

Ribbon Knots and Ribbon Disks

Kouhei ASANO (Kinki Univ.),

Yoshihiko MARUMOTO (Osaka City Univ.)

and Takaaki YANAGAWA (Kobe Univ.)

Given a ribbon knot, we will define, in § 1, the ribbon disk pair associated with it. On the other hand, J.F.P.Hudson and D.W.Sumners gave a method to construct a disk pair [3], [14]. In §1 and 2, we will generalize this construction and show that a ribbon disk pair is obtained by this construction and vice versa.

In [11], C.D.Papakyriakopoulos proved that the complement of a classical knot is aspherical. As an analogy of this, we will prove, in §3, that the complement of a ribbon disk is aspherical, and it follows from this that the fundamental group of a ribbon knot complement has no element of finite order. In the final section, we will calculate the higher homotopy groups of a higher-dimensional ribbon knot complement, and in Theorem 4.3 we show that a ribbon n -knot for $n \geq 3$ is unknotted if the fundamental group of the knot complement is the infinite cyclic group. This result is proved independently by A.Kawauchi and T.Matsumoto [6].

Throughout the paper, we work in the piecewise-linear category although the results remain valid in the smooth category.

§ 1. Preliminaries

1.1. By S^n denote an n -sphere, and by B^n or D^n an n -disk.

By ∂M , $\text{int } M$ and $\text{cl } M$ denote the boundary, the interior and the closure of a manifold M respectively. In this paper, every submanifold in a manifold is assumed to be locally flat. If $\partial M \neq \emptyset$, by $\mathcal{D}M$ we mean the double of M , i.e. $\mathcal{D}M$ is obtained from the disjoint union of two copies of M by identifying their boundaries via the identity map. For a subcomplex C in a manifold M , $N(C; M)$ is a regular neighbourhood of C in M . By a pair (M, W) denote a manifold M and a proper submanifold W in M , i.e. $W \cap \partial M = \partial W$. Two pairs (M_1, W_1) and (M_2, W_2) are equivalent if there exists a homeomorphism from M_1 to M_2 which maps W_1 to W_2 . Let $\partial(M, W) = (\partial M, \partial W)$ and $\mathcal{D}(M, W) = (\mathcal{D}M, \mathcal{D}W)$.

An n-knot will mean a pair (S^{n+2}, K^n) with K^n homeomorphic to an n -sphere, and we will usually refer to the n-knot K^n . An n -knot is unknotted if it bounds an $(n+1)$ -disk in S^{n+2} . For a proper disk D^n in a manifold M , (M, D^n) is unknotted, or D^n is unknotted in M , if there exists an $(n+1)$ -disk D^{n+1} in M such that $D^{n+1} \cap \partial M$ is an n -disk in ∂D^{n+1} and $\text{cl}(\partial D^{n+1} \cap \text{int } M) = D^n$. For terminologies in handle theory, we refer the readers to [12], and for knot theory, to [15].

1.2. Let $S_0^n, S_1^n, \dots, S_m^n$ be mutually disjoint n -spheres in a q -manifold M^q for $n \geq 1$ and $q \geq 3$. Suppose that an embedding β of $B^n \times I$ into M^q , where $I = [0, 1]$, satisfies the following :

- (1) $\beta(B^n \times I) \cap (S_0^n \cup \dots \cup S_m^n) = \beta(B^n \times \partial I)$, and
- (2) the orientation of $\beta(B^n \times I)$ is coherent with that of $S_0^n \cup \dots \cup S_m^n$.

Then we call β or $\beta(B^n \times I)$ a band compatible with $S_0^n \cup \dots \cup S_m^n$.

Let $\beta_1, \dots, \beta_m$ be bands compatible with $S_0^n \cup \dots \cup S_m^n$ such that

- (1) $\beta_i(B^n \times I) \cap \beta_j(B^n \times I) = \emptyset$ if $i \neq j$, and

(2) $\bigcup \{S_i^n; 0 \leq i \leq m\} \cup \bigcup \{\beta_j(B^n \times I); 1 \leq j \leq m\}$ is connected.*

Then

$$(\bigcup \{S_i^n; 0 \leq i \leq m\} - \bigcup \{\beta_j(B^n \times \partial I); 1 \leq j \leq m\}) \cup \bigcup \{\beta_j(\partial B^n \times I); 1 \leq j \leq m\}$$

is an n -sphere, and denoted by

$$\mathcal{F}(S_0^n, \dots, S_m^n; \beta_1, \dots, \beta_m).$$

Suppose that $M^q = S^{n+2}$ and there exist mutually disjoint $(n+1)$ -disks $B_0^{n+1}, \dots, B_m^{n+1}$ with $\partial B_i^{n+1} = S_i^n$ for $0 \leq i \leq m$. Then

$$K^n = \mathcal{F}(S_0^n, \dots, S_m^n; \beta_1, \dots, \beta_m)$$

is called a ribbon n -knot of type $(\beta_1, \dots, \beta_m)$.

