

THE SUCCESSIVE UNCONSTRAINED LEAST-SQUARES ESTIMATION TECHNIQUE  
( SULSET )

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Abstract : An algorithm of the class of the SUMT to solve the non-linear programming problems with inequality and/or equality constraints is described. We call this technique " the Successive Unconstrained Least-Squares Estimation Technique ( SULSET ) " . We propose the acceleration technique and derive the rate of convergence in this paper.

1. Introduction

In order to solve the nonlinear programming problem :

(A) minimize  $f(x)$  subject to  $g_i(x) \leq 0$ ,  $i=1, \dots, m$ , and  $x \in E^n$ ,

the SUMT ( Sequential Unconstrained Minimization Technique ) has been proposed. This method is based upon the property that , when the problem (A) is transformed into a sequence of unconstrained minimization problems of appropriate penalty functions , the convergent point of the sequence of the unconstrained minima is the optimal solution of the problem (A). Carrol first proposed the CRST ( Created Response Surface Technique ) in his well-known paper [4] and later Fiacco and McCormick established this technique theoretically and practically in [5] - [12]. They called it the SUMT.

Fiacco and McCormick first proposed the " interior point method " in [7]. This method can be stated by introducing the following penalty

function :

$$(1) \quad P(x, r_k) = f(x) - r_k \sum_{i=1}^m 1/g_i(x) \quad ,$$

where  $\{r_k\}$  is a monotonically decreasing scalar sequence which tends to zero. The sequence of the minima of (1) converges to the optimal solution  $x^*$  of the problem (A) as  $r_k \rightarrow 0$  ( $k \rightarrow \infty$ ).

On the other hand , Zangwill proposed the " exterior point method " in [28]. The penalty function can be defined as follows :

$$(2) \quad T(x, t_k) = f(x) + t_k \sum_{i=1}^m [ \max( 0, g_i(x) ) ]^{1+\varepsilon} , \varepsilon > 0 \quad .$$

The sequence of the minima of (2) converges to  $x^*$  as  $t_k \rightarrow \infty$  ( $k \rightarrow \infty$ ).

In the former method , the so-called " barriers " on the boundary of the feasible region , which the penalty functions construct , prevent the sequence of the minima  $\{x^k\}$  from jumping out of the interior of the feasible region. While in the latter method , the so-called " walls " are constructed outside the feasible region and the sequence of the minima  $\{x^k\}$  slips down to the optimal solution of the problem (A) along them.

Furthermore the generalizations of various types of penalty functions have been considered and the convergence properties have been also discussed by Fiacco and McCormick [11] , Fiacco [6] , Fiacco and Jones [12] and Stong [27].

Afterward Fiacco and McCormick proposed the SUMT without parameters in [9] and [11]. They called this method the " Q-function type SUMT ". The following penalty function was proposed :

$$(3) \quad Q(x, x^{k-1}) = 1/[ f(x^{k-1}) - f(x) ] - \sum_{i=1}^m 1/g_i(x) \quad .$$

This method can be classified to an interior point type.

On the contrary , the Q-function type exterior point methods

are proposed by Morrison [22] and Kowalik , Osborne and Ryan [18]. For the most part of this paper , we will discuss the method of this type and name this class of methods the " Successive Unconstrained Least-Squares Estimation Technique ( SULSET ) " .

Section 2 mainly gives the results of [22] and [18] , section 3 gives our acceleration technique , section 4 gives the comparison with the numerical results of [18] , section 5 gives the derivation of the rate of convergence , and section 6 gives the generalization of the SULSET.

For unconstrained minimizations , recently the conjugate gradient type methods are very prevalent ( [14] , [3] , [13] , [15] and [16] ). However for unconstrained least-squares estimations , we would like to recommend the Marquardt's Maximum Neighbourhood Method ( [2] , [17] , [20] and [24] ). Because this method converges rapidly and is very stable in characteristics .

## 2. The Successive Unconstrained Least-Squares Estimation Technique

Let us consider the following nonlinear programming problem :

$$(B) \quad \text{minimize } f(x) \quad \text{subject to } g_i(x)=0, i=1, \dots, m, \text{ and } x \in E^n.$$

Morrison suggested the algorithm to transform the problem (B) to successive least-squares problems :

$$(C) \quad \text{minimize } S(x, X_k) \quad \text{subject to } x \in E^n,$$

where

$$(1) \quad S(x, X_k) = [f(x) - X_k]^2 + \sum_{i=1}^m c_i g_i(x)^2$$

$c_i, i=1, \dots, m$  , are positive constants and

$\{X_k\}$  is a monotonically increasing scalar sequence which tends to the minimum value  $v^*$  of the problem (B).

The next theorem shows the validity of the above transformation.

THEOREM 2.1 ( Convergence Theorem of the SULSET ) If (a)  $f$  and  $g_i$ ,  $i=1, \dots, m$ , are continuous functions of  $x$ , (b)  $S(x, X_k)$  is the function as defined in (1), (c)  $\{X_k\}$  is a monotonically increasing sequence and converges to  $v^*$ , (d)  $A^*$  is the set of the optimal solutions of the problem (B), and (e)  $x^k$  is the minimum of  $S(x, X_k)$ , then

- (i)  $f(x^k) \leq v^*$ ,
- (ii)  $f(x^k) \leq f(x^{k+1})$ ,
- (iii) there exists a limit of the sequence  $\{x^k\}$  and this limit point belongs to  $A^*$ , and
- (iv)  $\lim_{k \rightarrow \infty} S(x^k, X_k) = 0$ .

Proof. See the ref. [22] and the proof of Theorem 6.1.

It is a question of how to construct the sequence  $\{X_k\}$  to converge towards  $v^*$  from below. The Morrison-parameter sequence is following :

$$(2) \quad X_{k+1}^M = X_k + [S(x^k, X_k)]^{1/2}.$$

In order to ensure that the Morrison-parameter sequence converges to  $v^*$ , the following assumption is necessary.

ASSUMPTION 2.1 ( Continuity Condition ) Let  $v^*(q)$  be the minimum value of the problem :

$$\text{minimize } f(x) \quad \text{subject to } g_i(x) = q_i, \quad i=1, \dots, m.$$

$v^*(q)$  is continuous at  $q=0$ .

