

Cardinal invariants associated with the combinatorics of the uniformity number of the ideal of meager-additive sets

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Abstract

In [CMRM24], it was proved that it is relatively consistent that *bounding number* $\mathfrak{b}$ is smaller than the uniformity of $\mathcal{MA}$, where $\mathcal{MA}$ denotes ideal of the meager-additive sets of 2^ω . To prove this, it was introduced certain cardinal invariant, which we call $\mathfrak{b}_b^{\text{eq}}$ regarding closely to Bartoszyński's and Judah's characterization of uniformity of $\mathcal{MA}$. In this survey, we will study this cardinal invariant and its dual (we call $\mathfrak{d}_b^{\text{eq}}$). In particular, we show its relation with the cardinals in Cichoń's diagram. Additionally, we present a number of open problems regarding these cardinals.

1 Introduction and preliminaries

We first review our terminology. Let $\mathcal{I}$ be an ideal of subsets of X such that $\{x\} \in \mathcal{I}$ for all $x \in X$. Throughout this paper, we demand that all ideals satisfy this latter requirement. We introduce the following four *cardinal invariants associated with $\mathcal{I}$* :

$$\begin{aligned} \text{add}(\mathcal{I}) &= \min \left\{ |\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} \notin \mathcal{I} \right\}, \\ \text{cov}(\mathcal{I}) &= \min \left\{ |\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} = X \right\}, \\ \text{non}(\mathcal{I}) &= \min \{ |A| : A \subseteq X, A \notin \mathcal{I} \}, \text{ and} \\ \text{cof}(\mathcal{I}) &= \min \{ |\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \forall A \in \mathcal{I} \exists B \in \mathcal{J} : A \subseteq B \}. \end{aligned}$$

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These cardinals are referred to as the *additivity*, *covering*, *uniformity* and *cofinality* of $\mathcal{I}$, respectively. The relationship between the cardinals defined above is illustrated in Figure 1.

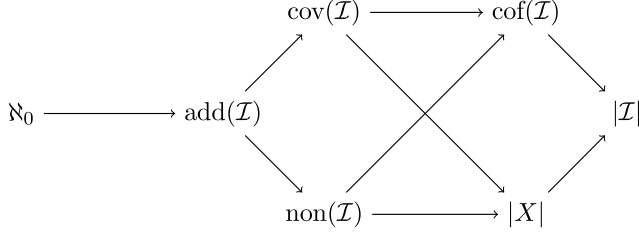


Figure 1: Diagram of the cardinal invariants associated with $\mathcal{I}$. An arrow $\mathfrak{x} \rightarrow \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$.

For $f, g \in \omega^\omega$ define

$$f \leq^* g \text{ iff } \exists m < \omega \forall n \geq m: f(n) \leq g(n).$$

Let

$$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega \text{ and } \neg \exists y \in \omega^\omega \forall x \in F: x \leq^* y\}$$

the *bounding number*, and let

$$\mathfrak{d} := \min\{|D| : D \subseteq \omega^\omega \text{ and } \forall x \in \omega^\omega \exists y \in D: x \leq^* y\}$$

the *dominating number*. As usual, $\mathfrak{c} := 2^\omega$ denotes the *size of the continuum*.

Definition 1.1. Let $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ be an ideal.

- (1) We say that $\mathcal{I}$ is *translation invariant*¹ if $A + x \in \mathcal{I}$ for each $A \in \mathcal{I}$ and $x \in 2^\omega$.
- (2) A set $X \subseteq 2^\omega$ is termed *$\mathcal{I}$ -additive* if, for every $A \in \mathcal{I}$, $A + X \in \mathcal{I}$. Denote by $\mathcal{IA}$ the collection of the $\mathcal{I}$ -additive subsets of 2^ω . Notice that $\mathcal{IA}$ is a (σ) -ideal and $\mathcal{IA} \subseteq \mathcal{I}$ when $\mathcal{I}$ is a translation invariant (σ) -ideal.

When $\mathcal{I}$ is either $\mathcal{M}$ or $\mathcal{N}$, the ideal $\mathcal{IA}$ has attracted a lot of attention. Bartoszyński and Judah [BJ94], Pawlikowski [Paw85], Shelah [She95], Zindulka [Zin19] and the author, Mejía, and Rivera-Madrid [CMRM24], for example, were among the many who looked into them.

Denote by $\mathbb{I}$ the set of partitions of ω into finite non-empty intervals.

Theorem 1.2 ([She95, Thm. 13]). *Let $X \subseteq 2^\omega$. Then $X \in \mathcal{NA}$ iff for all $I = \langle I_n : n \in \omega \rangle \in \mathbb{I}$ there is some $\varphi \in \prod_{n \in \omega} \mathcal{P}(2^{I_n})$ such that $\forall n \in \omega: |\varphi(n)| \leq n$ and $X \subseteq H_\varphi$, where*

$$H_\varphi := \{x \in 2^\omega : \forall^\infty n \in \omega: x \upharpoonright I_n \in \varphi(n)\}.$$

¹This paper considers the Cantor space 2^ω as a topological group with the standard modulo 2 coordinatewise addition.

The following lemma is an immediate consequence of [Definition 1.1](#).

Lemma 1.3 ([[CMRM24](#), Lem. 1.3]). *For any translation invariant ideal $\mathcal{I}$ on 2^ω , we have:*

- (1) $\text{add}(\mathcal{I}) \leq \text{add}(\mathcal{IA})$.
- (2) $\text{non}(\mathcal{IA}) \leq \text{non}(\mathcal{I})$.

The cardinal $\text{non}(\mathcal{IA})$ has been studied in [[Paw85](#), [Kra02](#)] under the different name *transitive additivity of $\mathcal{I}$* .²

$$\text{add}_t^*(\mathcal{I}) = \min\{|X| : X \subseteq 2^\omega \text{ and } \exists A \in \mathcal{I} : A + X \notin \mathcal{I}\}.$$

It is clear from the definition that $\text{non}(\mathcal{IA}) = \text{add}_t^*(\mathcal{I})$.

Pawlikowski [[Paw85](#)] characterized $\text{add}_t^*(\mathcal{N})$ (i.e. $\text{non}(\mathcal{NA})$) employing slaloms.

Definition 1.4. Given a sequence of non-empty sets $b = \langle b(n) : n \in \omega \rangle$ and $h : \omega \rightarrow \omega$, define

$$\prod b := \prod_{n \in \omega} b(n), \text{ and}$$

$$\mathcal{S}(b, h) := \prod_{n \in \omega} [b(n)]^{\leq h(n)}.$$

For two functions $x \in \prod b$ and $\varphi \in \mathcal{S}(b, h)$ write

$$x \in^* \varphi \text{ iff } \forall^\infty n \in \omega : x(n) \in \varphi(n).$$

Theorem 1.5 ([[Paw85](#), Lem. 2.2], see also [[CM23](#), Thm. 8.3]). *$\text{non}(\mathcal{NA})$ is the size of the minimal bounded family $F \subseteq \omega^\omega$ such that*

$$\forall \varphi \in \mathcal{S}(b, \text{id}_\omega) \exists x \in F : x \notin^* \varphi.$$

Stated differently, the uniformity of $\mathcal{NA}$ can be described using localization cardinals as follows.

For b and h as in [Definition 1.4](#), define

$$\mathfrak{b}_{b,h}^{\text{Lc}} := \min \left\{ |F| : F \subseteq \prod b \text{ and } \neg \exists \varphi \in \mathcal{S}(b, h) \forall x \in F : x \in^* \varphi \right\},$$

and set $\text{minLc} := \min\{\mathfrak{b}_{b, \text{id}_\omega}^{\text{Lc}} : b \in \omega^\omega\}$. Here, id_ω denotes the identity function on ω .

Hence, we obtain that $\text{non}(\mathcal{NA}) = \text{minLc}$. Another characterization of minLc is the following.

Lemma 1.6 ([[CM19](#), Lemma 3.8]). *$\text{minLc} = \min\{\mathfrak{b}_{b,h}^{\text{Lc}} : b \in \omega^\omega\}$ when h goes to infinity.*

Hence, we can infer:

²In [[BJ95](#)] is denoted by $\text{add}^*(\mathcal{I})$.

Corollary 1.7. $\text{non}(\mathcal{NA}) = \min\{\mathfrak{b}_{b,h}^{\text{Lc}} : b \in \omega^\omega\}$ when h goes to infinity.

Moreover, it recently was proved that

Lemma 1.8 ([CMRM24, Thm. A]). $\text{non}(\mathcal{NA}) = \text{add}(\mathcal{NA})$.

The characterization of $\text{add}(\mathcal{N})$ by Pawlikowski can be expressed as follows as a direct result of the previous result:

Theorem 1.9 ([Paw85, Lem. 2.3]). $\text{add}(\mathcal{N}) = \min\{\mathfrak{b}, \text{add}(\mathcal{NA})\}$.

We below focus on the σ -ideal of meager-additive sets and its uniformity. Just as in [Theorem 1.2](#), we have one characterization for $\mathcal{MA}$ due to Bartoszyński and Judah

Theorem 1.10 ([BJ94, Thm. 2.2]). *Let $X \subseteq 2^\omega$. Then $X \in \mathcal{MA}$ iff for all $I \in \mathbb{I}$ there are $J \in \mathbb{I}$ and $y \in 2^\omega$ such that*

$$\forall x \in X \forall^\infty n < \omega \exists k < \omega : I_k \subseteq J_n \text{ and } x \upharpoonright I_k = y \upharpoonright I_k.$$

Furthermore, Shelah [She95, Thm. 18] proved that J can be found coarser than I , i.e., all members of J are the union of members of I

They also established a characterization of the uniformity of the meager-additive ideal:

Theorem 1.11 ([BJ94, Thm. 2.2], see also [BJ95, Thm. 2.7.14]).

The cardinal $\text{non}(\mathcal{MA})$ is the largest cardinal κ such that, for every bounded family $F \subseteq \omega^\omega$ of size $< \kappa$,

$$(\clubsuit) \quad \exists r, h \in \omega^\omega \forall f \in F \exists n \in \omega \forall m \geq n \exists k \in [r(m), r(m+1)]: f(k) = h(k).$$

We below introduce two cardinal invariants motivated by $(\clubsuit)$, which were introduced by by the author along with Mejía and Rivera-Madrid in [CMRM24].