Our definition of a ribbon n -knot is equivalent to that of [2], [20].

1.3. Remark. In 1.2, it is easily seen that we can deform each band isotopically so that

$$\beta_i(B^n \times I) \cap S_j^n = \begin{cases} \beta_i(B^n \times 0) & \text{if } j = 0, \\ \beta_i(B^n \times 1) & \text{if } j = i, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

1.4. Let D^{n+3} be obtained from the disjoint union of $S^{n+2} \times I$ and B^{n+3} by identifying $S^{n+2} \times 1$ and ∂B^{n+3} . Let K^n be a ribbon n -knot of type $(\beta_1, \dots, \beta_m)$, then we can construct an $(n+1)$ -disk L^{n+1} in D^{n+3} which bounds $K^n \times 0$ as follows: Let $D_i^{n+1} = (S_i^n \times [0, 3/4]) \cup (B_i^{n+1} \times 3/4)$ in $S^{n+2} \times I$ for $0 \leq i \leq m$, where B_i^{n+1} and S_i^n are as in 1.2. For $1 \leq j \leq m$, let $\bar{\beta}_j : B^n \times I \times I \rightarrow S^{n+2} \times I$ be the product of β_j and a map from I into I which takes t to $t/2$, i.e.

$$\bar{\beta}_j(x, y, t) = (\beta_j(x, y), t/2)$$

for $x \in B^n$ and $y, t \in I$. Then

$$L^{n+1} = (\bigcup \{D_i^{n+1}; 0 \leq i \leq m\} - \bigcup \{\bar{\beta}_j(B^n \times \partial I \times I); 1 \leq j \leq m\}) \cup \bigcup \{\bar{\beta}_j(\partial B^n \times I \times I) \cup \bar{\beta}_j(B^n \times I \times 1); 1 \leq j \leq m\}$$

is an $(n+1)$ -disk and bounds $K^n \times 0$ in D^{n+3} . Note that the section of L^{n+1} by $S^{n+2} \times t$ is

- | | |
|---|-----------------------|
| (1) $K^n \times t$ | if $0 \leq t < 1/2$, |
| (2) $(\bigcup \{S_i^n; 0 \leq i \leq m\} \cup \bigcup \{\beta_j(B^n \times I); 1 \leq j \leq m\}) \times 1/2$ | if $t = 1/2$, |
| (3) $(S_0^n \cup \dots \cup S_m^n) \times t$ | if $1/2 < t < 3/4$, |
| (4) $(B_0^n \cup \dots \cup B_m^n) \times 3/4$ | if $t = 3/4$, |
| (5) $\emptyset$ | if $3/4 < t \leq 1$. |

(See Fig. 1.1.)

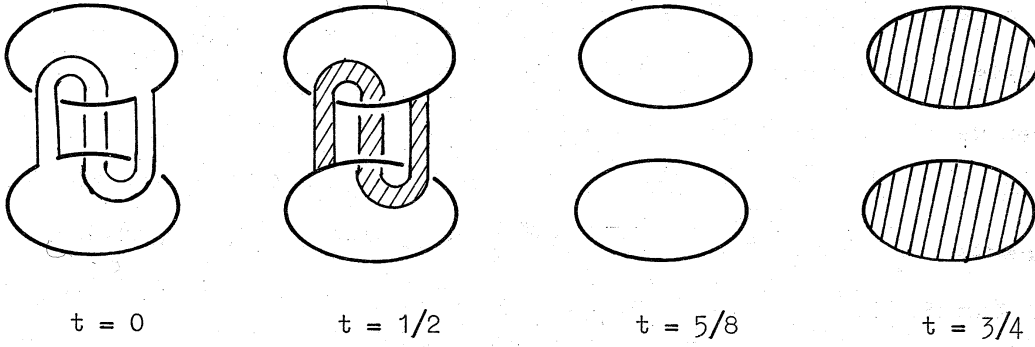


Fig. 1.1

We call L^{n+1} in D^{n+3} the ribbon $(n+1)$ -disk associated with a ribbon n -knot K^n , or (D^{n+3}, L^{n+1}) the ribbon $(n+1)$ -disk pair associated with K^n .