This assumption is quite natural, because it means that if the constraints changes only a little bit, then the minimum value changes only a little bit. The following theorem shows that the Morrison-parameter is effective.

THEOREM 2.2 ( Convergence Theorem of the Morrison-Parameter Sequence )

If the conditions of Theorem 2.1 and Assumption 2.1 are satisfied ,  
the the Morrison-parameter sequence  $\{X_k\}$  , such as defined in (2) ,  
converges to  $v^*$  .

Proof. See the proof of Theorem 6.2 .

The relation between dual feasibility and the SULSET is stated  
in the following theorem .

THEOREM 2.3 Let the conditions of Theorem 2.1 be satisfied and  $f(x)$   
and  $g_i(x)$  ,  $i=1, \dots, m$  , be continuously differentiable functions with  
respect to their arguments ; let  $(x^*, \lambda^*)$  be the stationary point of  
the Lagrangian

$$(3) \quad L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x)$$

corresponding to the solution  $x^*$  of the problem (B). If  $g_i(x^k)$  ,  
 $i=1, \dots, m$  , are linearly independent for each  $k$  large enough , then

$$(4) \quad c_i g_i(x^k) / [ f(x^k) - X_k ] \rightarrow \lambda_i^* \quad \text{as } k \rightarrow \infty \quad \text{for } i=1, \dots, m .$$

Proof. See the proof of LEMMA 1 in [18] .

Furthermore Kowalik , Osborne and Ryan proposed the tangent  
parameter sequence :

$$(5) \quad X_{k+1}^T = X_k + S(x^k, X_k) / | f(x^k) - X_k |$$

The next theorem indicates the validity and effectiveness of tan-  
gent parameter sequence.

THEOREM 2.4 Let the conditions of Theorem 2.1 be satisfied. Then

$$(i) \quad X_{k+1}^T \geq X_{k+1}^M$$

(ii) If the conditions of Theorem 2.3 hold and if there exists a set  $\Phi$  containing  $(x^*, \lambda^*)$  in its interior such that

(a)  $L$  has the saddle point property in  $\Phi$  so that

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*) \quad , \text{ and}$$

(b) for  $\bar{\lambda}$  such that  $(x, \bar{\lambda}) \in \Phi$  there exists an  $\bar{x}$  such that

$(\bar{x}, \bar{\lambda}) \in \Phi$  ,  $L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda})$  for all  $x$  such that  $(x, \bar{\lambda}) \in \Phi$  and  $(\bar{x}, \bar{\lambda})$  is the unique solution in  $\Phi$  to the system of equations

$$\nabla L(x, \bar{\lambda}) = 0$$

then  $v^* \geq X_{k+1}^T$  provided  $X_k$  is sufficiently close to  $v^*$ .

Proof. (i) It is obvious that  $X_{k+1}^T \geq X_k$ . From the definition of  $x^k$ ,

$$(X_{k+1}^T - X_k) |f(x^k) - X_k| \geq (f(x^k) - X_k)^2.$$

Therefore

$$(6) \quad (X_{k+1}^T - X_k) \geq |f(x^k) - X_k|$$

Since

$$\begin{aligned} (X_{k+1}^T - X_k) |f(x^k) - X_k| &= S(x^k, X_k) \quad \text{and} \\ (X_{k+1}^M - X_k)^2 &= S(x^k, X_k) \end{aligned}$$

from (6),

$$(X_{k+1}^T - X_k)^2 \geq (X_{k+1}^M - X_k)^2.$$

Thus

$$X_{k+1}^T \geq X_{k+1}^M.$$

(ii) See the proof of Theorem 1 in [18].

Q.E.D.

For an inequality constraint  $h(x) \leq 0$ , we will use the equivalent equality constraint

$$g(x) \triangleq \max(0, h(x)) = 0$$

By the above stated discussion, we can solve any constrained nonlinear programming problem.

### 3. The Accelerated Successive Unconstrained Least-Squares Estimation Technique

Morrison proposed the subfunctions such as (2.1) and gave  $X_{k+1}$  such as (2.2). This situation can be illustrated as Fig.3.1 .

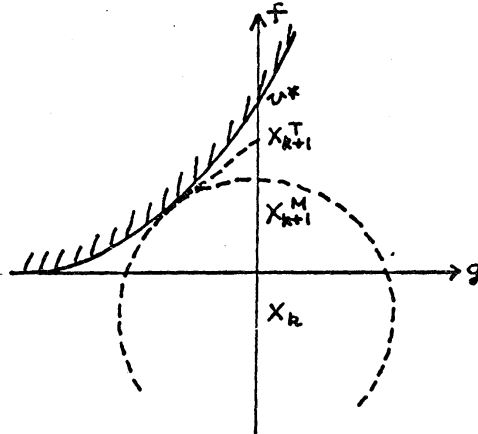


Fig.3.1

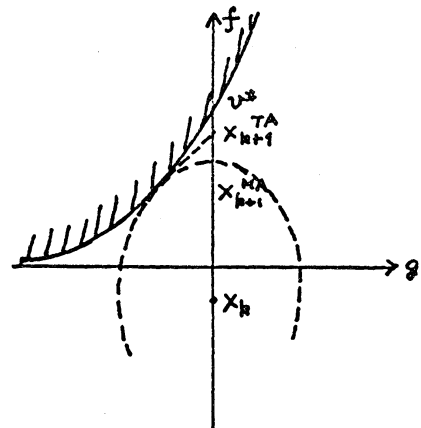


Fig.3.2

That is , the contours  $(f-x)^2 + g^2 = \text{const.}$  represent circles with the center  $(0, X)$  . If  $S(x, X) = (f-X)^2 + t g^2$  ( $t > 1$ ) , the contours represent ellipses which are long in the direction  $f$ -axis such as Fig. 3.2 . Since the discussion in section 2 are all valid even if  $c_i$  ,  $i=1, \dots, m$  , are different at each  $k$  , we will take the following system which is expected to have the more rapidly convergent property. We will propose an algorithm to transform the problem (B) to the successive least-squares estimation problems :

$$(D) \quad \text{minimize} \quad S(x, X_k, t_k)$$

where

$$(1) \quad S(x, X_k, t_k) = [f(x) - X_k]^2 + t_k \sum_{i=1}^m c_i \varepsilon_i(x)^2 .$$

Let

$$(2) \quad X_{k+1}^{MA} = X_k + [S(x_A^k, X_k, t_k)]^{1/2}$$

$$(3) \quad t_{k+1} = c t_k \quad (c > 1, t_k > 0) \quad , \text{ and}$$

$$(4) \quad x_{k+1}^{TA} = x_k + S(x_A^k, x_k, t_k) / |f(x_A^k) - x_k|$$

where  $x_A^k$  is the minimum of the problem (D).