Definition 1.12. Let $b \in \omega^\omega$. For $I \in \mathbb{I}$, and for $f, h \in \prod b$, define

$$f \sqsubset^\bullet (I, h) \text{ iff } \forall^\infty n \in \omega \exists k \in I_n : f(k) = h(k).$$

We define the following cardinal invariants associated with $\sqsubset^\bullet$.

$$\mathfrak{b}_b^{\text{eq}} := \min\{|F| : F \subseteq \prod b \text{ and } \neg \exists I \in \mathbb{I} \exists h \in \prod b \forall f \in F : f \sqsubset^\bullet (I, h)\}$$

and

$$\mathfrak{d}_b^{\text{eq}} := \min\{|D| : D \subseteq \mathbb{I} \times \prod b \text{ and } \forall f \in \prod b \exists (I, h) \in D : f \sqsubset^\bullet (I, h)\}.$$

The study of uniformity of $\mathcal{MA}$ was better understood due to these cardinals, which for instance, were utilized by the author along with Mejía and Rivera-Madrid [CMRM24] to prove the consistency of $\text{non}(\mathcal{MA}) > \mathfrak{b}$ and $\text{cov}(\mathcal{MA}) < \text{non}(\mathcal{N})$.

It also turns out that [Theorem 1.11](#) can be reformulated as

$$(\spadesuit) \quad \text{non}(\mathcal{MA}) = \min\{\mathfrak{b}_b^{\text{eq}} : b \in \omega^\omega\}.$$

To be thorough, we provide a proof of $(\spadesuit)$ (see [Lemma 2.8](#) and [Lemma 2.9](#)).

This survey aims to study the cardinals invariants $\mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}_b^{\text{eq}}$, so one of the goal of this article is to establish:

Theorem A. *The following relations in Figure 2 hold, where $\mathfrak{x} \rightarrow \mathfrak{y}$ means $\mathfrak{x} \leq \mathfrak{y}$.*

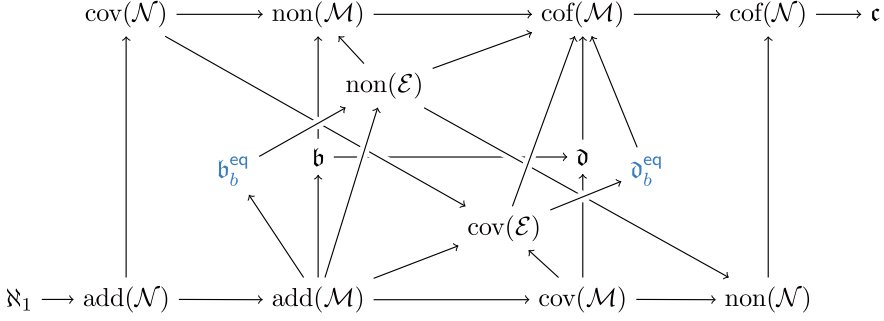


Figure 2: Including $\mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}_b^{\text{eq}}$ to Cichoń's diagram.

Theorem B. *The following relations in Figure 3 hold, where $\mathfrak{x} \rightarrow \mathfrak{y}$ means $\mathfrak{x} \leq \mathfrak{y}$.*

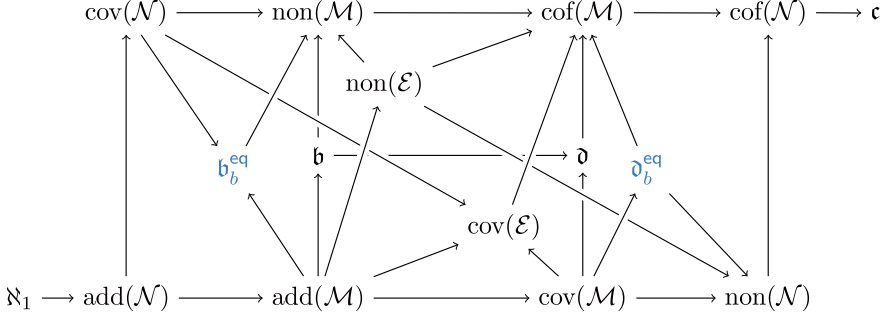


Figure 3: Including $\mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}_b^{\text{eq}}$ to Cichoń's diagram. Additionally, if $\sum_{k < \omega} \frac{1}{b(k)} = \infty$ then $\text{cov}(\mathcal{N}) \leq \mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}_b^{\text{eq}} \leq \text{non}(\mathcal{N})$.

Theorem A-B will be proved in Section 2. In Section 3, we present one forcing notion closely related to the $\mathfrak{b}_b^{\text{eq}}$, which we call $\mathbb{P}_b$ and illustrate the effect of iterating $\mathbb{P}_b$ on Cichoń's diagram. In addition, we prove the following:

Theorem C (Theorem 3.23). *Let $\aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ be regular cardinals, and assume λ_5 is a cardinal such that $\lambda_5 \geq \lambda_4$, $\lambda_5 = \lambda_5^{\aleph_0}$ and $\text{cf}([\lambda_5]^{<\lambda_i}) = \lambda_5$ for $i = 1, \dots, 3$. Then, we can construct a FS iteration of length λ_5 of ccc posets forcing Figure 4.*

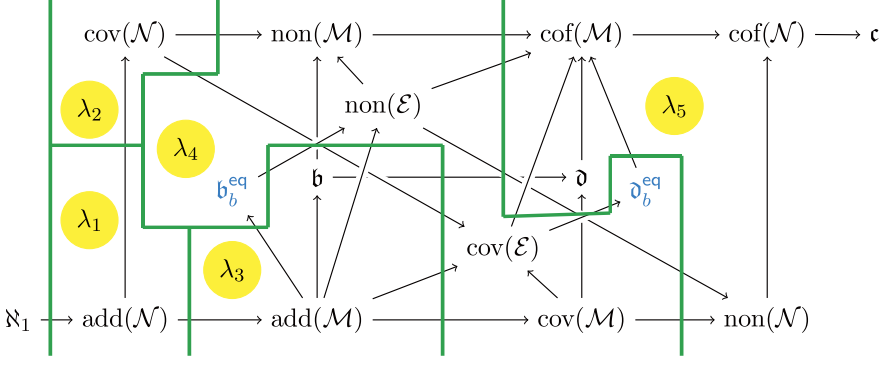


Figure 4: The constellation of Cichoń's diagram forced in [Theorem C](#).

We close this section by reviewing everything needed to develop our targets.

Definition 1.13. We say that $R = \langle X, Y, \sqsubset \rangle$ is a *relational system* if it consists of two non-empty sets X and Y and a relation $\sqsubset$.

(1) A set $F \subseteq X$ is *R-bounded* if $\exists y \in Y \forall x \in F: x \sqsubset y$.

(2) A set $E \subseteq Y$ is *R-dominating* if $\forall x \in X \exists y \in E: x \sqsubset y$.

We associate two cardinal invariants with this relational system R :

$\mathfrak{b}(R) := \min\{|F| : F \subseteq X \text{ is } R\text{-unbounded}\}$ the *unbounding number* of R , and

$\mathfrak{d}(R) := \min\{|D| : D \subseteq Y \text{ is } R\text{-dominating}\}$ the *dominating number* of R .

Note that $\mathfrak{d}(R) = 1$ iff $\mathfrak{b}(R)$ is undefined (i.e. there are no R -unbounded sets, which is the same as saying that X is R -bounded). Dually, $\mathfrak{b}(R) = 1$ iff $\mathfrak{d}(R)$ is undefined (i.e. there are no R -dominating families).

Directed preorders provide a very representative broad example of relational systems.

Definition 1.14. We say that $\langle S, \leq_S \rangle$ is a *directed preorder* if it is a preorder (i.e. $\leq_S$ is a reflexive and transitive relation on S) such that

$$\forall x, y \in S \exists z \in S: x \leq_S z \text{ and } y \leq_S z.$$

A directed preorder $\langle S, \leq_S \rangle$ is seen as the relational system $S = \langle S, S, \leq_S \rangle$, and its associated cardinal invariants are denoted by $\mathfrak{b}(S)$ and $\mathfrak{d}(S)$. The cardinal $\mathfrak{d}(S)$ is actually the *cofinality* of S , typically denoted by $\text{cof}(S)$ or $\text{cf}(S)$.

Example 1.15. Define the following relation on $\mathbb{I}$:

$$I \sqsubseteq J \text{ iff } \forall^\infty n < \omega \exists m < \omega: I_m \subseteq J_n.$$

Note that $\sqsubseteq$ is a directed preorder on $\mathbb{I}$, so we think of $\mathbb{I}$ as the relational system with the relation $\sqsubseteq$. In Blass [Bla10], it is proved that $\mathbb{I} \cong_{\text{T}} \omega^\omega$. Hence, $\mathfrak{b} = \mathfrak{b}(\mathbb{I})$ and $\mathfrak{d} = \mathfrak{d}(\mathbb{I})$.

Example 1.16. We consider the following relational systems for any ideal $\mathcal{I}$ on X .

(1) $\mathcal{I} := \langle \mathcal{I}, \subseteq \rangle$ is a directed partial order. Note that

$$\begin{aligned}\mathfrak{b}(\mathcal{I}) &= \text{add}(\mathcal{I}) \\ \mathfrak{d}(\mathcal{I}) &= \text{cof}(\mathcal{I})\end{aligned}$$

(2) $\mathbf{C}_{\mathcal{I}} := \langle X, \mathcal{I}, \in \rangle$. When $\bigcup \mathcal{I} = X$,

$$\begin{aligned}\mathfrak{b}(\mathbf{C}_{\mathcal{I}}) &= \text{non}(\mathcal{I}) \\ \mathfrak{d}(\mathbf{C}_{\mathcal{I}}) &= \text{cov}(\mathcal{I})\end{aligned}$$

Example 1.17. For $b \in \omega^\omega$ define the relational system $\mathbf{R}_b := \langle \prod b, \mathbb{I} \times \prod b, \sqsubset^\bullet \rangle$. Notice that

(1) $\mathbf{R}_b \cong_{\mathbf{T}} \langle \prod b, \mathbb{I} \times \omega^\omega, \sqsubset^\bullet \rangle$. Then $\mathfrak{b}(\mathbf{R}_b) = \mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}(\mathbf{R}_b) = \mathfrak{d}_b^{\text{eq}}$.

(2) If $b' \in \omega^\omega$ and $b \leq^* b'$, then $\mathbf{R}_b \preceq_{\mathbf{T}} \mathbf{R}_{b'}$. In particular, $\mathfrak{b}_{b'}^{\text{eq}} \leq \mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}_{b'}^{\text{eq}} \leq \mathfrak{d}_b^{\text{eq}}$.

Remark 1.18. If $b \not\leq^* 2$ then we can find some $(I, h) \in \mathbb{I} \times \prod b$ such that $f \sqsubset^\bullet (I, h)$ for all $f \in \prod b$, so $\mathfrak{d}(\mathbf{R}_b) = 1$ and $\mathfrak{b}(\mathbf{R}_b)$ is undefined.

We now review the products of relational systems.