The double $\mathfrak{D}(D^{n+3}, L^{n+1})$ of a ribbon $(n+1)$ -disk pair is an $(n+1)$ -knot in the $(n+3)$ -sphere $\mathfrak{S}D^{n+3}$. Since $\mathfrak{D}(D_0^{n+1} \cup \dots \cup D_m^{n+1})$ is trivial $(n+1)$ -link and each $\mathfrak{D}(\beta_i(B^n \times I \times I))$ is a band, L^{n+1} is a ribbon $(n+1)$ -knot. We say that $\mathfrak{D}(D^{n+3}, L^{n+1})$ is an equatorial knot of $\mathfrak{D}(D^{n+3}, L^{n+1})$. (See [20].)

1.5. We will generalize the construction of $(n+1)$ -disk in [3] and [14], for $n \geq 1$. Let D_0^{n+1} be an unknotted $(n+1)$ -disk in B^{n+3} . Adding

m 1-handles $h_1^1, \dots, h_m^1$ to B^{n+3} such that $h_i^1 \cap D_0^{n+1} = \emptyset$ for each i , we obtain an $(n+3)$ -disk with m 1-handles, say V . We take mutually disjoint oriented 1-spheres $\alpha_1, \dots, \alpha_m$ on $\partial V - \partial D_0^{n+1}$ such that α_i intersects the belt sphere of h_i^1 at only one point and $\alpha_i \cap h_j^1 = \emptyset$ for $i \neq j$. Then we call α_i a standard curve on ∂V . Let Δ_0 be a proper 2-disk in $N(\partial D_0^{n+1}; \partial V)$ such that Δ_0 intersects ∂D_0^{n+1} at only one point, then we call Δ_0 a meridian disk of ∂D_0^{n+1} in ∂V and $\alpha_0 = \partial \Delta_0$ a meridian of ∂D_0^{n+1} in ∂V , where we give an orientation to α_0 . Let u_i be a simple closed curve in $\partial V - \partial D_0^{n+1}$ for $1 \leq i \leq m$ such that there exists an ambient isotopy of ∂V which carries u_i to α_i for all i . Then we add m 2-handles $h_1^2, \dots, h_m^2$ to V along $u_1, \dots, u_m$ such that $h_i^2 \cap D_0^{n+1} = \emptyset$ for each i . By the handle cancelling theorem, h_i^2 cancels h_i^1 for each i . Thus $V \cup h_1^2 \cup \dots \cup h_m^2$ is an $(n+3)$ -disk D^{n+3} . In general, D_0^{n+1} is not unknotted in D^{n+3} , so we rewrite D_0^{n+1} in D^{n+3} as L^{n+1} . We say that the pair (D^{n+3}, L^{n+1}) is of S-type, or simply L^{n+1} is of S-type when no confusion can arise.

Let Δ_{0i} , for $1 \leq i \leq m$, be mutually disjoint meridian disks of ∂D_0^{n+1} in ∂V , and γ_i a band in ∂V compatible with α_i and $\alpha_{0i} = \partial \Delta_{0i}$ such that

- (1) $\gamma_i(B^1 \times I) \cap \gamma_j(B^1 \times I) = \emptyset$ for $i \neq j$, and
- (2) $\gamma_i(B^1 \times I) \cap N(\partial D_0^{n+1}; \partial V) = \gamma_i(B^1 \times 0)$ for $1 \leq i \leq m$.

Then there exists an ambient isotopy of ∂V which carries v_i to α_i for $1 \leq i \leq m$, where $v_i = \mathcal{F}(\alpha_i, \alpha_{0i}; \gamma_i)$ for each i . Thus the $(n+3)$ -manifold obtained from V by adding m 2-handles with v_i , for $1 \leq i \leq m$, as the attaching spheres is an $(n+3)$ -disk which contains D_0^{n+1} as a proper $(n+1)$ -disk, then this disk pair is said to be of S*-type. Clearly, a disk pair of S*-type is of S-type.

1.6. Remark. In 1.5, if $n \geq 2$, then the attaching sphere u_i of h_i^2 is uniquely determined up to ambient isotopy by its homotopy class $[u_i]$ in $\pi_1(\partial V - \partial D_0^{n+1})$. For $n = 1$, this is false.

1.7. Let C_0 be a bouquet of $m+1$ 1-spheres $e_0^1, e_1^1, \dots, e_m^1$. Let z_i be the element of $\pi_1(C_0)$ represented by e_i^1 for $0 \leq i \leq m$. By C denote a 2-dimensional cell complex obtained from C_0 by attaching 2-cells $e_1^2, \dots, e_m^2$ with the boundary formulae corresponding to $w_1, \dots, w_m$, where $w_i = w_i(z_0, z_1, \dots, z_m)$ is an element of $\pi_1(C_0)$ such that $w_i(1, z_1, \dots, z_m) = z_i$ for $1 \leq i \leq m$. Then we call C a cell complex of S-type.