The following theorem shows that the above system has an accelerated convergence property.

THEOREM 3.1 If  $t_k > 1$ , then  $x_{k+1}^{MA} \geq x_{k+1}^M$ .

$$\text{Proof.} \quad x_{k+1}^{MA} - x_{k+1}^M = S(x_A^k, x_k)^{1/2} - S(x_A^k, x_k, t_k)^{1/2}$$

$$0 \leq S(x_A^k, x_k) \leq S(x_A^k, x_k) \leq S(x_A^k, x_k, t_k) \quad \text{Q.E.D.}$$

We use the following algorithm which is like that of Kowalik et al..

#### ALGORITHM

##### 0. Preparation Phase .

- (i) Choose an arbitrary  $x^0 \in E^n$ .
- (ii) Move the solution to  $x^0 \in R$  by appropriate algorithms such that of [5], where  $R$  is the feasible region.

##### I. Initial Phase .

- (i) Set  $X_0 = f(x^0)$ ,  $k=1$  and  $\text{step} = \text{steplb}$ .
- (ii) Minimize  $S(x, x_k, t_k)$  to find  $x^k$ . Here  $x_k = x_{k-1} - \text{step}$  and  $t_k = 1$ .
- (iii) If  $[S(x^k, x_k, t_k)]^{1/2} < \text{eps}$  then  $\text{step} = 2 * \text{step}$ ;  $k=k+1$ ; go to (ii); else  $BU = x_{k-1}$ ; proceed II.

##### II. Iteration Phase .

- (i) Set  $BL = x_k$  and  $t_{k+1} = c t_k$ .
- (ii) Compute  $x_{k+1}^{MA}$ ,  $x_{k+1}^{TA}$ .
- (iii)  $k=k+1$ ;  $BL = x_k^{MA}$ . If  $x_k^{TA} < BU$ , then  $x_k = x_k^{TA}$  else  $x_k = x_k^{MA}$ .
- (iv) Minimize  $S(x, x_k, t_k)$  to find  $x^k$ .
- (v) If  $[S(x^k, x_k, t_k)]^{1/2} \geq \text{eps}$ , then go to II(ii) else  $BU = x_k$ ; If  $BU - BL < \text{eps}$  then go to end else  $x_k = BL$  and go to II(iv).

It can be proved that this algorithm converges .

#### 4. Numerical Results

In order to illustrate the acceleration performance stated in section 3 , let us show the numerical results of the following problems. In each table , (b) , (c) and (d) are the results showed in [18].

(i) Rosen-Suzuki problem :

$$f = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$$

subject to

$$-x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + 8 \geq 0$$

$$-x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 + 10 \geq 0$$

$$-2x_1^2 - x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_2 + x_4 + 5 \geq 0$$

The function has a minimum  $f=-44$  at  $x=(0,1,2,-1)$ .

(ii) Beale problem :

$$f = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$

subject to  $x_1 \geq 0$  ,  $x_2 \geq 0$  ,  $x_3 \geq 0$  ,  $x_4 \leq 3$  and  $x_4 = x_1 + x_2 + 2x_3$ .

The function has a minimum  $f=1/9$  at  $x=(4/3, 7/9, 4/9, 3)$  .

(iii) Post office parcel problem :

$$f = -x_1x_2x_3 \quad \text{subject to } 0 \leq x_1 \leq 20 , 0 \leq x_2 \leq 11 ,$$

$$0 \leq x_3 \leq 42 , 0 \leq x_4 \leq 72 \text{ and } x_4 = x_1 + 2x_2 + 2x_3 .$$

The function has a minimum  $f=-3300$  at  $x=(20, 11, 15, 72)$  .

Table 4.1

THE ROSEN-SUZUKI PROBLEM  
(a) The accelerated tangent parameter sequence (c=256)

| k | $f(x^k)$   | $x_1$      | $x_2$     | $x_3$     | $x_4$       |
|---|------------|------------|-----------|-----------|-------------|
| 0 | -46.190235 | 0.09807343 | 1.0508771 | 2.1330896 | -0.97308568 |
| 1 | -44.392404 | 0.02013741 | 1.0103326 | 2.0257672 | -0.98555903 |
| 2 | -44.037648 | 0.00197659 | 1.0010217 | 2.0025327 | -0.99835910 |
| 3 | -44.002755 | 0.00014774 | 1.0000768 | 2.0001833 | -0.99988065 |
| 4 | -44.000182 | 0.00001172 | 1.0000049 | 2.0000108 | -0.99999372 |
| 5 | -44.000012 | 0.00000026 | 1.0000007 | 2.0000009 | -0.99999925 |
| 6 | -44.000000 | 0.00000007 | 1.0000000 | 2.0000000 | -1.00000000 |

(b) The tangent parameter sequence

| k  | $f(x^k)$ | $x_1$   | $x_2$  | $x_3$  | $x_4$   |
|----|----------|---------|--------|--------|---------|
| 0  | -44.810  | 0.05034 | 1.0236 | 2.0555 | -0.9555 |
| 1  | -44.029  | 0.00200 | 1.0010 | 2.0023 | -0.9971 |
| 2  | -44.000  | 0.00010 | 1.0000 | 1.9999 | -1.0001 |
| 3  | -44.010  | 0.00110 | 1.0005 | 2.0011 | -0.9985 |
| 4  | -44.009  | 0.00063 | 1.0003 | 2.0007 | -0.9991 |
| 5  | -44.005  | 0.00039 | 1.0002 | 2.0004 | -0.9995 |
| 6  | -44.003  | 0.00023 | 1.0001 | 2.0002 | -0.9997 |
| 8  | -44.001  | 0.00003 | 0.9999 | 2.0002 | -0.9998 |
| 10 | -44.000  | 0.00000 | 1.0000 | 2.0000 | -0.9999 |