Definition 1.19. Let $\bar{\mathbf{R}} = \langle \mathbf{R}_i : i \in K \rangle$ be a sequence of relational systems $\mathbf{R}_i = \langle X_i, Y_i, \sqsubset_i \rangle$. Define $\prod \bar{\mathbf{R}} = \prod_{i \in K} \mathbf{R}_i := \langle \prod_{i \in K} X_i, \prod_{i \in K} Y_i, \sqsubset^\times \rangle$ where $x \sqsubset^\times y$ iff $x_i \sqsubset_i y_i$ for all $i \in K$.

For two relational systems $\mathbf{R}$ and $\mathbf{R}'$, write $\mathbf{R} \times \mathbf{R}'$ to denote their product, and when $\mathbf{R}_i = \mathbf{R}$ for all $i \in K$, we write $\mathbf{R}^K := \prod \bar{\mathbf{R}}$.

Fact 1.20 ([CM25]). Let $\bar{\mathbf{R}}$ be as in [Definition 1.19](#). Then $\sup_{i \in K} \mathfrak{d}(\mathbf{R}_i) \leq \mathfrak{d}(\prod \bar{\mathbf{R}}) \leq \prod_{i \in K} \mathfrak{d}(\mathbf{R}_i)$ and $\mathfrak{b}(\prod \bar{\mathbf{R}}) = \min_{i \in K} \mathfrak{b}(\mathbf{R}_i)$.

We use the composition of relational systems to prove [Lemma 2.5](#).

Definition 1.21 ([Bla10, Sec. 4]). Let $\mathbf{R}_e = \langle X_e, Y_e, \sqsubset_e \rangle$ be a relational system for $e \in \{0, 1\}$. The *composition of $\mathbf{R}_0$ with $\mathbf{R}_1$* is defined by $(\mathbf{R}_0; \mathbf{R}_1) := \langle X_0 \times X_1^{Y_0}, Y_0 \times Y_1, \sqsubset_* \rangle$ where

$$(x, f) \sqsubset_* (y, b) \text{ iff } x \sqsubset_0 y \text{ and } f(y) \sqsubset_1 b.$$

Fact 1.22. Let $\mathbf{R}_i$ be a relational system for $i < 3$. If $\mathbf{R}_0 \preceq_{\mathbf{T}} \mathbf{R}_1$, then $\mathbf{R}_0 \preceq_{\mathbf{T}} \mathbf{R}_1 \times \mathbf{R}_2 \preceq_{\mathbf{T}} (\mathbf{R}_1, \mathbf{R}_2)$ and $\mathbf{R}_1 \times \mathbf{R}_2 \cong_{\mathbf{T}} \mathbf{R}_2 \times \mathbf{R}_1$.

The following theorem describes the effect of the composition on cardinal invariants.

Theorem 1.23 ([Bla10, Thm. 4.10]). Let $\mathbf{R}_e$ be a relational system for $e \in \{0, 1\}$. Then $\mathfrak{b}(\mathbf{R}_0; \mathbf{R}_1) = \min\{\mathfrak{b}(\mathbf{R}_0), \mathfrak{b}(\mathbf{R}_1)\}$ and $\mathfrak{d}(\mathbf{R}_0; \mathbf{R}_1) = \mathfrak{d}(\mathbf{R}_0) \cdot \mathfrak{d}(\mathbf{R}_1)$.

Instead of dealing with all meager sets, we will consider a suitably chosen cofinal family below.

Definition 1.24. Let $I \in \mathbb{I}$ and let $x \in 2^\omega$. Define

$$B_{x,I} := \{y \in 2^\omega : \forall^\infty n \in \omega : y \upharpoonright I_n \neq x \upharpoonright I_n\}.$$

For $n \in \omega$, define

$$B_{x,I}^n := \{y \in 2^\omega : \forall m \geq n : x \upharpoonright I_m \neq y \upharpoonright I_m\}.$$

Then $B_{x,I}^m \subseteq B_{x,I}^n$ whenever $m < n < \omega$. Thus, $B_{x,I} = \bigcup_{n \in \omega} B_{x,I}^n$.

Denote by B_I the set $B_{0,I} = \{y \in 2^\omega : \forall^\infty n \in \omega : y \upharpoonright I_n \neq 0\}$.

A pair $(x, I) \in 2^\omega \times \mathbb{I}$ is known as a *chopped real*, and these are used to produce a cofinal family of meager sets. It is clear that $B_{x,I}$ is a meager subset of 2^ω (see, e.g. [Bla10]).

Theorem 1.25 (Talagrand [Tal80], see also e.g. [BJS93, Prop. 13]). *For each meager set $F \subseteq 2^\omega$ and $I \in \mathbb{I}$ there are $x \in 2^\omega$ and $I' \in \mathbb{I}$ such that $F \subseteq B_{I',x}$ and each I'_n is the union of finitely many I_k 's.*

Lemma 1.26 ([BJS93, Prop 9]). *For $x, y \in 2^\omega$ and for $I, J \in \mathbb{I}$, the following statements are equivalent:*

- (1) $B_{I,x} \subseteq B_{J,y}$.
- (2) $\forall^\infty n < \omega \exists k < \omega : I_k \subseteq J_n$ and $x \upharpoonright I_k = y \upharpoonright I_k$.

Definition 1.27. Given a sequence $b = \langle b(n) : n \in \omega \rangle$ of non-empty sets, denote

$$\text{seq}_{<\omega} b := \bigcup_{n < \omega} \prod_{i < n} b(i).$$

For each $\sigma \in \text{seq}_{<\omega}(b)$ define

$$[s] := [s]_b := \{x \in \prod b : s \subseteq x\}.$$

As a topological space, $\prod b$ has the product topology with each $b(n)$ endowed with the discrete topology. Note that $\{[s]_b : s \in \text{seq}_{<\omega} b\}$ forms a base of clopen sets for this topology. When each $b(n)$ is countable we have that $\prod b$ is a Polish space and, in addition, if $|b(n)| \geq 2$ for infinitely many n , then $\prod b$ is a perfect Polish space. The most relevant instances are:

- The Cantor space 2^ω , when $b(n) = 2$ for all n , and
- The Baire space ω^ω , when $b(n) = \omega$ for all n .

We now review the Lebesgue measure on $\prod b$ when each $b(n) \leq \omega$ is an ordinal. For any ordinal $0 < \eta \leq \omega$, the probability measure μ_η on the power set of η is defined by:

- when $\eta = n < \omega$, μ_n is the measure such that, for all $k < n$, $\mu_n(\{k\}) = \frac{1}{n}$, and
- when $\eta = \omega$, μ_ω is the measure such that, for $k < \omega$, $\mu_\omega(\{k\}) = \frac{1}{2^{k+1}}$.

Denote by $\mathbf{Lb}_b$ the product measure of $\langle \mu_{b(n)} : n < \omega \rangle$, which we call *the Lebesgue Measure on $\prod b$* , so $\mathbf{Lb}_b$ is a probability measure on the Borel σ -algebra of $\prod b$. More concretely, $\mathbf{Lb}_b$ is the unique measure on the Borel σ -algebra such that, for any $s \in \text{seq}_{<\omega} b$, $\mathbf{Lb}_b([s]) = \prod_{i < |s|} \mu_{b(i)}(\{s(i)\})$. In particular, denote by $\mathbf{Lb}$, $\mathbf{Lb}_2$ and $\mathbf{Lb}_\omega$ the Lebesgue measure on $\mathbb{R}$, on 2^ω , and on ω^ω , respectively.

Let X be a topological space. Denote by $\mathcal{M}(X)$ the collection of all meager subsets of X , and let $\mathcal{M} := \mathcal{M}(\mathbb{R})$. If X is a perfect Polish space, then $\mathcal{M}(X) \cong_{\text{T}} \mathcal{M}(\mathbb{R})$ and $\text{Cv}_{\mathcal{M}(X)} \cong_{\text{T}} \text{Cv}_{\mathcal{M}(\mathbb{R})}$ (see [Kec95, Ex. 8.32 & Thm. 15.10]). Therefore, the cardinal invariants associated with the meager ideal are independent of the perfect Polish space used to calculate it. When the space is clear from the context, we write $\mathcal{M}$ for the meager ideal.

On the other hand, denote by $\mathcal{B}(X)$ the σ -algebra of Borel subsets of X , and assume that $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ is a σ -finite measure such that $\mu(X) > 0$ and every singleton has measure zero. Denote by $\mathcal{N}(\mu)$ the ideal generated by the μ -measure zero sets, which is also denoted by $\mathcal{N}(X)$ when the measure on X is clear. Then $\mathcal{N}(\mu) \cong_{\text{T}} \mathcal{N}(\mathbf{Lb})$ and $\text{Cv}_{\mathcal{N}(\mu)} \cong_{\text{T}} \text{Cv}_{\mathcal{N}(\mathbf{Lb})}$ where $\mathbf{Lb}$ is the Lebesgue measure on $\mathbb{R}$ (see [Kec95, Thm. 17.41]). Therefore, the four cardinal invariants associated with both measure zero ideals are the same. When $b = \langle b(n) : n < \omega \rangle$, each $b(n) \leq \omega$ is a non-zero ordinal, and $\prod b$ is perfect, we have that $\mathbf{Lb}_b$ satisfies the properties of μ above. When the measure space is understood, we just write $\mathcal{N}$ for the null ideal.

Definition 1.28. For b as above, denote by $\mathcal{E}(\prod b)$ the ideal generated by the F_σ measure zero subsets of $\prod b$. Likewise, define $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}([0, 1])$. When $\prod b$ is perfect, the map $F_b: \prod b \rightarrow [0, 1]$ defined by

$$F_b(x) := \sum_{n < \omega} \frac{x(n)}{\prod_{i \leq n} b(i)}$$

is a continuous onto function, and it preserves measure. Hence, this map preserves sets between $\mathcal{E}(\prod b)$ and $\mathcal{E}([0, 1])$ via images and pre-images. Therefore, $\mathcal{E}(\prod b) \cong_{\text{T}} \mathcal{E}([0, 1])$ and $\text{Cv}_{\mathcal{E}(\prod b)} \cong_{\text{T}} \text{Cv}_{\mathcal{E}([0, 1])}$. We also have $\mathcal{E}(\mathbb{R}) \cong_{\text{T}} \mathcal{E}([0, 1])$ and $\text{Cv}_{\mathbb{R}} \cong_{\text{T}} \text{Cv}_{\mathcal{E}([0, 1])}$, as well as $\mathcal{E}(\omega^\omega) \cong_{\text{T}} \mathcal{E}(2^\omega)$ and $\text{Cv}_{\mathcal{E}(\omega^\omega)} \cong_{\text{T}} \text{Cv}_{\mathcal{E}(2^\omega)}$.

When the space is clear, we write $\mathcal{E}$. Therefore, the cardinal invariants of $\mathcal{E}$ do not depend on the previous spaces.

2 ZFC results

This section aims to display the new arrows that appear in Cichon's diagram. All of the contents in this section are taken almost verbatim from [CMRM24, Sec. 2].