In 1.5, $cl(B^{n+3} - N(D_0^{n+1}; B^{n+3}))$ has a 1-dimensional spine. Hence, by the assumption on the attaching spheres u_i of h_i^2 , we have the following :

1.8. Proposition. Let (D^{n+3}, L^{n+1}) be an $(n+1)$ -disk pair of S-type for $n \geq 1$. Then $cl(D^{n+3} - N(L^{n+1}; D^{n+3}))$ collapses to a cell complex of S-type.

§ 2. Ribbon disks and disk pairs of S-type

2.1. Lemma. Let $w = w(z_0, z_1, \dots, z_m)$ be a word in F , the free group on $z_0, z_1, \dots, z_m$. Then $w(1, z_1, \dots, z_m) = z_i$ in F if and only if there exist a word t_j in F and an integer ε_j such that

$$w = \prod_j (t_j z_0 t_j^{-1})^{\varepsilon_j} z_i.$$

Proof. The sufficiency is trivial. The necessity is proved by easy calculation, see [1].

2.2. Lemma. Let D_0^{n+1} , V , α_i and u_i be as in 1.5. Then there exist disjoint meridian disks Δ_{0ij} 's of ∂D_0^{n+1} in ∂V , a band γ_{ij} in ∂V compatible with α_i and $\tilde{\alpha}_{ij} = \partial \Delta_{0ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq r(i)$, and an ambient isotopy $\{\varphi_t; 0 \leq t \leq 1\}$ of ∂V such that

- (1) $\gamma_{ij}(B^1 \times I) \cap \gamma_{kl}(B^1 \times I) = \emptyset$ if $(i, j) \neq (k, l)$,
- (2) $\gamma_{ij}(B^1 \times I) \cap N(\partial D_0^{n+1}; \partial V) = \gamma_{ij}(B^1 \times 0)$,
- (3) $\gamma_{ij}(B^1 \times I) \cap \alpha_k = \emptyset$ if $i \neq k$,
- (4) $\varphi_t|_{\partial D_0^{n+1}}$ is the identity map for $0 \leq t \leq 1$,
- (5) φ_0 is the identity map, and
- (6) $\varphi_1(u_i) = \mathcal{K}(\alpha_i, \tilde{\alpha}_{i1}, \dots, \tilde{\alpha}_{ir(i)}; \gamma_{i1}, \dots, \gamma_{ir(i)})$
for $1 \leq i \leq m$. (See Fig. 2.1.)

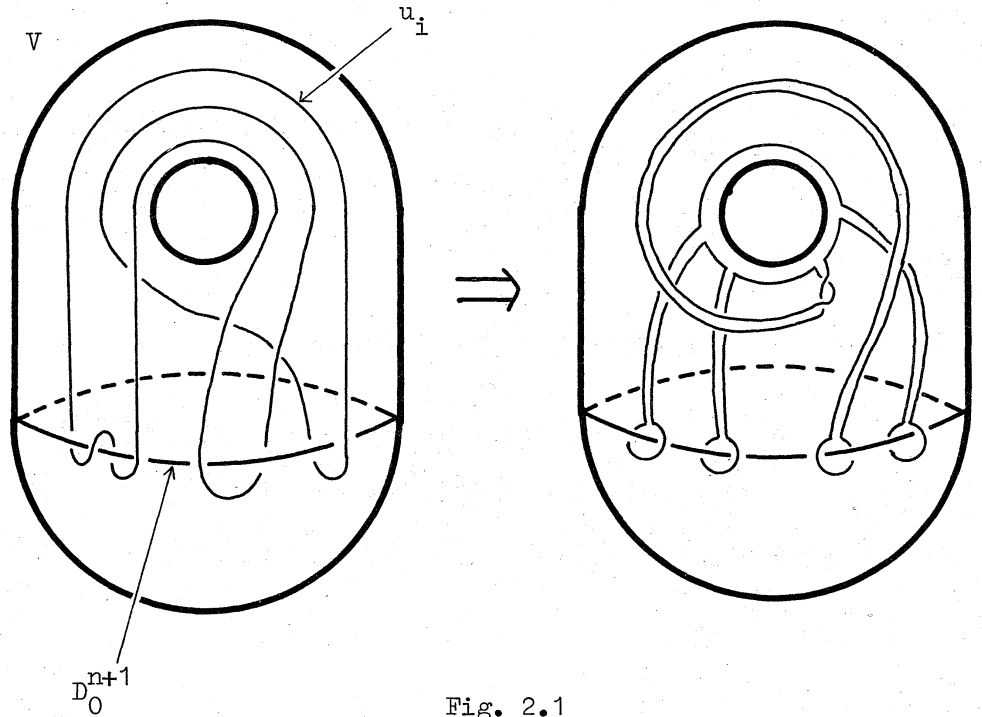


Fig. 2.1

Proof of Lemma 2.2. For $n = 1$, the assertion is easily shown by the modification as in Fig. 2.2.