(c) The Morrison-parameter sequence

| k  | $f(x^k)$ | $x_1$    | $x_2$  | $x_3$  | $x_4$   |
|----|----------|----------|--------|--------|---------|
| 0  | -44.810  | 0.050340 | 1.0236 | 2.0555 | -0.9555 |
| 1  | -44.460  | 0.031718 | 1.0147 | 2.0337 | -0.9663 |
| 2  | -44.273  | 0.019428 | 1.0090 | 2.0203 | -0.9772 |
| 3  | -44.160  | 0.011706 | 1.0054 | 2.0122 | -0.9855 |
| 10 | -44.004  | 0.000301 | 1.0001 | 2.0003 | -0.9996 |
| 20 | -44.000  | 0.000024 | 1.0002 | 1.9999 | -1.0001 |

(d) The SUMT transformation

| r         | $f(x(r))$ | $x_1$    | $x_2$  | $x_3$  | $x_4$   |
|-----------|-----------|----------|--------|--------|---------|
| $10^0$    | -41.468   | -0.01564 | 0.0125 | 1.8992 | -0.8350 |
| $10^{-1}$ | -43.326   | -0.00874 | 0.9709 | 1.9674 | -0.9572 |
| $10^{-2}$ | -43.758   | -0.00369 | 0.9933 | 1.9888 | -0.9886 |
| $10^{-3}$ | -43.924   | -0.00127 | 0.9985 | 1.9963 | -0.9966 |
| $10^{-4}$ | -43.976   | -0.00041 | 0.9996 | 1.9988 | -0.9989 |
| $10^{-5}$ | -43.992   | -0.00013 | 0.9998 | 1.9996 | -0.9996 |
| $10^{-6}$ | -43.998   | -0.00004 | 0.9995 | 1.9999 | -0.9998 |
| $10^{-7}$ | -43.999   | -0.00000 | 0.9999 | 2.0000 | -0.9999 |

Table 4.2  
THE BEALE PROBLEM

(a) The accelerated tangent parameter sequence (  $c=256$  )

| $k$ | $f(x^k)$   | $x_1$     | $x_2$      | $x_3$      | $x_4$     |
|-----|------------|-----------|------------|------------|-----------|
| 0   | 0.08902419 | 1.2983708 | 0.80108687 | 0.50271733 | 3.1027244 |
| 1   | 0.11101519 | 1.3331901 | 0.77787364 | 0.44468401 | 3.0004219 |
| 2   | 0.11111105 | 1.3333333 | 0.77777774 | 0.44444461 | 3.0000003 |

(b) The tangent parameter sequence

| $k$ | $f(x^k)$ | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|-----|----------|--------|--------|--------|--------|
| 0   | 0.0423   | 1.2054 | 0.8629 | 0.6572 | 3.3829 |
| 1   | 0.1104   | 1.3322 | 0.7785 | 0.4414 | 3.0034 |
| 2   | 0.1111   | 1.3336 | 0.7779 | 0.4442 | 3.0000 |
| 3   | 0.1111   | 1.3334 | 0.7778 | 0.4444 | 3.0000 |

(c) The Morrison-parameter sequence

| $k$ | $f(x^k)$ | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|-----|----------|--------|--------|--------|--------|
| 0   | 0.0423   | 1.2054 | 0.8629 | 0.6572 | 3.3829 |
| 1   | 0.1091   | 1.3303 | 0.7798 | 0.4494 | 3.0089 |
| 2   | 0.1110   | 1.3333 | 0.7778 | 0.4446 | 3.0002 |
| 3   | 0.1111   | 1.3333 | 0.7779 | 0.4444 | 3.0001 |
| 4   | 0.1111   | 1.3333 | 0.7778 | 0.4445 | 3.0000 |

(d) The SUMT transformation

| $r$       | $f(x(r))$ | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|-----------|-----------|--------|--------|--------|--------|
| $10^0$    | 0.7037    | 0.8952 | 0.7052 | 0.4285 | 2.4576 |
| $10^{-1}$ | 0.2328    | 1.2929 | 0.6973 | 0.3314 | 2.6503 |
| $10^{-2}$ | 0.1459    | 1.3793 | 0.7375 | 0.3516 | 2.8202 |
| $10^{-5}$ | 0.1126    | 1.3356 | 0.7763 | 0.4407 | 2.9933 |
| $10^{-7}$ | 0.1113    | 1.3336 | 0.7776 | 0.4441 | 2.9993 |
| $10^{-9}$ | 0.1111    | 1.3333 | 0.7777 | 0.4444 | 2.9999 |

Table 4.3  
THE POST OFFICE PARCEL PROBLEM

(a) The accelerated tangent parameter sequence (  $C=64$  )

| k | $f(x^k)$   | $x_1$     | $x_2$     | $x_3$     | $x_4$     |
|---|------------|-----------|-----------|-----------|-----------|
| 0 | -3538.8889 | 20.381124 | 11.496137 | 15.103793 | 72.790492 |
| 1 | -3537.1789 | 20.378465 | 11.492997 | 15.102620 | 72.784848 |
| 2 | -3503.6666 | 20.325985 | 11.430291 | 15.080436 | 72.673719 |
| 3 | -3317.4940 | 20.028554 | 11.041133 | 15.001921 | 72.057332 |
| 4 | -3300.0180 | 20.000029 | 11.000043 | 15.000001 | 72.000059 |
| 5 | -3300.0000 | 20.000000 | 11.000000 | 15.000000 | 72.000000 |

(b) The tangent parameter sequence

| k | $f(x^k)$ | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|---|----------|--------|--------|--------|--------|
| 0 | -3305.0  | 20.013 | 11.018 | 14.988 | 72.026 |
| 1 | -3300.0  | 20.000 | 11.000 | 15.000 | 72.000 |