Lemma 2.1. $\text{Cv}_{\mathcal{M}} \preceq_{\text{T}} \mathbf{R}_b$ whenever $b \geq^* 2$. In particular, $\mathbf{b}_b^{\text{eq}} \leq \text{non}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq \mathfrak{d}_b^{\text{eq}}$.

Proof. We work with $\mathcal{M}(\prod b)$ instead of $\mathcal{M}$ (see Section 1). Let $F: \prod b \rightarrow \prod b$ be the

identity function and define $G: \mathbb{I} \times \prod b \rightarrow \mathcal{M}(\prod b)$ as follows.

$$\begin{aligned} G: \mathbb{I} \times \prod b &\rightarrow \mathcal{M}(\prod b) \\ (J, h) &\mapsto \{x \in \prod b : x \sqsubset^\bullet (J, h)\} \end{aligned}$$

Observe that $\{x \in \prod b : x \sqsubset^\bullet (J, h)\} \in \mathcal{M}(\prod b)$, since

$$\{x \in \prod b : x \sqsubset^\bullet (J, h)\} = \bigcup_{m < \omega} \bigcap_{n \geq m} \bigcup_{k \in J_n} A_k^{h(k)},$$

where $A_k^\ell := \{x \in \prod b : x(k) = \ell\}$ for $\ell < b(k)$, and each A_k^ℓ is clopen. In fact, it is F_σ -set. It is clear that if $x \sqsubset^\bullet (J, h)$, then $x \in \{x \in \prod b : x \sqsubset^\bullet (J, h)\}$. $\square$

We below present connections between R_b and measure zero.

Lemma 2.2. *Let $b \in \omega^\omega$.*

- (1) *If $\sum_{k < \omega} \frac{1}{b(k)} < \infty$ then $\text{Cv}_\mathcal{E} \preceq_T R_b$. In particular, $\mathfrak{b}_b^{\text{eq}} \leq \text{non}(\mathcal{E})$ and $\text{cov}(\mathcal{E}) \leq \mathfrak{d}_b^{\text{eq}}$.*
- (2) *If $\sum_{k < \omega} \frac{1}{b(k)} = \infty$. Then $\text{Cv}_\mathcal{N} \preceq_T R_b^\perp$. As a consequence, $\text{cov}(\mathcal{N}) \leq \mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}_b^{\text{eq}} \leq \text{non}(\mathcal{N})$.*

Proof. To prove (1)–(2), we work with $\mathcal{N}(\prod b)$ instead of $\mathcal{N}$ (see Section 1).

(1): Observe that

$$\text{Lb}_b([s]_b) = \text{Lb}\left(\{x \in \prod b : s \subseteq x\}\right) = \prod_{i < |s|} \frac{1}{|b(i)|}$$

for any $s \in \text{seq}_{<\omega} b$. Let F and G be as in Lemma 2.1. To complete the proof it suffices to prove

$$\text{Lb}_b(\{x \in \prod b : x \sqsubset^\bullet (J, h)\}) = 0.$$

Recall

$$\{x \in \prod b : x \sqsubset^\bullet (J, h)\} = \bigcup_{m < \omega} \bigcap_{n \geq m} \bigcup_{k \in J_n} A_k^{h(k)}.$$

Notice that $\text{Lb}_b(A_k^\ell) = \frac{1}{b(k)}$, so we obtain

$$\text{Lb}_b\left(\{x \in \prod b : x \sqsubset^\bullet (J, h)\}\right) \leq \lim_{m \rightarrow \infty} \prod_{n \geq m} \sum_{k \in J_n} \frac{1}{b(k)}.$$

This limit above is 0 because $\sum_{k < \omega} \frac{1}{b(k)} < \infty$.

(2): Since $\sum_{k < \omega} \frac{1}{b(k)} = \infty$, find $J \in \mathbb{I}$ such that $\sum_{k \in J_n} \frac{1}{b(k)} \geq n$ for all $n < \omega$. Observe that

$$\prod b \setminus \{x \in \prod b : x \sqsubset^\bullet (J, h)\} = \bigcap_{m < \omega} \bigcup_{n \geq m} \bigcap_{k \in J_n} (\prod b \setminus A_k^{h(k)}),$$

so

$$\begin{aligned} \text{Lb}_b \left(\prod b \setminus \{x \in \prod b : x \sqsubset^\bullet (J, h)\} \right) &\leq \lim_{m \rightarrow \infty} \sum_{n \geq m} \prod_{k \in J_n} \left(1 - \frac{1}{b(k)} \right) \\ &\leq \lim_{m \rightarrow \infty} \sum_{n \geq m} e^{-\sum_{k \in J_n} \frac{1}{b(k)}}. \end{aligned}$$

Since $\sum_{k \in J_n} \frac{1}{b(k)} \geq n$, $\text{Lb}_b (\prod b \setminus \{x \in \prod b : x \sqsubset^\bullet (J, h)\}) = 0$.

Now we define the functions Ψ_- and Ψ_+ as follows.

$$\begin{aligned} \Psi_- : \prod b &\longrightarrow \mathbb{I} \times \prod b \text{ and} \\ \Psi_+ : \prod b &\longmapsto \mathcal{N}(\prod b) \end{aligned}$$

for each $h \in \prod b$ and $x \in \prod b$ by the assignments

$$\begin{aligned} \Psi_- : x &\longmapsto (J, x) \\ \Psi_+ : h &\longmapsto \prod b \setminus \{x \in \prod b : x \sqsubset^\bullet (J, h)\} \end{aligned}$$

It is not hard to see that for any $x \in \prod b$ and for any $h \in \prod b$, if $h \not\sqsubset^\bullet (J, x)$ then $x \in \prod b \setminus \{x \in \prod b : x \sqsubset^\bullet (J, h)\}$. $\square$

We introduce the following relational system for combinatorial purposes.

Definition 2.3. Let $b := \langle b(n) : n < \omega \rangle$ be a sequence of non-empty sets. Define the relational system $\text{Ed}_b := \langle \prod b, \prod b, \neq^\infty \rangle$ where $x =^\infty y$ means $x(n) = y(n)$ for infinitely many n . The relation $x \neq^\infty y$ means that x and y are eventually different. Denote $\mathfrak{b}_{b,1}^{\text{aLc}} := \mathfrak{b}(\text{Ed}_b)$ and $\mathfrak{d}_{b,1}^{\text{aLc}} := \mathfrak{d}(\text{Ed}_b)$.

Recall the following characterization of the cardinal invariants associated with $\mathcal{M}$. The one for $\text{add}(\mathcal{M})$ is due to Miller [Mil81].

Theorem 2.4 ([CM19, Sec. 3.3]).

$$\text{add}(\mathcal{M}) = \min(\{\mathfrak{b}\} \cup \{\mathfrak{d}_{b,1}^{\text{aLc}} : b \in \omega^\omega\}) \text{ and } \text{cof}(\mathcal{M}) = \sup(\{\mathfrak{d}\} \cup \{\mathfrak{b}_{b,1}^{\text{aLc}} : b \in \omega^\omega\})$$

Following, we are establishing a connection between R_b and $(\text{Ed}_b^\perp, \mathbb{I})$.

Lemma 2.5. For $b \in \omega^\omega$, $\text{R}_b \preceq_{\text{T}} (\text{Ed}_b^\perp, \mathbb{I})$. As a consequence, $\mathfrak{d}_b^{\text{eq}} \leq \max\{\mathfrak{b}_{b,1}^{\text{aLc}}, \mathfrak{d}\}$ and $\min\{\mathfrak{d}_{b,1}^{\text{aLc}}, \mathfrak{b}\} \leq \mathfrak{b}_b^{\text{eq}}$.

Proof. Define $\Psi_- : \prod b \rightarrow \prod b \times \mathbb{I}^{\prod b}$ by $\Psi_-(x) := (x, F_x)$ where, for $y \in \mathbb{I}$, if $y =^\infty x$ then $F_x(y) := I_x^y \in \mathbb{I}$ is chosen such that $\forall k < \omega \exists i \in I_{x,k}^y : y(i) = x(i)$; otherwise, $F_x(y)$ can be anything (in $\mathbb{I}$).

Define $\Psi_+ : \prod b \times \mathbb{I} \rightarrow \mathbb{I} \times \prod b$ by $\Psi_+(y, J) = (J, y)$. We check that (Ψ_-, Ψ_+) is a Tukey connection. Assume that $x, y \in \prod b$, $J \in \mathbb{I}$ and that $\Psi_-(x) \sqsubset_* (y, J)$, i.e. $x =^\infty y$ and $I_x^y \sqsubseteq J$. Since each $I_{x,k}^y$ contains a point where x and y coincide, $I_x^y \sqsubseteq J$ implies that, for all but finitely many $n < \omega$, J_n contains a point where x and y coincide, which means that $x \sqsubset^\bullet (J, y) = \Psi_+(y, J)$. $\square$

[Theorem 2.4](#) and [Lemma 2.5](#) together yield:

Corollary 2.6. *For all $b \in \omega^\omega$, $\mathfrak{d}_b^{\text{eq}} \leq \text{cof}(\mathcal{M})$.*

Note that $\text{add}(\mathcal{M}) \leq \min\{\mathfrak{b}_b^{\text{eq}} : b \in \omega^\omega\}$ already follows from [Lemma 1.3](#) and [\(2.1\)](#).

We close this section with the proof of [\(2.2\)](#). Before, a natural question regarding [\(2.3\)](#) that arises is

Question 2.7. *Does $\text{cov}(\mathcal{MA}) = \sup\{\mathfrak{d}_b^{\text{eq}} : b \in \omega^\omega\}$ hold?*

One negative answer to the prior question was given by the author along with Mejía and Rivera-Madrid [[CMRM24](#)]. Concretely, they proved the consistency of

$$\sup\{\mathfrak{d}_b^{\text{eq}} : b \in \omega^\omega\} < \text{cov}(\mathcal{MA}).$$

We prove [\(2.4\)](#) by using the subsequent two lemmas.

Lemma 2.8. *Let $b \in \omega^\omega$. Then $R_b \preceq_T \text{Cv}_{\mathcal{MA}}$. As a consequence,*

$$\text{non}(\mathcal{MA}) \leq \min\{\mathfrak{b}_b^{\text{eq}} : b \in \omega^\omega\} \text{ and } \sup\{\mathfrak{d}_b^{\text{eq}} : b \in \omega^\omega\} \leq \text{cov}(\mathcal{MA}).$$

Proof. Given $b \in \omega^\omega$, thanks to [Example 1.17](#) we may assume that there is some $I^b \in \mathbb{I}$ such that $b(n) = 2^{|I_n^b|}$. Then, we can identify numbers $< b(n)$ with 0-1 sequences of length $|I_n^b|$. We will find maps $\Psi_- : \prod b \rightarrow 2^\omega$ and $\Psi_+ : \mathcal{MA} \rightarrow \mathbb{I} \times \prod b$ such that, for any $f \in \prod b$ and for any $X \in \mathcal{MA}$, $\Psi_-(f) \in X$ implies $f \sqsubset^\bullet \Psi_+(X)$.