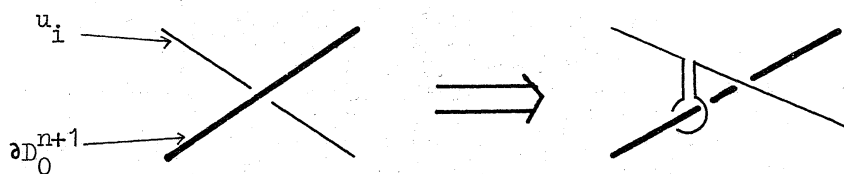


Fig. 2.2

For $n \geq 2$, the assertion is proved by using Remark 1.6 and Lemma 2.1. For details, see [1].

Using Lemma 2.2, we have the following Proposition 2.3, and we omit the proof [1].

2.3. Proposition. For $n \geq 1$, an $(n+1)$ -disk pair of S-type is of S^* -type. (See Fig. 2.3.)

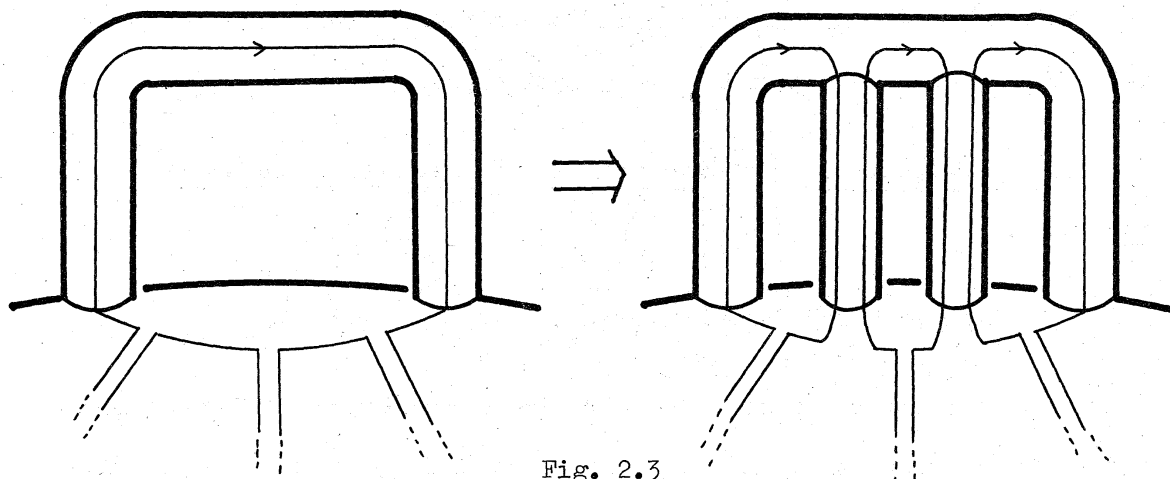


Fig. 2.3

Then we have the following theorem :

2.4. Theorem. Suppose $n \geq 1$. Then a ribbon $(n+1)$ -disk pair is

of S-type, and conversely an $(n+1)$ -disk pair of S-type is a ribbon disk pair. (For the proof, see [1].)

2.5. Remark. A.Omae [9] proved that the boundary pair of a 3-disk pair of S-type is a ribbon 2-knot for a special case, and L.R.Hitt [2] announced that he proved that the boundary pair of an $(n+1)$ -disk pair of S-type is a ribbon n -knot and the converse.

By Proposition 1.8, Lemma 2.1, Lemma 2.2 and Theorem 2.4, we have the following Corollary 2.6 :

2.6. Corollary. Let (D^{n+3}, L^{n+1}) be a ribbon $(n+1)$ -disk pair for $n \geq 1$, then $cl(D^{n+3} - N(L^{n+1}; D^{n+3}))$ collapses to a cell complex of S-type. Conversely, let C be a cell complex of S-type. Then there exists a ribbon $(n+1)$ -disk pair (D^{n+3}, L^{n+1}) for $n \geq 1$ such that C is a spine of the exterior of L^{n+1} in D^{n+3} .

In [20], the third author proved the following Proposition 2.7, and using Theorem 2.4 we can give an alternative proof [1] :

2.7. Proposition. Every ribbon n -knot has an equatorial knot, for $n \geq 2$.

§ 3. Asphericity of ribbon disks

In this section, we will show that the complement of a higher dimensional ribbon disk is aspherical which is an analogy to the case of classical knots [11].

