{* \cdots *}^{N-w^*}) \mid (\# \text{ of } 1\text{s}) = D\},$$

i.e., the induced subgraph on C is isomorphic to the Johnson graph $J(N - w^, D - w^*)$.*

Let $C = \{(\overbrace{1 \cdots 1}^{w^*} \overbrace{* \cdots *}^{N-w^*}) \mid (\# \text{ of } 1\text{s}) = D\}$, and $\Gamma^{(1)} = H(w^*, 2)$, $\Gamma^{(2)} = H(N - w^*, 2)$. Then

$$C = \Gamma_{w^*}^{(1)}(\mathbf{0}) \times \Gamma_w^{(2)}(\mathbf{0}),$$

and we also have

$$\Gamma_i(C) = \Gamma_{w^*-i}^{(1)}(\mathbf{0}) \times \Gamma_{w+i}^{(2)}(\mathbf{0}).$$

Let $\mathcal{T}_1(\mathbf{0})$ be the Terwilliger algebra of $H(w^*, 2)$ with respect to $\mathbf{0}$, where $\mathbf{0}$ denotes the all zero word, and $\mathcal{T}_2(\mathbf{0})$ the Terwilliger algebra of $H(N - w^*, 2)$ with respect to $\mathbf{0}$. Let $\mathcal{T}(C)$ be the Terwilliger algebra of $J(N, D)$ with respect to C . Let $\tilde{X}$ denote the vertex set of $H(N, 2)$. Recall that the vertex set X of $J(N, D)$ is a subset of $\tilde{X}$. For a subset $\mathcal{A}$ of $\text{Mat}_{\tilde{X}}(C)$, let $\mathcal{A}|_{X \times X} \subset \text{Mat}_X(C)$ denote the set of principal submatrices of matrices in $\mathcal{A}$. The following is the key lemma.

Lemma 6

$$\mathcal{T}(C) \subseteq \mathcal{T}_1(\mathbf{0}) \otimes \mathcal{T}_2(\mathbf{0})|_{X \times X} \quad (\subset \text{Mat}_X(C))$$

Let W_i be an irreducible $\mathcal{T}_i(\mathbf{0})$ -module ($i = 1, 2$). Let

$$W := W_1 \otimes W_2|_X \subset C^X,$$

where the right hand side denotes the set of vectors from $W_1 \otimes W_2$ whose indices are restricted on X . Then

Lemma 7 *W is a $\mathcal{T}(C)$ -module.*

Go [4] gave an explicit description of W_1, W_2 . We will make use of results in [4] for the characterization of W .

Lemma 8 *Let $\mathcal{B}_1, \mathcal{B}_2$ be standard bases for W_1, W_2 (see [4]). Then*

- (i) $\mathcal{B} := \{u \otimes u' \mid u \in \mathcal{B}_1, u' \in \mathcal{B}_2, u \otimes u'|_X \neq 0\}$ is a basis for W .
- (ii) $\text{Span}\{u \otimes u'\} = E_i^* W$ for some i .
- (iii) W is thin.

We can determine the endpoint of W by comparing supports of W_1 and W_2 . For determination of the dual endpoint of W , the following will be useful:

Proposition 9 [11] *Let $\mathcal{T}(\mathbf{0})$ be the Terwilliger algebra of the binary Hamming graph $H(N, 2)$ with respect to $\mathbf{0}$. Let U be an irreducible $\mathcal{T}(\mathbf{0})$ -module of endpoint r . Then $v(\neq \mathbf{0}) \in U|_X$ is an eigenvector of $J(N, D)$ for eigenvalue θ_r .*

Next we will check that W is irreducible. To see that it is so, we consider a tridiagonal matrix. Let $[A]_{\mathcal{B}}$ be the matrix representing A with respect to the basis $\mathcal{B}$. Then $[A]_{\mathcal{B}}$ is tridiagonal since W is thin. Moreover, by calculation we can verify that the off-diagonal entries of $[A]_{\mathcal{B}}$ are nonzero. Hence we have the following:

Lemma 10 *W is an irreducible $\mathcal{T}(C)$ -module.*

4 Main results

Let $\Gamma = J(N, D)$ and $C \subset X$. Suppose C satisfies $w + w^* = D$. Let $\mathcal{T}(C)$ be the Terwilliger algebra of Γ with respect to C . Let W be an irreducible $\mathcal{T}(C)$ -module of endpoint ν , dual endpoint μ , diameter d .

Theorem 11 *There exist integers e, f satisfying*

$$\begin{aligned} 0 \leq e \leq \left\lfloor \frac{w^*}{2} \right\rfloor, \quad 0 \leq f \leq \left\lfloor \frac{N - w^*}{2} \right\rfloor, \\ \nu = \max\{e, f - w\}, \quad \mu = e + f, \\ d = \begin{cases} w^* - 2\nu & \text{if } \nu = e, \\ \min\{D - \mu, N - D - 2\nu - w\} & \text{if } \nu = f - w. \end{cases} \end{aligned}$$

Remarks. e, f comes from endpoints of W_1, W_2 .

Remarks. If $N \neq 2D$, then e, f are uniquely determined for given ν, μ, d . In this case,

$T(C) = T_1 \otimes T_2|_{X \times X}$ in Lemma 6.

Theorem 12 *W has a basis $\mathcal{B} = \{v_0, \dots, v_d\}$ satisfying*

$$v_i \in E_{i+\nu}^* W \quad (0 \leq i \leq d),$$

and with respect to which the matrix representing A is tridiagonal with entries

$$\begin{aligned} c_i(W) &= i(i + 2\nu - \mu + w), \\ a_i(W) &= D(N - D) + \mu(\mu + d - N - 1) + d(d - N \\ &\quad + 2\nu + w) + i(N - 4\nu - 2i - 2w), \\ b_i(W) &= (d - i)(N - d - 2\nu - \mu - i - w). \end{aligned}$$

Remarks. $c_i(W) + a_i(W) + b_i(W) = \theta_\mu$.

Remarks. If $w = 0$, the above $c_i(W), a_i(W), b_i(W)$ coincide with the results by Terwilliger [10].

Corollary 13 *Isomorphism classes are determined by (ν, μ, d) .*

5 Remark

Let $A^* = \text{diag}(E_1 \chi)$. Then (A, A^*) acts on W as a *Leonard pair* with parameter array $(h, r, s, s^*, r, d, \theta_0, \theta_0^*)$ (*Dual Hahn*):

$$\begin{aligned} \theta_i &= \theta_0 + hi(i + 1 + s), \\ \theta_i^* &= \theta_0^* + s^*i, \\ \varphi_i &= hs^*i(i - d - 1)(i + r), \\ \phi_i &= hs^*i(i - d - 1)(i + r - s - d - 1). \end{aligned}$$

Especially, we have

$$s = -N - 2 + 2\mu,$$

$$r = -N + d + 2\nu + \mu - 1 + w.$$

See [9] for details on Leonard pairs. If $w = 0$, the above parameters coincide with the results by Terwilliger [10].

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