Define $\Psi_- : \prod b \rightarrow 2^\omega$ as follows.

$$\begin{aligned} \Psi_- : \prod b &\rightarrow 2^\omega \\ x &\mapsto x_f^{I^b} = \underbrace{f(0)}_{\text{length } |I_0^b|} \frown \dots \frown \underbrace{f(n)}_{\text{length } |I_n^b|} \frown \dots \end{aligned}$$

For $X \in \mathcal{MA}$, $X + B_{I^b} \in \mathcal{M}$. Note that

$$X + B_{I^b} = \bigcup_{x \in X} B_{x, I^b}.$$

Then, by [Theorem 1.25](#), there are $y \in 2^\omega$ and $J \in \mathbb{I}$ such that

$$\bigcup_{x \in X} B_{x, I^b} \subseteq B_{y, J}.$$

Let $h \in \prod b$ such that $y = x_h^{I^b}$ (recall that $b(n) = 2^{|I_n^b|}$), so put $\Psi_+(X) := (J', h)$ where

$$k \in J'_n \text{ iff } \min J_n < \max I_k^b \leq \max J_n.$$

It remains to prove that, for any $f \in \prod b$ and for any $X \in \mathcal{MA}$, $\Psi_-(f) \in X$ implies $f \sqsubset^\bullet \Psi_+(X)$. Suppose that $x_f^{I^b} \in X$ and $\Psi_+(X) = (J', h)$. Then $B_{x_f^{I^b}, I^b} \subseteq B_{x_h^{I^b}, J'}$. Hence, by using [Lemma 1.26](#),

$$\forall^\infty n \exists k : I_k^b \subseteq J_n \text{ and } x_f^{I^b} \upharpoonright I_k^b = x_h^{I^b} \upharpoonright I_k^b.$$

Since $I_k^b \subseteq J_n$ implies $k \in J'_n$, the equation above implies that $f \sqsubset^\bullet (J', h)$. □

Lemma 2.9. *For any dominating family $D \subseteq \omega^\omega$, $\mathbf{C}_{\mathcal{MA}} \preceq_T \prod_{b \in D} \mathbf{R}_b$. In particular, $\min_{b \in D} \mathfrak{b}_b^{\text{eq}} \leq \text{non}(\mathcal{MA})$ and $\text{cov}(\mathcal{MA}) \leq \prod_{b \in D} \mathfrak{d}_b^{\text{eq}}$.*

Proof. Without loss of generality, we may assume that there is some $\mathbb{I}$ -dominating family D_0 , i.e. $\forall I \in \mathbb{I} \exists J \in D_0: I \subseteq J$, such that for each $b \in D$ there is some $I \in D_0$ such that $b = 2^I$, i.e. $b(n) = 2^{I_n}$ for all $n < \omega$.

Define $\Psi_-: 2^\omega \rightarrow \prod_{I \in D_0} 2^I$ by $\Psi_-(x)(I) := \langle x \upharpoonright I_n : n < \omega \rangle$; and define $\Psi_+: \prod_{I \in D_0} \mathbb{I} \times 2^I \rightarrow \mathcal{MA}$ such that, for $z = \langle (J^I, z^I) : I \in \mathbb{I} \rangle$,

$$\Psi_+(z) := \{x \in 2^\omega : \forall I \in D_0 \forall^\infty n < \omega \exists k \in J_k^I : x \upharpoonright I_k = z^I(k)\}.$$

For each $I \in D_0$ let $I'_n := \bigcup_{k \in J_n^I} I_k$ and $y^I \in 2^\omega$ the concatenation of all the $z^I(k) \in 2^{I_k}$ for $k < \omega$, i.e., $y^I \upharpoonright I_k = z^I(k)$. Then $I' := \langle I'_n : n < \omega \rangle \in \mathbb{I}$, $I \subseteq I'$ and

$$\forall^\infty n < \omega \exists k < \omega : I_k \subseteq I'_n \text{ and } x \upharpoonright I_k = y \upharpoonright I_k.$$

Therefore, by [Theorem 1.10](#), $\Psi_+(z) \in \mathcal{MA}$. (Ψ_-, Ψ_+) is clearly the required Tukey connection. $\square$

3 Consistent results

The main goal of this section is to establish [Theorem C](#), which is based on [\[CMRM24, Sec. 3\]](#).

We now present one forcing notion closely related to $\mathfrak{b}_b^{\text{eq}}$, that is to say, that increases $\mathfrak{b}_b^{\text{eq}}$.

Definition 3.1 ([\[CMRM24, Def. 3.20\]](#)). Given $b \in \omega^\omega$, the poset $\mathbb{P}_b$ is defined as follows: A condition $p = (s, t, F) \in \mathbb{P}_b$ if it fulfills the following:

- $s \in \omega^{<\omega}$ is increasing with $s(0) > 0$ (when $|s| > 0$),
- $t \in \text{seq}_{<\omega}(b) := \bigcup_{n < \omega} \prod_{i < n} b(i)$, and
- $F \in [\prod b]^{<\aleph_0}$.

We order $\mathbb{P}_b$ by setting $(s', t', F') \leq (s, t, F)$ iff $s \subseteq s'$, $t \subseteq t'$, $F \subseteq F'$ and,

$$\forall f \in F \forall n \in |s'| \setminus |s| \exists k \in [s'(n-1), s'(n)) : f(k) = t'(k). \text{ (Here } s'(-1) := 0.)$$

Fact 3.2. *Let $b \in \omega^\omega$. Then $\mathbb{P}_b$ is σ -centered.*

Proof. For $s \in \omega^{<\omega}$ increasing, and for $t \in \text{seq}_{<\omega}(b)$, set

$$P_{s,t} := \{(s', t', F) \in \mathbb{P}_b : s' = s \text{ and } t' = t\}$$

It is not hard to verify that $P_{s,t}$ is centered and $\bigcup_{s \in \omega^{<\omega}, t \in \text{seq}_{<\omega}(b)} P_{s,t} = \mathbb{P}_b$. $\square$

Let G be a $\mathbb{P}_b$ -generic filter over V . In $V[G]$, define

$$r_{\text{gn}} := \bigcup \{s : \exists t, F : (s, t, F) \in G\} \text{ and } h_{\text{gn}} := \bigcup \{t : \exists s, F : (s, t, F) \in G\}.$$

Then $(r_{\text{gn}}, h_{\text{gn}}) \in \omega^\omega \times \prod b$ and, for every $f \in \prod b \cap V$, and for all but finitely many $n \in \omega$ there is some $k \in [r_{\text{gn}}(n), r_{\text{gn}}(n+1)]$ such that $f(k) = h_{\text{gn}}(k)$. We can identify the generic real with $(J_{\text{gn}}, h_{\text{gn}}) \in \mathbb{I} \times \prod b$ where $J_{\text{gn}, n} := [r_{\text{gn}}(n-1), r_{\text{gn}}(n))$, which satisfies that, for every $f \in \prod b \cap V$, $f \sqsubset^\bullet (J_{\text{gn}}, h_{\text{gn}})$.

Definition 3.3 ([Mej19, BCM21]). Let $F \subseteq \mathcal{P}(\omega)$ be a filter. We assume that all filters are *free*, i.e. they contain the *Frechet filter* $\text{Fr} := \{\omega \setminus a : a \in [\omega]^{<\aleph_0}\}$. A set $a \subseteq \omega$ is *F-positive* if it intersects every member of F . Denote by F^+ the collection of F -positive sets.

Given a poset $\mathbb{P}$ and $Q \subseteq \mathbb{P}$, Q is *F-linked* if, for any $\langle p_n : n < \omega \rangle \in Q^\omega$, there is some $q \in \mathbb{P}$ such that

$$q \Vdash \{n < \omega : p_n \in \dot{G}\} \in F^+, \text{ i.e. it intersects every member of } F.$$

Note that, in the case $F = \text{Fr}$, the previous equation is “ $q \Vdash \{n < \omega : p_n \in \dot{G}\}$ is infinite”.

We say that Q is *uf-linked* (*ultrafilter-linked*) if it is F -linked for any filter F on ω containing the *Frechet filter* Fr .

For an infinite cardinal μ , $\mathbb{P}$ is μ -*F-linked* if $\mathbb{P} = \bigcup_{\alpha < \mu} Q_\alpha$ for some F -linked Q_α ($\alpha < \mu$). When these Q_α are *uf-linked*, we say that $\mathbb{P}$ is μ -*uf-linked*.

Note that if $F \subseteq F'$ are filters on ω , then σ -*uf-linked* $\Rightarrow \sigma$ - F' -linked $\Rightarrow \sigma$ - F -linked $\Rightarrow \sigma$ -*Fr-linked*. For ccc posets, however, we have:

Lemma 3.4 ([Mej19, Lem 5.5]). *If $\mathbb{P}$ is ccc then any subset of $\mathbb{P}$ is uf-linked iff it is Fr-linked.*

Below are presented a few well-known and basic instances of σ -uf-linked posets.

Example 3.5.

- (1) Let $\mathbb{P}$ be a poset and $Q \subseteq \mathbb{P}$. Note that a sequence $\langle p_n : n < \omega \rangle$ in Q witnesses that Q is not *Fr-linked* iff the set $\{q \in \mathbb{P} : \forall^\infty n < \omega : q \perp p_n\}$ is dense.
- (2) Any singleton is *uf-linked*. Hence, any poset $\mathbb{P}$ is $|\mathbb{P}|$ -*uf-linked*. In particular, Cohen forcing is σ -*uf-linked*.
- (3) Random forcing $\mathbb{B}$ is σ -*uf-linked* [Mej19].
- (4) The forcing eventually different real forcing $\mathbb{E}$ (see [Mej19]) is σ -*uf-linked*. This poset satisfies a stronger property see Example 3.9 (2).

The upcoming lemma indicates that σ -*Fr-linked* poset does not add dominating reals.

Lemma 3.6 ([Mej19]). *Any μ -Fr-linked poset is $\mu^+ \cdot \omega^\omega$ -good.*

We now focus on reviewing one linkedness property stronger than ultrafilter linkedness.

Definition 3.7 ([GMS16, BCM21, CMRM24]). Given a (non-principal) ultrafilter D on ω and $Q \subseteq \mathbb{P}$, say that Q is D -lim-linked if there are a $\mathbb{P}$ -name $\dot{D}'$ of an ultrafilter on ω extending D and a map $\lim^D: Q^\omega \rightarrow \mathbb{P}$ such that, whenever $\bar{p} = \langle p_n: n < \omega \rangle \in Q^\omega$,

$$\lim^D \bar{p} \Vdash \{n < \omega: p_n \in \dot{G}\} \in \dot{D}'.$$

A set $Q \subseteq \mathbb{P}$ has uf -lim-linked if it is D -lim-linked for any ultrafilter D .