(c) The Morrison-parameter sequence

| k   | $f(x^k)$ | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|-----|----------|--------|--------|--------|--------|
| 0   | -3305.0  | 20.013 | 11.018 | 14.988 | 72.026 |
| 5   | -3304.8  | 20.012 | 11.018 | 14.988 | 72.025 |
| 30  | -3304.1  | 20.010 | 11.015 | 14.990 | 72.021 |
| 100 | -3302.5  | 20.006 | 11.009 | 14.994 | 72.013 |
| 200 | -3301.8  | 20.004 | 11.007 | 14.996 | 72.010 |

(d) The SUMT transformation

| r         | $f(x(r))$ | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|-----------|-----------|--------|--------|--------|--------|
| $10^0$    | -3283.6   | 19.868 | 10.892 | 15.175 | 72.000 |
| $10^{-1}$ | -3294.8   | 19.959 | 10.964 | 15.057 | 72.000 |
| $10^{-2}$ | -3298.1   | 19.989 | 10.984 | 15.022 | 72.000 |
| $10^{-3}$ | -3299.0   | 19.997 | 10.989 | 15.012 | 72.000 |
| $10^{-5}$ | -3299.3   | 20.000 | 10.991 | 15.009 | 72.000 |
| $10^{-7}$ | -3300.0   | 20.000 | 11.000 | 15.000 | 72.000 |

From these numerical results , it can be said that our algorithm has the more rapidly convergent property than that of Kowalik et al. .

### 5. The Rate of Convergence

Let us derive the rates of convergence of the ordinary and accelerated Morrison-parameter sequences in the neighbourhood of the optimal solution  $x^*$  of the problem (B) .

If the sequence  $\{a_n\}$  converges to  $\alpha$  and if there exists real number  $\rho$  and  $c \neq 0$  such that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1} - \alpha|}{|a_n - \alpha|^\rho} = c$$

then  $\rho$  is called the rate of convergence of  $\{a_n\}$  and  $c$  is called the asymptotic error constant.

In order to prove the asymptotic convergence theorems of the SULSET , the next lemma is necessary.

LEMMA 5.1 Let  $P$  be an  $n \times k$  - matrix , the rank of which is  $m$  ,  $H$  be a positive definite  $n \times n$  - matrix and  $A = P P^T$  . Then there exists

$$(1) \quad V = \lim_{t \rightarrow \infty} [tA + H]^{-1}$$

and the rank of  $V$  is  $(n-1)$  , where  $1 = \min(n, m)$  . And

$$(2) \quad V P = 0 \quad .$$

Furthermore , if  $t \rightarrow \infty$  , the approach of  $[tA + H]^{-1}P$  towards null-matrix is elementwisely the order of  $1/t$  .

Proof. If  $m \geq n$  , the rank of  $A$  is  $n$  and  $A$  is positive definite. Therefore

$$(3) \quad V = \lim_{t \rightarrow \infty} [tA + H]^{-1} = \lim_{t \rightarrow \infty} \frac{1}{t} [A + \frac{1}{t} H]^{-1} = \lim_{t \rightarrow \infty} \frac{1}{t} A^{-1} = 0 ,$$

by which the conclusions are satisfied.

If  $m < n$  , let us form an  $n \times m$  - matrix  $\tilde{P}$  of rank  $m$  which consists

of a set of  $m$  linearly independent column vectors of  $P$  and an  $n \times (n-m)$  - matrix  $N$  which consists of a set of  $(n-m)$  linearly independent column vectors which are orthogonal to all the column vectors of  $\tilde{P}$ . Then the  $n \times n$  - matrix  $[\tilde{P} \ N]$  is nonsingular. Let

$$(4) \quad \tilde{V}(t) = [\tilde{P} \ N]^{-1} [tA + H]^{-1} \begin{bmatrix} \tilde{P}^T \\ N^T \end{bmatrix}^{-1} \\ = \left[ t \begin{bmatrix} \tilde{P}^T \\ N^T \end{bmatrix} [P \ P^T] [\tilde{P} \ N] + \begin{bmatrix} \tilde{P}^T \\ N^T \end{bmatrix} H [\tilde{P} \ N] \right]^{-1} \\ = \left[ t \begin{bmatrix} \tilde{P}^T P & P^T \tilde{P} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{P}^T H P & \tilde{P}^T H N \\ N^T H \tilde{P} & N^T H N \end{bmatrix} \right]^{-1}$$

Let

$$(5) \quad Q = \tilde{P}^T P P^T \tilde{P}, \quad Q_1 = \tilde{P}^T H \tilde{P}, \quad Q_2 = \tilde{P}^T H N, \quad Q_3 = \tilde{P}^T H N \text{ and } Q_4 = N^T H N.$$

Then by using the partial inversion of matrix,

$$(6) \quad \tilde{V}(t) = \begin{bmatrix} [tQ + Q_1 - Q_2 Q_4^{-1} Q_3]^{-1} & -[tQ + Q_1]^{-1} Q_2 [Q_4 - Q_3 [tQ + Q_1]^{-1} Q_2]^{-1} \\ -Q_4^{-1} Q_3 [tQ + Q_1 - Q_2 Q_4^{-1} Q_3]^{-1} & [Q_4 - Q_3 [tQ + Q_1]^{-1} Q_2]^{-1} \end{bmatrix}$$

Since  $[tA + H]$  is positive definite,  $\tilde{V}(t)$  is positive definite.

Therefore  $[tQ + Q_1 - Q_2 Q_4^{-1} Q_3]$  and  $[Q_4 - Q_3 [tQ + Q_1]^{-1} Q_2]$  are positive definite and invertible. And since  $Q$  and  $Q_1$  are positive definite, from the discussion of the case  $m \geq n$ ,

$$(7) \quad \lim_{t \rightarrow \infty} [tQ + Q_1 - Q_2 Q_4^{-1} Q_3]^{-1} = 0, \text{ and}$$

$$(8) \quad \lim_{t \rightarrow \infty} [tQ + Q_1]^{-1} = 0.$$

Thus

$$(9) \quad \tilde{V} = \lim_{t \rightarrow \infty} \tilde{V}(t) = \begin{bmatrix} 0 & 0 \\ 0 & Q_4^{-1} \end{bmatrix}$$

Since  $Q_4$  is an  $(n-m) \times (n-m)$  - positive definite matrix, the rank of  $\tilde{V}$  is  $(n-m)$ . And since  $[\tilde{P} \ N]$  is nonsingular, the rank of  $V$  is  $(n-m)$ .