3.1. Regarding S^4 as a one-point compactification of R^4 , we may consider that a 2-knot is in R^4 . Let R_t^3 be the hyperplane of R^4

whose 4-th coordinate is t , R_+^4 the half space of R^4 whose 4-th coordinate is non-negative, and $R_-^4 = \text{cl}(R^4 - R_+^4)$. By Proposition 2.7, we can assume that a ribbon 2-knot K^2 satisfies the followings :

- (1) K^2 is symmetric with respect to R_0^3 ,
- (2) $K^2 \cap R_+^4$ has elliptic critical points only in R_2^3 , and
- (3) $K^2 \cap R_+^4$ has hyperbolic critical points only in R_1^3 (Fig. 3.1).

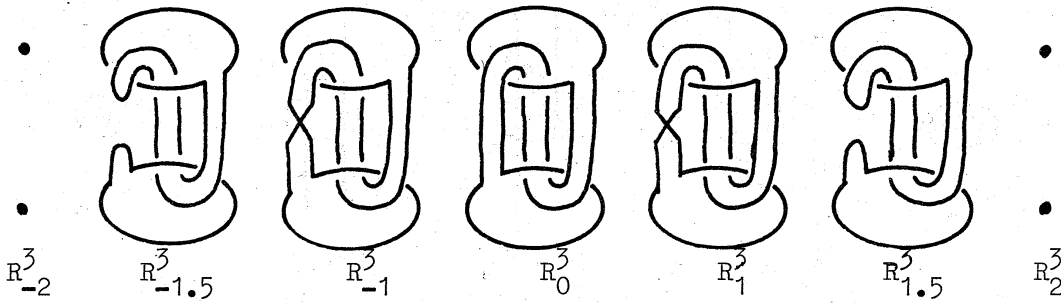


Fig. 3.1

On the other hand, a ribbon 2-knot K^2 can be described as follows :

- (1) all elliptic critical points occur at R_2^3 or R_{-2}^3 , and
- (2) all hyperbolic critical points occur at R_0^3 (Fig. 3.2).

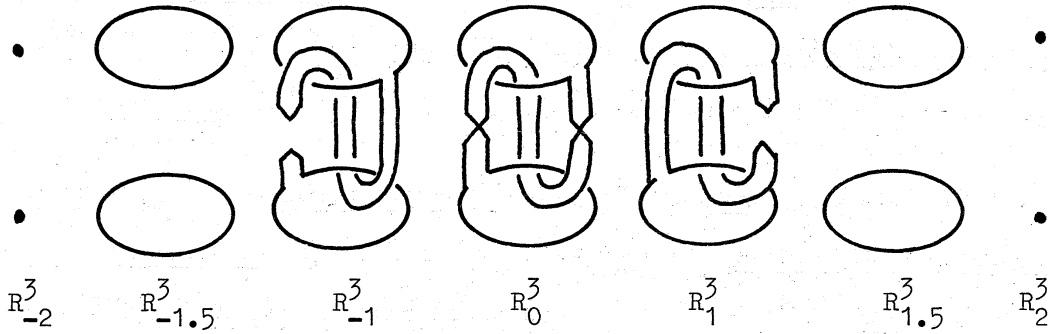


Fig. 3.2

The latter description of a 2-knot is adopted by S.J.Lomonaco [8], then he has stated the following in Theorem 3.2 in [8] :

3.2. Proposition. Let K^2 be a ribbon 2-knot of type $(\beta_1, \dots, \beta_m)$, and d_i^3 a 3-disk in $N(\beta_i; S^4)$ such that the intersection of d_i^3 and $\beta_i(B^2 \times I)$ is $\beta_i(B^2 \times 1/2)$ for $1 \leq i \leq m$. Let $[\partial d_i^3]$ be the element of $\pi_2(S^4 - K^2)$ represented by ∂d_i^3 . Then $\pi_2(S^4 - K^2)$ is generated by $[\partial d_1^3], \dots, [\partial d_m^3]$ as a $\mathbb{Z}\pi_1$ -module, where $\mathbb{Z}\pi_1$ is the integral group ring of $\pi_1 = \pi_1(S^4 - K^2)$. (Fig. 3.3)