In addition, for an infinite cardinal θ , the poset $\mathbb{P}$ is *uniformly μ - D -lim-linked* if $\mathbb{P} = \bigcup_{\alpha < \theta} Q_\alpha$ where each Q_α is D -lim-linked and the $\mathbb{P}$ -name $\dot{D}'$ above mentioned only depends on D (and not on Q_α , although we have different limits for each Q_α . When these Q_α are uf -lim-linked, we say that $\mathbb{P}$ is *uniformly μ - uf -lim-linked*

Remark 3.8. Any uf -lim-linked set $Q \subseteq \mathbb{P}$ is clearly uf -linked, which implies that it is Fr -linked.

Example 3.9.

- (1) Any singleton is uf -lim-linked. As a consequence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$ - uf -lim-linked, witnessed by its singletons.
- (2) $\mathbb{E}$ is uniformly σ - uf -lim-linked (see [GMS16], see also [Mil81]).
- (3) $\mathbb{B}$ is not σ - uf -lim-linked (see [BCM21, Rem. 3.10]).

Next, we show another example of uniformly σ - uf -lim-linked.

Lemma 3.10 ([CMRM24, Thm. 3.21]). *Let $b \in \omega^\omega$. Then $\mathbb{P}_b$ is uniformly σ - uf -lim-linked.*

Proof. For $s \in \omega^{<\omega}$, $t \in \text{seq}_{<\omega}(b)$ and $m < \omega$

$$P_{s,t,m} := P_b(s, t, m) = \{(s', t', F) \in \mathbb{P}_b: s' = s, t' = t \text{ and } |F| \leq m\}.$$

For an ultrafilter D on ω , and $\bar{p} = \langle p_n: n \in \omega \rangle \in P_{s,t,m}$, we show how to define $\lim^D \bar{p}$. Let $p_n = (s, t, F_n) \in P_{s,t,m}$. Considering the lexicographic order $\triangleleft$ of $\prod b$, and let $\{x_{n,k}: k < m_n\}$ be a $\triangleleft$ -increasing enumeration of F_n where $m_n \leq m$. Next find an unique $m_* \leq m$ such that $A := \{n \in \omega: m_n = m_*\} \in D$. For each $k < m_*$, define $x_k := \lim_n^D x_{n,k}$ in $\prod b$ where $x_k(i)$ is the unique member of $b(i)$ such that $\{n \in A: x_{n,k}(i) = x_k(i)\} \in D$ (this coincides with the topological D -limit). Therefore, we can think of $F := \{x_k: k < m_*\}$ as the D -limit of $\langle F_n: n < \omega \rangle$, so we define $\lim^D \bar{p} := (s, t, F)$. Note that $\lim^D \bar{p} \in P_{s,t,m}$.

The sequence $\langle P_{s,t,m}: s \in \omega^{<\omega}, t \in \text{seq}_{<\omega}(b), m < \omega \rangle$ witnesses that $\mathbb{P}_b$ is uniformly σ - D -lim-linked for any ultrafilter D on ω . This is a consequence of the following claim:

Claim 3.11 ([CMRM24, Claim. 3.22]). *The set*

$$D \cup \bigcup_{s,m} \{ \{n < \omega: p_n \in G\}: \bar{p} \in P_{s,t,m}^\omega \cap V, \lim^D \bar{p} \in G \}$$

has the finite intersection property whenever G is $\mathbb{P}$ -generic over V .

□

We below review briefly the preservation theory of unbounded families presented in [CM19, Sect. 4]. This a generalization of Judah and Shelah's [JS90] and Brendle's [Bre91] preservation theory.

Definition 3.12. Let $R = \langle X, Y, \sqsubset \rangle$ be a relational system and let θ be a cardinal.

- (1) For a set M ,
 - (i) An object $y \in Y$ is *R-dominating over M* if $x \sqsubset y$ for all $x \in X \cap M$.
 - (ii) An object $x \in X$ is *R-unbounded over M* if it $R^\perp$ -dominating over M , that is, $x \not\sqsubset y$ for all $y \in Y \cap M$.
- (2) A family $\{x_i : i \in I\} \subseteq X$ is *strongly θ -R-unbounded* if $|I| \geq \theta$ and, for any $y \in Y$, $|\{i \in I : x_i \sqsubset y\}| < \theta$.

We look at the following type of well-defined relational systems.

Definition 3.13. Say that $R = \langle X, Y, \sqsubset \rangle$ is a *Polish relational system (Prs)* if

- (1) X is a Perfect Polish space,
- (2) Y is a non-empty analytic subspace of some Polish Z , and
- (3) $\sqsubset = \bigcup_{n < \omega} \sqsubset_n$ where $\langle \sqsubset_n : n \in \omega \rangle$ is some increasing sequence of closed subsets of $X \times Z$ such that, for any $n < \omega$ and for any $y \in Y$, $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is closed nowhere dense.

Remark 3.14. By Definition 3.13 (3), $\langle X, \mathcal{M}(X), \in \rangle$ is Tukey below R where $\mathcal{M}(X)$ denotes the σ -ideal of meager subsets of X . Therefore, $\mathfrak{b}(R) \leq \text{non}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq \mathfrak{d}(R)$.

For the rest of this section, fix a Prs $R = \langle X, Y, \sqsubset \rangle$ and an infinite cardinal θ .

Definition 3.15 (Judah and Shelah [JS90], Brendle [Bre91]). A poset $\mathbb{P}$ is *θ -R-good* if, for any $\mathbb{P}$ -name $\dot{h}$ for a member of Y , there is a non-empty set $H \subseteq Y$ (in the ground model) of size $< \theta$ such that, for any $x \in X$, if x is R -unbounded over H then $\Vdash x \not\sqsubset \dot{h}$.

We say that $\mathbb{P}$ is *R-good* if it is $\aleph_1$ -R-good.

The previous is a standard property associated with preserving $\mathfrak{b}(R)$ small and $\mathfrak{d}(R)$ large after forcing extensions.

Remark 3.16. Notice that $\theta < \theta_0$ implies that any θ -R-good poset is θ_0 -R-good. Also, if $\mathbb{P} < \mathbb{Q}$ and $\mathbb{Q}$ is θ -R-good, then $\mathbb{P}$ is θ -R-good.

Lemma 3.17 ([CM19, Lemma 2.7]). *Assume that θ is a regular cardinal. Then any poset of size $< \theta$ is θ -R-good. In particular, Cohen forcing $\mathbb{C}$ is R-good.*

We now present the instances of Prs and the corresponding good posets that we use in our applications.

Example 3.18.

- (1) Define $\Omega_n := \{a \in [2^{<\omega}]^{<\aleph_0} : \text{Lb}(\bigcup_{s \in a} [s]) \leq 2^{-n}\}$ (endowed with the discrete topology) and put $\Omega := \prod_{n < \omega} \Omega_n$ with the product topology, which is a perfect Polish space. For every $x \in \Omega$ denote

$$N_x := \bigcap_{n < \omega} \bigcup_{m \geq n} \bigcup_{s \in x(m)} [s],$$

which is clearly a Borel null set in 2^ω .

Define the Prs $\mathbf{Cn} := \langle \Omega, 2^\omega, \sqsubset^n \rangle$ where $x \sqsubset^n z$ iff $z \notin N_x$. Recall that any null set in 2^ω is a subset of N_x for some $x \in \Omega$, so $\mathbf{Cn}$ and $\mathbf{C}_\mathcal{N}^\perp$ are Tukey-Galois equivalent. Hence, $\mathfrak{b}(\mathbf{Cn}) = \text{cov}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Cn}) = \text{non}(\mathcal{N})$.

Any μ -centered poset is μ^+ - $\mathbf{Cn}$ -good ([Bre91]). In particular, σ -centered posets are $\mathbf{Cn}$ -good.

- (2) The relational system $\mathbf{Ed}_b$ is Polish when $b = \langle b(n) : n < \omega \rangle$ is a sequence of non-empty countable sets such that $|b(n)| \geq 2$ for infinitely many n . Consider $\mathbf{Ed} := \langle \omega^\omega, \omega^\omega, \neq^\infty \rangle$. By [BJ95, Thm. 2.4.1 & Thm. 2.4.7] (see also [CM23, Thm. 5.3]), $\mathfrak{b}(\mathbf{Ed}) = \text{non}(\mathcal{M})$ and $\mathfrak{d}(\mathbf{Ed}) = \text{cov}(\mathcal{M})$.
- (3) The relational system $\omega^\omega = \langle \omega^\omega, \omega^\omega, \leq^* \rangle$ is Polish. Any μ -Fr-linked poset (see Definition 3.3) is μ^+ - ω^ω -good (see Lemma 3.6).
- (4) For each $k < \omega$, let $\text{id}^k : \omega \rightarrow \omega$ such that $\text{id}^k(i) = i^k$ for all $i < \omega$ and $\mathcal{H} := \{\text{id}^{k+1} : k < \omega\}$. Let $\mathbf{Lc}^* := \langle \omega^\omega, \mathcal{S}(\omega, \mathcal{H}), \in^* \rangle$ be the Polish relational system where

$$\mathcal{S}(\omega, \mathcal{H}) := \{\varphi : \omega \rightarrow [\omega]^{<\aleph_0} : \exists h \in \mathcal{H} \forall i < \omega : |\varphi(i)| \leq h(i)\},$$

and recall that $x \in^* \varphi$ iff $\forall^\infty n : x(n) \in \varphi(n)$. As a consequence of [BJ95, Thm. 2.3.9] (see also [CM23, Thm. 4.2]), $\mathfrak{b}(\mathbf{Lc}^*) = \text{add}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Lc}^*) = \text{cof}(\mathcal{N})$.

Any μ -centered poset is μ^+ - $\mathbf{Lc}^*$ -good (see [Bre91, JS90]) so, in particular, σ -centered posets are $\mathbf{Lc}^*$ -good. Besides, Kamburelis [Kam89] showed that any Boolean algebra with a strictly positive finitely additive measure is $\mathbf{Lc}^*$ -good (in particular, any subalgebra of random forcing).

- (5) For $b \in \omega^\omega$, $\mathbf{R}_b$ is a Polish relational system when $b \geq^* 2$ (see Example 1.17).
- (6) Let $\mathbf{M} := \langle 2^\omega, \mathbb{I} \times 2^\omega, \sqsubset^m \rangle$ where

$$x \sqsubset^m (I, y) \text{ iff } \forall^\infty n : x \restriction I_n \neq y \restriction I_n.$$

This is a Polish relational system and $\mathbf{M} \cong_{\mathbf{T}} \mathbf{C}_\mathcal{M}$ (by Theorem 1.25).