From (4),

$$(10) \quad V = \begin{bmatrix} \tilde{P} & N \end{bmatrix} \tilde{V} \begin{bmatrix} \tilde{P}^T \\ N^T \end{bmatrix} \\ = N \begin{bmatrix} N^T H N \end{bmatrix}^{-1} N^T.$$

Therefore, from the definition of  $N$ ,

$$V \tilde{P} = 0, \text{ that is, } V P = 0.$$

Since

$$(11) \quad V(t) = \begin{bmatrix} tA + H \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{P} & N \end{bmatrix} \tilde{V}(t) \begin{bmatrix} \tilde{P}^T \\ N^T \end{bmatrix} \\ = \tilde{P} \begin{bmatrix} tQ + Q_1 - Q_2 Q_4^{-1} Q_3 \end{bmatrix}^{-1} \tilde{P}^T - N Q_4^{-1} Q_3 \begin{bmatrix} tQ + Q_1 - Q_2 Q_4^{-1} Q_3 \end{bmatrix}^{-1} \tilde{P}^T \\ - \tilde{P} \begin{bmatrix} tQ + Q_1 \end{bmatrix}^{-1} Q_2 \begin{bmatrix} Q_4 - Q_3 \begin{bmatrix} tQ + Q_1 \end{bmatrix}^{-1} Q_2 \end{bmatrix}^{-1} N^T + \\ N \begin{bmatrix} Q_4 - Q_3 \begin{bmatrix} tQ + Q_1 \end{bmatrix}^{-1} Q_2 \end{bmatrix}^{-1} N^T,$$

$$(12) \quad V(t) P = \tilde{P} \begin{bmatrix} tQ + Q_1 - Q_2 Q_4^{-1} Q_3 \end{bmatrix}^{-1} \tilde{P}^T P - N Q_4^{-1} Q_3 \begin{bmatrix} tQ + Q_1 - Q_2 Q_4^{-1} Q_3 \end{bmatrix}^{-1} \tilde{P}^T P.$$

Therefore

$$\lim_{t \rightarrow \infty} t \begin{bmatrix} tA + H \end{bmatrix}^{-1} P = \tilde{P} Q^{-1} \tilde{P}^T P - N Q_4^{-1} Q_3 Q^{-1} \tilde{P}^T P,$$

that is, if  $t \rightarrow \infty$ , the approach of  $\begin{bmatrix} tA + H \end{bmatrix}^{-1} P$  towards null matrix is elementwisely the order of  $1/t$ . Q.E.D.

**THEOREM 5.1** ( Asymptotic Convergence Theorem of the Morrison-parameter Sequence System ) If the conditions of Theorem 2.1, 2.2, and 2.3 are satisfied and  $\sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*)$  is positive definite or  $P P^T$  is positive definite, then for  $k$  large enough, the rate of convergence of the Morrison-parameter sequence of the SULSET (C) ( $c_i = 1, i=1, \dots, m$ ),

$$(13) \quad R_M = \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|} \approx \frac{1}{2} K, \text{ where}$$

$$(14) \quad K = \lim_{t \rightarrow \infty} \nabla f(x^*)^T t \left[ t \begin{bmatrix} P P^T + \nabla f(x^*) \nabla f(x^*)^T \end{bmatrix} + \sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*) \right]^{-1} \nabla f(x^*)$$

and

$$(15) \quad P = [\nabla g_1(x^*) \dots \nabla g_m(x^*)] \quad .$$

Proof. From the Taylor's theorem ,

$$(16) \quad f(x) = f(x^*) + \nabla f(x^* + \theta \delta)^T \delta \quad \text{and}$$

$$(17) \quad g_i(x) = g_i(x^*) + 2 g_i(x^*) \nabla g_i(x^*)^T \delta + \frac{1}{2} \delta^T [ 2 \nabla g_i(x^* + \theta_i \delta) \nabla g_i(x^* + \theta_i \delta)^T + 2 g_i(x^* + \theta_i \delta) \nabla^2 g_i(x^* + \theta_i \delta) ] \delta$$

where  $x = x^* + \delta$  ,  $0 \leq \theta$  ,  $\theta_i \leq 1$  and  $i=1, \dots, m$  .

Since  $k$  is large enough , from Theorem 2.1 , 2.2 and 2.3 ,  $x^k$  is near enough to  $x^*$ . Therefore

$$\nabla f(x^* + \theta \delta) \cong \nabla f(x^*)$$

$$\nabla g_i(x^* + \theta_i \delta) \cong \nabla g_i(x^*) , i=1, \dots, m ,$$

$$\nabla^2 g_i(x^* + \theta_i \delta) \cong \nabla^2 g_i(x^*) , i=1, \dots, m , \text{ and}$$

$$g_i(x^* + \theta_i \delta) / [ f(x^*) - X_k ] \cong \lambda_i^* , i=1, \dots, m .$$

From (2.1) , (15) and (17) ,

$$(18) \quad \delta^k = -[\tau_k [ PP^T + \nabla f(x^*) \nabla f(x^*)^T ] + \sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*) ]^{-1} \nabla f(x^*)$$

where

$$(19) \quad \tau_k = 1 / [ f(x^*) - X_k ] \quad .$$

Since  $\tau_k$  is large enough , from Lemma 5.1 ,

$$(20) \quad R_M \cong \frac{\|\delta^{k+1}\|}{\|\delta^k\|} \cong \frac{\tau_k}{\tau_{k+1}} = \frac{f(x^*) - X_{k+1}}{f(x^*) - X_k} \\ = 1 - [ S(x^k, X_k) ]^{1/2} / [ f(x^*) - X_k ] \\ = 1 - [ 1 - \nabla f(x^*)^T \tau_k [ \tau_k [ PP^T + \nabla f(x^*) \nabla f(x^*)^T ] + \sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*) ]^{-1} \nabla f(x^*) ]^{1/2} \\ \cong \frac{1}{2} K \quad \text{Q.E.D.}$$