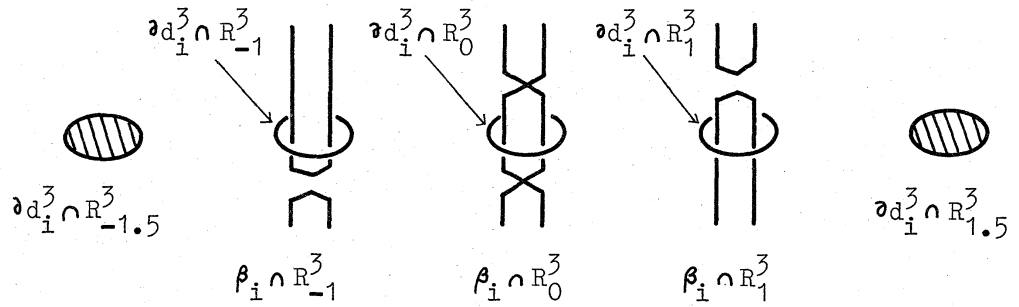


Fig. 3.3

The following Proposition 3.3 has been proved by the third author [21] using the Van-Kampen theorem :

3.3. Proposition. Let (D^{n+3}, L^{n+1}) be the ribbon $(n+1)$ -disk pair associated with a ribbon n -knot K^n , and $(S^{n+3}, K^{n+1}) = \mathcal{D}(D^{n+3}, L^{n+1})$. Then $\pi_1(S^{n+2} - K^n) \cong \pi_1(D^{n+3} - L^{n+1}) \cong \pi_1(S^{n+3} - K^{n+1})$ provided $n \geq 2$.

3.4. Lemma. Let (D^{n+3}, L^{n+1}) be the ribbon $(n+1)$ -disk pair associated with a ribbon n -knot K^n . If $n \geq 2$, then the inclusion from $S^{n+2} - K^n$ into $D^{n+3} - L^{n+1}$ induces an onto-homomorphism $\pi_2(S^{n+2} - K^n) \rightarrow \pi_2(D^{n+3} - L^{n+1})$ as $\mathbb{Z}\pi_1$ -modules.

Proof. Let K^n be of type $(\beta_1, \dots, \beta_m)$, then we will use the notations in 1.4.

Note that, by the general position arguments, we can choose a base for

$\pi_2(D^{n+3} - L^{n+1})$ consisting of embedded 2-spheres. Let Σ^2 be a 2-sphere in $D^{n+3} - L^{n+1}$, then we may assume that Σ^2 is in $S^{n+2} \times I - L^{n+1}$. We can move Σ^2 into general position so that Σ^2 does not intersect $p_i \times [3/4, 1]$ where $p_i \in \text{int } B_i^{n+1}$. Since $B_i^{n+1} \times [3/4, 1]$ is regarded as a regular neighbourhood of $p_i \times [3/4, 1]$ in $S^{n+2} \times [3/4, 1]$, we have $\Sigma^2 \cap B_i^{n+1} \times [3/4, 1] = \emptyset$ for $0 \leq i \leq m$. This implies that Σ^2 can be pushed into $S^{n+2} \times [0, 3/4] - L^{n+1}$. For $q \in \text{int } B^n$, let $W_i = \beta_i(q \times I) \times [1/2, 3/4]$. Again by the general position arguments, we can have $\Sigma^2 \cap W_i = \emptyset$ for each i . Hence Σ^2 does not intersect $\beta_i(B^n \times I) \times [1/2, 3/4]$ for each i , i.e. we can deform Σ^2 isotopically into $S^{n+2} \times [0, 1/2] - L^{n+1}$. Therefore Σ^2 can be moved isotopically into $S^{n+2} \times 0 - L^{n+1}$, i.e. into $S^{n+2} - K^n$. By this fact and Proposition 3.3 the required result is implied.

3.5. Lemma. For a ribbon 3-disk pair (D^5, L^3) , we have
 $\pi_2(D^5 - L^3) = 0$.

Proof. Let (D^5, L^3) be associated with a ribbon 2-knot of type $(\beta_1, \dots, \beta_m)$. We will use the notations in 1.4 for $n = 2$ and in 3.2. The 2-sphere ∂d_i^3 bounds the 3-disk $(\partial d_i^3 \times [0, 3/4]) \cup (d_i^3 \times 3/4)$ in $D^5 - L^3$ for each i . This implies that $\pi_2(D^5 - L^3) = 0$ by Lemma 3.4.

The following Theorem 3.6 is a generalization of [19] :

3.6. Theorem. Let (D^{n+3}, L^{n+1}) be a ribbon $(n+1)$ -disk pair with
 $n \geq 1$, then $D^{n+3} - L^{n+1}$ is aspherical.

Proof. By Corollary 2.6, there exists a cell complex C of S -type such that $D^{n+3} - L^{n+1}$ is homotopy equivalent to C . Again by 2.6, there exists a ribbon 3-disk pair (D^5, L^3) such that $D^5 - L^3$ is homotopy

equivalent to C . It follows from Lemma 3.5 that $\pi_2(C) = 0$. Let $\tilde{C}$ be the universal covering space of C . Then $H_1(\tilde{C}) = 0$ for $i \geq 3$, since $\tilde{C}$ is 2-dimensional. Thus, by Hurewicz theorem, $\tilde{C}$ is aspherical, because $\pi_2(\tilde{C}) \cong \pi_2(C) = 0$. Therefore C is aspherical. This completes the proof.