Note that, whenever M is a transitive model of ZFC, $c \in 2^\omega$ is a Cohen real over M iff c is $\mathbf{M}$ -unbounded over M .

- (7) Define the relational system $\mathbf{Ce} = \langle 2^\omega, \mathbf{NE}, \sqsubset^* \rangle$ where $\mathbf{NE}$ is the collection of sequences $\bar{T} = \langle T_n : n < \omega \rangle$ such that each T_n is a subtree of ${}^{<\omega}2$ (not necessarily well-pruned), $T_n \subseteq T_{n+1}$ and $\text{Lb}([T_n]) = 0$, i.e. $\lim_{n \rightarrow \infty} \frac{|T \cap {}^{n+1}2|}{2^n} = 0$, and $x \sqsubset^* \bar{T}$ iff $x \in [T_n]$ for some $n < \omega$.

Good posets are preserved along FS iterations as follows.

Lemma 3.19 ([BCM23, Sec. 4]). *Let $\langle \mathbb{P}_\xi, \dot{Q}_\xi : \xi < \pi \rangle$ be a FS iteration such that, for $\xi < \pi$, $\mathbb{P}_\xi$ forces that $\dot{Q}_\xi$ is a non-trivial θ -cc θ -R-good poset. Let $\{\gamma_\alpha : \alpha < \delta\}$ be an increasing enumeration of 0 and all limit ordinals smaller than π (note that $\gamma_\alpha = \omega\alpha$), and for $\alpha < \delta$ let $\dot{c}_\alpha$ be a $\mathbb{P}_{\gamma_{\alpha+1}}$ -name of a Cohen real in X over V_{γ_α} .*

Then $\mathbb{P}_\pi$ is θ -R-good. Moreover, if $\pi \geq \theta$ then $\mathbf{C}_{[\pi]^{<\theta}} \preceq_T \mathbf{R}$, $\mathfrak{b}(\mathbf{R}) \leq \theta$ and $|\pi| \leq \mathfrak{d}(\mathbf{R})$.

To force a lower bound of $\mathfrak{b}(\mathbf{R})$, we use:

Lemma 3.20 ([CM22, Thm. 2.12]). *Let $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ be a Polish relational system, θ an uncountable regular cardinal, and let $\mathbb{P}_\pi = \langle \mathbb{P}_\xi, \dot{Q}_\xi : \xi < \pi \rangle$ be a FS iteration of θ -cc posets with $\text{cf}(\pi) \geq \theta$. Assume that, for all $\xi < \pi$ and any $A \in [X]^{<\theta} \cap V_\xi$, there is some $\eta \geq \xi$ such that $\dot{Q}_\eta$ adds an $\mathbf{R}$ -dominating real over A . Then $\mathbb{P}_\pi$ forces $\theta \leq \mathfrak{b}(\mathbf{R})$, i.e. $\mathbf{R} \preceq_T \mathbf{C}_{[X]^{<\theta}}$.*

The following results illustrates the effect of adding cofinally many $\mathbf{R}$ -dominating reals along a FS iteration.

Lemma 3.21 ([CM22, Lem. 2.9]). *Let $\mathbf{R}$ be a definable relational system of the reals, and let λ be a limit ordinal of uncountable cofinality. If $\mathbb{P}_\lambda = \langle \mathbb{P}_\xi, \dot{Q}_\xi : \xi < \lambda \rangle$ is a FS iteration of $\text{cf}(\lambda)$ -cc posets that adds $\mathbf{R}$ -dominating reals cofinally often, then $\mathbb{P}_\lambda$ forces $\mathbf{R} \preceq_T \lambda$.*

In addition, if $\mathbf{R}$ is a Prs and all iterands are non-trivial, then $\mathbb{P}_\lambda$ forces $\mathbf{R} \cong_T \mathbf{M} \cong_T \lambda$. In particular, $\mathbb{P}_\lambda$ forces $\mathfrak{b}(\mathbf{R}) = \mathfrak{d}(\mathbf{R}) = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \text{cf}(\lambda)$.

Next, we illustrate the effect of iterating $\mathbb{P}_b$ on Cichoń's diagram.

Theorem 3.22. *Let π be an ordinal of uncountable cofinality such that $|\pi|^{\aleph_0} = |\pi|$. The FS iteration of $\mathbb{P}$ of length π (i.e. the FS iteration $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \pi \rangle$ where each $\dot{Q}_\alpha$ is a $\mathbb{P}_\alpha$ -name of $\mathbb{P}_b$) forces $\mathfrak{c} = |\pi|$, $\text{Cv}_{\mathcal{M}} \cong_T \pi$ and $\text{Cv}_{\mathcal{N}}^\perp \cong_T \omega^\omega \cong_T \mathbf{C}_{[\mathbb{R}]^{<\aleph_1}}$. In particular, it forces $\text{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_1$, $\mathfrak{b}_b^{\text{eq}} = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \mathfrak{d}_b^{\text{eq}} = \text{cf}(\pi)$ and $\text{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c} = |\pi|$ (see Figure 5).*

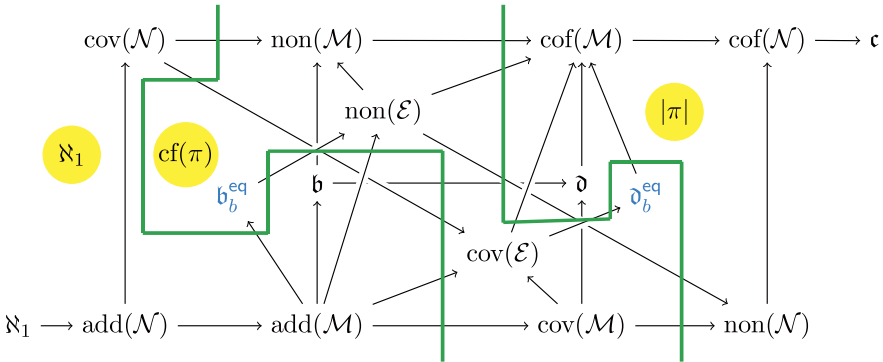


Figure 5: Cichoń's diagram after adding π -many generic reals with $\mathbb{P}_b$, where π has uncountable cofinality and $|\pi|^{\aleph_0} = |\pi|$.

The proof of the above theorem is a consequence of [Theorem C](#), so we proceed to prove [Theorem C](#).

Theorem 3.23. *Let $\aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ be regular cardinals, and assume λ_5 is a cardinal such that $\lambda_5 > \lambda_4$ and $\lambda_5 = \lambda_5^{\aleph_0}$ and $\text{cf}([\lambda_5]^{<\lambda_i}) = \lambda_5$ for $i = 1, \dots, 4$. Then, we can construct a FS iteration of length λ of ccc posets forcing $\mathfrak{c} = \lambda_5$, $\text{Lc}^* \cong_{\text{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_1}}$, $\text{Cv}_{\mathcal{N}}^\perp \cong_{\text{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_3}}$, $\omega^\omega \cong_{\text{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_4}}$, and $\mathbf{R}_b \cong_{\text{T}} \mathbf{M} \cong_{\text{T}} \lambda_4$. In particular, it forced [Figure 4](#).*

Proof. Make a FS iteration $\mathbb{P} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \lambda \rangle$ of length $\lambda := \lambda_5 \lambda_4$ (ordinal product) as follows. Fix a partition $\langle C_i : 1 \leq i \leq 3 \rangle$ of $\lambda_5 \setminus \{0\}$ where each set has size λ_5 . For each $\rho < \lambda_4$ denote $\eta_\rho := \lambda_5 \rho$. We define the iteration at each $\xi = \eta_\rho + \varepsilon$ for $\rho < \lambda_4$ and $\varepsilon < \lambda_5$ as follows:

$$\dot{\mathbb{Q}}_\xi := \begin{cases} \mathbb{P}_b & \text{if } \varepsilon = 0, \\ \text{LOC}^{\dot{N}_\xi} & \text{if } \varepsilon \in C_1, \\ \mathbb{B}^{\dot{N}_\xi} & \text{if } \varepsilon \in C_2, \\ \mathbb{D}^{\dot{N}_\xi} & \text{if } \varepsilon \in C_3, \end{cases}$$

where $\dot{N}_\xi$ is a $\mathbb{P}_\xi$ -name of a transitive model of ZFC of size $< \lambda_i$ when $\varepsilon \in C_i$.

Additionally, by a book-keeping argument, we make sure that all such models N_ξ are constructed such that, for any $\rho < \lambda_4$:

- (•₁) if $A \in V_{\eta_\rho}$ is a subset of ω^ω of size $< \lambda_1$, then there is some $\varepsilon \in C_1$ such that $A \subseteq N_{\eta_\rho + \varepsilon}$;
- (•₂) if $A \in V_{\eta_\rho}$ is a subset of Ω of size $< \lambda_2$, then there is some $\varepsilon \in C_2$ such that $A \subseteq N_{\eta_\rho + \varepsilon}$; and
- (•₃) if $A \in V_{\eta_\rho}$ is a subset of ω^ω of size $< \lambda_3$, then there is some $\varepsilon \in C_3$ such that $A \subseteq N_{\eta_\rho + \varepsilon}$.

We prove that $\mathbb{P}$ is as required. Clearly, $\mathbb{P}$ forces $\mathfrak{c} = \lambda_5$.

On the one side, notice that all iterands are λ_1 - Lc^* -good (see [Lemma 3.17](#) and [Example 3.18 \(4\)](#)), λ_2 - $\text{Cv}_{\mathcal{N}}^\perp$ -good (see [Lemma 3.17](#) and [Example 3.18 \(1\)](#)) and λ_3 - ω^ω -good (see [Lemma 3.17](#), [Lemma 3.10](#), and [Lemma 3.6](#)), so by [Lemma 3.19](#) we obtain $\mathbb{P}$ forces $\mathbf{C}_{[\lambda_5]^{<\lambda_1}} \preceq_{\text{T}} \text{Lc}^*$, $\mathbf{C}_{[\lambda_5]^{<\lambda_1}} \preceq_{\text{T}} \text{Cv}_{\mathcal{N}}^\perp$ and $\mathbf{C}_{[\lambda_5]^{<\lambda_1}} \preceq_{\text{T}} \omega^\omega$. On the other hand, by using (•₁) and [Lemma 3.20](#), $\mathbb{P}$ forces $\text{Lc}^* \preceq_{\text{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_1}}$.

In a similar way to the previous argument, $\mathbb{P}$ forces $\text{Cv}_{\mathcal{N}}^\perp \preceq_{\text{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_3}}$, $\omega^\omega \preceq_{\text{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_4}}$.