**THEOREM 5.2** ( Asymptotic Convergence Theorem of the Accelerated Morrison-parameter Sequence System of the SULSET ) If the conditions of Theorem

5.1 are satisfied, for  $k$  large enough, the asymptotic convergence property of the accelerated Morrison-parameter sequence system of the SULSET (D) can be described such as

$$(21) \quad R_A = \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|} \approx \frac{M}{2c t_k}$$

where

$$(22) \quad M = \lim_{t \rightarrow \infty} \nabla f(x^*)^T t [t PP^T + \sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*)]^{-1} \nabla f(x^*)$$

Proof. By using the same formulation of the proof of Theorem 5.1,

$$(23) \quad \delta^k = -[\tau_k [PP^T + \frac{1}{t_k} \nabla f(x^*) \nabla f(x^*)^T] + \sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*)]^{-1} \nabla f(x^*)$$

where

$$(24) \quad \tau_k = \frac{t_k}{f(x^*) - X_k}$$

Therefore

$$(25) \quad R_A \approx \frac{\|\delta^{k+1}\|}{\|\delta^k\|} = \frac{\tau_k}{\tau_{k+1}} = \frac{1}{c} \cdot \frac{f(x^*) - X_{k+1}}{f(x^*) - X_k}$$

where

$$(26) \quad \frac{f(x^*) - X_{k+1}}{f(x^*) - X_k} = 1 - [1 - \frac{1}{t_k} \nabla f(x^*)^T \tau_k [\tau_k [PP^T + \frac{1}{t_k} \nabla f(x^*) \nabla f(x^*)^T] + \sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*)]^{-1} \nabla f(x^*)]^{1/2}$$

$$\approx \frac{M}{2 t_k}$$

Q.E.D.

## 6. The Generalization

Let us give the generalized formulation for the SULSET. For the problem (A), we will take the following function:

$$(1) \quad S(x, X_k) = |f(x) - X_k|^\alpha + W(x), \quad \alpha > 0,$$

where  $W(x)$  is a continuous function with respect to  $x$  and

$$(2) \quad W(x) = \begin{cases} 0 & (\text{if } g_i(x) \leq 0, i=1, \dots, m) \\ >0 & (\text{otherwise}) \end{cases}$$

Let the feasible region be denoted by  $R = \{x \mid g_i(x) \leq 0, i=1, \dots, m\}$  and

$$\text{if } \inf_{x \in R} f(x) < X_k < \sup_{x \in R} f(x),$$

$$\text{then } \inf_{x \in E^n} S(x, X_k) = 0.$$

It can be proved that if there exists a solution of the problem

(A) and the sequence  $\{X_k\}$  converges to the minimum value  $v^*$  from below, then under appropriate conditions, the sequence of minima,  $\{x^k\}$ , of  $S(x, X_k)$  converges to the solution.

In order to prove the convergence property, the following definition and lemma are necessary.

DEFINITION 6.1 A nonempty set  $M^* \subset M$  is called an isolated set of  $M$  if there exists a closed set  $E$  such that  $E^0 \supset M^*$  and such that if  $x \in E - M^*$  then  $x \notin M$ .

LEMMA 6.1 If a set of local minima  $A^*$  corresponding to the local minimum value  $v^*$  of the problem (A) is a nonempty isolated compact set, then there exists a compact set  $S$  such that  $A^* \subset S^0$ , and for any point  $y \in R \cap S$ , if  $y \notin A^*$ , then  $f(y) > v^*$ .

Proof. See the ref. [11], p.47.

The proof of convergence follows the format used by Fiacco and McCormick.

THEOREM 6.1 (Convergence Theorem of the Generalized SULSET) If (a)  $f, g_1, \dots, g_m$  are continuous functions of  $x$ , (b)  $S(x, X_k)$  is the fun-

ction as defined in (1), (c) a set of points  $A^*$  that are local minima of the problem (A) corresponding to the local minimum value  $v^*$  is a non-empty isolated compact set, and (d)  $\{X_k\}$  is a monotonically increasing sequence which converges to  $v^*$  from below, then

- (i)  $f(x^k) \leq v^*$ ,
- (ii) there exists a compact set  $S$ , given in Lemma 6.1, such that  $A^* \subset S^0$ , and for  $k$  large enough, the unconstrained minima  $x^k$  of  $S(x, X_k)$  in  $S^0$  exists and every limit point of any subsequence of  $\{x^k\}$  is in  $A^*$ .
- (iii)  $\lim_{k \rightarrow \infty} W(x^k) = 0$ ;
- (iv)  $\lim_{k \rightarrow \infty} f(x^k) = v^*$ ;
- (v)  $\lim_{k \rightarrow \infty} S(x^k, X_k) = 0$ .

Proof. Since

$|f(x^k) - X_k|^\alpha \leq S(x^k, X_k) \leq S(x^*, X_k) = (v^* - X_k)^\alpha$ , where  $x^* \in A^*$ , if  $f(x^k) > X_k$  then  $f(x^k) - X_k \leq v^* - X_k$ , that is,  $f(x^k) \leq v^*$  else  $f(x^k) \leq X_k \leq v^*$ . This proves part (i).

From the conditions (a) and (c), there exists a compact set  $S$  where  $A^* \subset S^0$  such that  $f(y) > v^*$  for all  $y \in R \cap S$  and  $y \notin A^*$ .

Let  $x^k$  be a minimum of  $S(x, X_k)$  in  $S$ . Since  $\{x^k\}$  is on the compact set  $S$ , there exists a convergent subsequence. For simplicity, let it also be denoted by  $\{x^k\}$ .

Let  $\hat{x}$  be the convergent point of  $\{x^k\}$  and let us assume  $\hat{x} \notin A^*$ . Then  $S(x, v^*) > 0$ . Since  $\lim_{k \rightarrow \infty} X_k = v^*$ , if  $x^0 \in A^*$ , then  $\lim_{k \rightarrow \infty} S(x^k, X_k) = S(\hat{x}, v^*) > 0 = \lim_{k \rightarrow \infty} S(x^0, X_k)$ . Therefore for  $k$  large enough, because of the continuity of  $S(x, X_k)$  with respect to  $x$  and  $X_k$ , this contradicts the fact that  $x^k$  is the minimum of  $S(x, X_k)$ . Hence  $\hat{x} \in A^*$ . But since  $A^* \subset S^0$ , for  $k$  large enough,  $x^k$  must be in  $S^0$ . This proves part (ii). Parts (iii) - (v) follow from the fact that  $\lim_{k \rightarrow \infty} f(x^k) = v^*$ .

The Morrison's type parameter sequence is following in this case :

$$(3) \quad X_{k+1}^M = X_k + [S(x^k, X_k)]^{1/\alpha}.$$

In order to prove the convergence of the Morrison-parameter sequence to  $v^*$  from below, let us provide the following assumptions.

ASSUMPTION 6.1 Let the minimum value of the problem :

minimize:  $f(x)$  subject to  $g_i(x) \leq q_i, q_i \geq 0, i=1, \dots, m$ ,

be a function of  $m$ -vector  $q = (q_1, \dots, q_m)^T$  and be denoted by  $v^*(q)$ .

Then  $v^*(q)$  is continuous at  $q = 0$  in the region  $\{q | q_i \geq 0, i=1, \dots, m\}$ .

ASSUMPTION 6.2 For  $\forall \varepsilon > 0$  there exists  $\Delta(\varepsilon) > 0$  such that

$$\sum_{i=1}^m [\max(0, g_i(x))]^2 < \varepsilon \text{ for } \forall x \in \{x | W(x) < \Delta(\varepsilon)\}.$$

THEOREM 6.2 (Convergence Theorem of the Generalized Morrison-parameter

Sequence) If the conditions of Theorem 6.1, Assumption 6.1 and 6.2

are satisfied, then the Morrison-parameter sequence  $\{X_k\}$ , such as defined in (3), converges to  $v^*$  ( $= v^*(0)$ ) from below.

Proof. If  $X_k \leq v^*$ , then  $X_{k+1} \leq v^*$ , because  $X_{k+1} \geq X_k$  and

$$\begin{aligned} (X_{k+1} - X_k)^\alpha &= S(x^k, X_k) \\ &\leq |f(x^*) - X_k|^\alpha + W(x^*) \quad , \quad \forall x^* \in A^* \\ &= |v^* - X_k|^\alpha \end{aligned}$$

Since  $\{X_k\}$  is monotonically increasing and bounded above, it converges to  $X$ . From Assumption 6.1, for  $\forall \varepsilon > 0$  there exists  $\delta > 0$  such

that  $|v^*(q) - v^*(0)| < \varepsilon$  for  $\forall q = \{q | \|q\| < \delta, q_i \geq 0, i=1, \dots, m\}$ .

Let  $k$  be large enough so that  $X_{k+1} - X_k < [\Delta(\delta)]^{1/\alpha}$ . Then  $S(x^k, X_k) <$

$\Delta(\delta)$ . Therefore  $W(x^k) < \Delta(\delta)$ . From Assumption 6.2,  $\|q^k\| < \delta$

where  $q_i^k = \max(0, g_i(x))$ ,  $i=1, \dots, m$ . So  $|v^*(q^k) - v^*(0)| < \varepsilon$ .

Since  $v^*(q^k) \leq v^*(0)$ ,  $v^*(q^k) > v^*(0) - \varepsilon$ .

From the definition of  $v^*(q^k)$  ,  $v^*(q^k) \leq f(x^k)$  .

Hence  $0 \leq v^*(0) - f(x^k) < \varepsilon$  .

Therefore  $f(x^k) \rightarrow v^*$  . Thus  $X_k \rightarrow v^*$  as  $k \rightarrow \infty$  , i.e.  $X = v^*$  .

Q.E.D.

For the accelerated SULSET of section 3 , a generalized expression can be described as follows :

$$(4) \quad S(x, X_k, t_k) = |f(x) - X_k|^\alpha + p(t_k) W(x) , \alpha > 0$$

where  $p(t)$  has the property that if  $0 < t_1 < t_2$  , then  $0 < p(t_1) < p(t_2)$  and if  $\{t_k\}_k$  is a monotonically increasing sequence of positive values where  $\lim_{k \rightarrow \infty} t_k = \infty$  , then  $\lim_{k \rightarrow \infty} p(t_k) = \infty$  .

Here it can be said that if  $X_k \rightarrow v^*$  and  $t_k \rightarrow \infty$  , the sequence of minima of  $S(x, X_k, t_k)$  converges into  $A^*$  and its convergence is accelerated.

## 7. Conclusions

Comparing with the Fiacco and McCormick's algorithm , this type of constrained minimization technique is better. The rate of convergence depends mainly upon the strategies of how the parameter  $X_k$  being chosen and the types of the penalty functions being used. The suggested acceleration technique performs excellently in practice. Our SULSET routine can solve any nonlinear programming problem with an arbitrary starting point. It automatically adjusts the solution to jump into the feasible region. From that point , in the case that the optimal solution is on the boundary of the feasible region , it moves the solution towards the boundary , where the optimal solution is located , until the solution begins to function until the global optimal solution is obtained. In the

case that the optimal solution is inside the feasible region , after the value of  $X_k$  is less than the optimal solution value , as in the former case , the parameter sequence routine functions to give the optimal solution in a single iteration.

Several techniques are under consideration to improve the convergence and numerical stability property of the SULSET routine. An extension of applying the convergence theorem of Lagrange multiplier to our technique is also under consideration.

And our algorithm uses the Maximum Neighbourhood Method which is believed to be the best least-squares estimator. Therefore the computation is very stable.

At last , we would like to point out that the SULSET will be the leading technique for constrained minimization problems.

#### Acknowledgements

We would like to thank Professor Sigeiti Moriguti , Assistant Professor Masao Iri , Mr. T. Takasawa and members of Moriguti Laboratory of the University of Tokyo for their valuable suggestions and helps.

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