Finally, since $\text{cf}(\lambda) = \lambda_4$, by [Lemma 3.21](#) $\mathbb{P}$ forces $\mathbf{R}_b \cong_{\text{T}} \mathbf{M} \cong_{\text{T}} \lambda_4$ because $\mathbb{P}_b$ adds $\mathbf{R}_b$ -dominating reals. $\square$

According to the preceding theorem, $\mathfrak{b}_b^{\text{eq}} > \text{cov}(\mathcal{N})$ is consistent. However, what about $\text{cov}(\mathcal{N}) > \mathfrak{b}_b^{\text{eq}}$? We then give a positive answer to this question.

A notion proceeding ultrafilter-limits, which is more powerful, is finitely additive measures (fams)-limits introduced implicitly in the proof of the consistency of $\text{cf}(\text{cov}(\mathcal{N})) = \omega$ by Shelah [[She00](#)] and was formalized in [[KST19](#)]. Recently, the author refined this in general setting along with Mejía, and Uribe-Zapata [[CMUZ24](#)].

Definition 3.24 ([CMUZ24]). Let $\mathbb{P}$ be a poset and let $\Xi: \mathcal{P}(\omega) \rightarrow [0, 1]$ be a fam (with $\Xi(\omega) = 1$ and $\Xi(\{n\}) = 0$ for all $n < \omega$), $I = \langle I_n: n < \omega \rangle$ be a partition of ω into finite sets, and $\varepsilon > 0$.

- (1) A set $Q \subseteq \mathbb{P}$ is (Ξ, I, ε) -linked if there is a function $\lim: Q^\omega \rightarrow \mathbb{P}$ and a $\mathbb{P}$ -name $\dot{\Xi}'$ of a fam on $\mathcal{P}(\omega)$ extending Ξ such that, for any $\bar{p} = \langle p_\ell: \ell < \omega \rangle \in Q^\omega$,

$$\lim \bar{p} \Vdash \int_\omega \frac{|\{\ell \in I_k: p_\ell \in \dot{G}\}|}{|I_k|} d\dot{\Xi}' \geq 1 - \varepsilon.$$

- (2) The poset $\mathbb{P}$ is μ -FAM-linked, witnessed by $\langle Q_{\alpha, \varepsilon}: \alpha < \mu, \varepsilon \in (0, 1) \cap \mathbb{Q} \rangle$, if:

- (i) Each $Q_{\alpha, \varepsilon}$ is (Ξ, I, ε) -linked for any Ξ and I .
- (ii) For $\varepsilon \in (0, 1) \cap \mathbb{Q}$, $\bigcup_{\alpha < \omega} Q_{\alpha, \varepsilon}$ is dense in $\mathbb{P}$.

- (3) The poset $\mathbb{P}$ is *uniformly* μ -FAM-linked if there is some $\langle Q_{\alpha, \varepsilon}: \alpha < \mu, \varepsilon \in (0, 1) \cap \mathbb{Q} \rangle$ as above, such that in (1) the name $\dot{\Xi}'$ only depends on Ξ (and not on any $Q_{\alpha, \varepsilon}$).

Example 3.25.

- (1) Any singleton is (Ξ, I, ε) -linked. Hence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$ -FAM-linked. In particular, Cohen forcing is uniformly σ -FAM-linked.
- (2) Shelah [She00] proved implicitly that random forcing is uniformly σ -FAM-linked. More generally, any measure algebra of Maharam type μ is uniformly μ -FAM-linked [MUZ24].
- (3) The creature ccc forcing from [HS16] adding eventually different reals is (uniformly) σ -FAM-linked. This is proved in [KST19], with more general setting in [Mej24].

The author with Mejía proved that fam-limits below help to control $\text{non}(\mathcal{E})$. Concretely, they proved:

Lemma 3.26 ([CMUZ24]). *σ -FAM-linked posets are Ce-good.*

The following results answered our question.

Lemma 3.27 ([BS92], see also [Car23]). *Assume $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular and assume that $b \in \omega^\omega$ satisfies $\sum_{k < \omega} \frac{1}{b(k)} < \infty$. Let $\mathbb{B}_\pi$ be a FS iteration of random forcing of length $\pi = \lambda\kappa$. Then, in $V^{\mathbb{B}_\pi}$, $\text{Lc}^* \cong_{\text{T}} \omega^\omega \cong_{\text{T}} \text{Cv}_\mathcal{E} \cong_{\text{T}} \text{C}_{[\lambda]^{<\aleph_1}}$ and $\text{Cv}_\mathcal{N}^\perp \cong_{\text{T}} \mathbf{M} \cong_{\text{T}} \kappa$.*

Proof. Since $\mathbb{B}$ adds random reals, these are $\text{Cv}_\mathcal{N}$ -unbounded reals, which are precisely the $\text{Cn}^\perp$ -dominating reals. So by Lemma 3.21 $\mathbb{B}_\pi$ forces $\text{Cv}_\mathcal{N}^\perp \cong_{\text{T}} \mathbf{M} \cong_{\text{T}} \lambda_4$ because $\text{cf}(\lambda) = \lambda_4$.

Notice that $\mathbb{B}$ is σ -uf-linked and σ -FAM-linked (see Example 3.5 (3) and Example 3.25 (2), respectively), so by Lemma 3.6 and Lemma 3.26 $\mathbb{B}$ is ω^ω -good and Ce-good, respectively. Thus, $\mathbb{B}$ is Lc^* -good by Example 3.18 (4). Hence, by Lemma 3.19, $\mathbb{B}_\pi$ forces $\text{C}_{[\lambda_5]^{<\aleph_1}} \preceq_{\text{T}} \text{Lc}^*$, $\text{C}_{[\lambda_5]^{<\aleph_1}} \preceq_{\text{T}} \omega^\omega$, $\text{C}_{[\lambda_5]^{<\aleph_1}} \preceq_{\text{T}} \text{Cv}_\mathcal{E}$.

On the other hand, clearly $\text{Lc}^* \preceq_{\text{T}} \text{C}_{[\lambda_5]^{<\aleph_1}}$, $\omega^\omega \preceq_{\text{T}} \text{C}_{[\lambda_5]^{<\aleph_1}}$, $\text{Cv}_\mathcal{E} \preceq_{\text{T}} \text{C}_{[\lambda_5]^{<\aleph_1}}$ are forced. Consequently, $\mathbb{B}_\pi$ forces $\text{Lc}^* \cong_{\text{T}} \omega^\omega \cong_{\text{T}} \text{Cv}_\mathcal{E} \cong_{\text{T}} \text{C}_{[\lambda]^{<\aleph_1}}$. $\square$

4 Open problems

We know the consistency $\mathfrak{b}_b^{\text{eq}} > \mathfrak{b}$, but the following is not known:

Problem 4.1. *Is $\mathfrak{b}_b^{\text{eq}} < \mathfrak{b}$ consistently. Dually, Is $\mathfrak{d} < \mathfrak{d}_b^{\text{eq}}$ consistent?*

Notice that for $b \in \omega^\omega$, $\mathfrak{b}_{b,1}^{\text{alc}} \leq \text{non}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq \mathfrak{d}_{b,1}^{\text{alc}}$. On the other hand, after a FS (finite support) iteration of uncountable cofinality length of ccc non-trivial posets, $\text{non}(\mathcal{M}) \leq \text{cov}(\mathcal{M})$, which implies by [Lemma 2.5](#) that $\mathfrak{b} \leq \mathfrak{b}_b^{\text{eq}}$ and $\mathfrak{d}_b^{\text{eq}} \leq \mathfrak{d}$. Hence, FS iterations cannot solve [Problem 4.1](#).

Despite the fact that $\mathfrak{b}_b^{\text{eq}} \leq \text{non}(\mathcal{E})$ ([Lemma 2.2 \(1\)](#)), we do not know the following:

Problem 4.2. *Is $\mathfrak{b}_b^{\text{eq}} < \text{non}(\mathcal{E})$ consistent for any (some) b ?*

Brendle [[Bre99](#)] (see also [[Car24](#), Lem. 2.6]) proved the consistency of $\text{non}(\mathcal{E}) > \mathfrak{d}$, so we ask:

Problem 4.3. *Is $\mathfrak{b}_b^{\text{eq}} > \mathfrak{d}$ consistent for any (some) b ?*

In relation to $\mathfrak{b}_b^{\text{eq}}$ and $\text{non}(\mathcal{E})$ when $\sum_{k < \omega} \frac{1}{b(k)} = \infty$, we do not know the following:

Problem 4.4. *Are each of the following statements consistent with ZFC?*

- (1) $\text{non}(\mathcal{E}) < \mathfrak{b}_b^{\text{eq}}$ for any (some) b . Dually, $\mathfrak{d}_b^{\text{eq}} < \text{cov}(\mathcal{E})$ for any (some) b .
- (2) $\mathfrak{b}_b^{\text{eq}} < \text{non}(\mathcal{E})$ for any (some) b . Dually, $\text{cov}(\mathcal{E}) < \mathfrak{d}_b^{\text{eq}}$ for any (some) b .

Recently, Yamazoe used uf-limits on intervals (introduced by Mejía [[Mej24](#)]) along FS iterations to construct a poset to force

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b} < \text{non}(\mathcal{E}) < \text{non}(\mathcal{M}) < \\ < \text{cov}(\mathcal{M}) < \mathfrak{d} < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}).$$

The above model can be modified to get the following:

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{b}_b^{\text{eq}} = \text{non}(\mathcal{E}) < \text{non}(\mathcal{M}) < \\ < \text{cov}(\mathcal{M}) < \mathfrak{d} < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}).$$

The point is that $\mathfrak{b}_b^{\text{eq}} \leq \text{non}(\mathcal{E})$ when $\sum_{k < \omega} \frac{1}{b(k)} < \infty$ and the forcing that increases $\mathfrak{b}_b^{\text{eq}}$ has uniformly σ -uf-lim-linked ([Lemma 3.10](#)). So we ask:

Problem 4.5. (1) *Is it consistent ZFC with*

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{b}_b^{\text{eq}} < \text{non}(\mathcal{E}) < \text{non}(\mathcal{M}) < \\ < \text{cov}(\mathcal{M}) < \mathfrak{d} < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}).$$

(2) *Is it consistent ZFC with*

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b}_b^{\text{eq}} < \mathfrak{b} < \text{non}(\mathcal{E}) < \text{non}(\mathcal{M}) < \\ < \text{cov}(\mathcal{M}) < \mathfrak{d} < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}).$$

Notice that FS iterations cannot solve (2) of Problem 4.5 (see discussion after Problem 4.1). A positive answer of Problem 4.4 could help solve the following:

Problem 4.6. (1) *Is it consistent ZFC with*

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b} < \text{non}(\mathcal{E}) < \mathfrak{b}_b^{\text{eq}} < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < \mathfrak{d} < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}).$$

(2) *Is it consistent ZFC with*

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b}_b^{\text{eq}} < \mathfrak{b} < \text{non}(\mathcal{E}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < \mathfrak{d} < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}).$$

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