3.7. Remark. A cell complex of S-type is a subcomplex of a contractible 2-complex, hence the proof of Theorem 3.6 gives a partial answer to a problem of J.H.C.Whitehead : Is any subcomplex of an aspherical 2-complex aspherical ?

3.8. Corollary. Let K^n be a ribbon n -knot for $n \geq 1$, then $\pi_1(S^{n+2} - K^n)$ has no element of finite order.

Proof. For $n = 1$, the assertion is a special case of [11]. For $n \geq 2$, this is true by Proposition 3.3, Theorem 3.6 and a result due essentially to P.A.Smith (p.216 in [4]), namely : The fundamental group of an aspherical polyhedron of finite dimension has no element of finite order.

T.Yajima characterized the knot group of ribbon 2-knots in [17], then by Corollary 3.8 and [17] we have the following :

3.9. Corollary. Let G be a finitely presented group having a Wirtinger presentation of deficiency 1 with $G/G' \cong \mathbb{Z}$. Then G has no element of finite order.

§ 4. Unknotting ribbon knots

4.1. Theorem. Let K^n be a ribbon n -knot for $n \geq 3$, then we have $\pi_1(S^{n+2} - K^n)$ for $2 \leq i \leq n-1$.

Proof. By Proposition 2.7, there exists a ribbon n -disk pair

(D^{n+2}, L^n) such that $\mathcal{Q}(D^{n+2}, L^n) = (S^{n+2}, K^n)$. Let $(D_\epsilon^{n+2}, L_\epsilon^n)$ be a copy of (D^{n+2}, L^n) for $\epsilon = \pm$ such that (D^{n+2}, L^n) is obtained from the disjoint union of (D_+^{n+2}, L_+^n) and (D_-^{n+2}, L_-^n) by identifying their boundaries via the identity map. Let $(S^{n+1}, K_0^{n-1}) = (D_+^{n+2}, L_+^n)$. Let $\tilde{X}$ be the universal covering space of $S^{n+2} - K^n$, $\tilde{X}_\epsilon$ the lift of $D_\epsilon^{n+2} - L_\epsilon^n$ in $\tilde{X}$ for $\epsilon = \pm$, and $\tilde{X}_0$ the lift of $S^{n+1} - K_0^{n-1}$ in $\tilde{X}$. By Proposition 3.3, all of $\tilde{X}_+$, $\tilde{X}_-$ and $\tilde{X}_0$ are also universal covering spaces. By the Mayer-Vietoris theorem, we have the following exact sequence :

$$\dots \rightarrow H_j(\tilde{X}_+) \oplus H_j(\tilde{X}_-) \rightarrow H_j(\tilde{X}) \rightarrow H_{j-1}(\tilde{X}_0) \rightarrow H_{j-1}(\tilde{X}_+) \oplus H_{j-1}(\tilde{X}_-) \rightarrow \dots$$

By Theorem 3.6, $H_j(\tilde{X}_\epsilon) = 0$ for $j \geq 1$ and $\epsilon = \pm$. Therefore it follows that $H_j(\tilde{X}) \cong H_{j-1}(\tilde{X}_0)$ for $j \geq 2$.

Suppose $n = 3$, then $\pi_2(S^5 - K^3) \cong H_2(\tilde{X}) \cong H_1(\tilde{X}_0) = 0$. By induction on the dimension n , it is easily seen that the fact $H_j(\tilde{X}) \cong H_{j-1}(\tilde{X}_0)$ and $H_1(\tilde{X}_0) = 0$ imply $H_i(\tilde{X}) = 0$ for $1 \leq i \leq n-1$. This implies the required result.

The following Proposition 4.2 is due to A.Kawauchi ([5] or p.331 in [15]) :

4.2. Proposition. For a 2-knot K^2 , $S^4 - K^2$ is homotopy equivalent to S^1 if $\pi_1(S^4 - K^2) \cong \mathbb{Z}$.

4.3. Theorem. Let K^n be a ribbon n -knot for $n \geq 3$. If $\pi_1(S^{n+2} - K^n) \cong \mathbb{Z}$, then K^n is unknotted.

Proof. We can use the notations in the proof of Theorem 4.1. In the proof of Theorem 4.1, we have $H_j(\tilde{X}) \cong H_{j-1}(\tilde{X}_0)$ for $j \geq 2$.

Suppose $n = 3$ and $\pi_1(S^5 - K^3) \cong \mathbb{Z}$, then by Proposition 3.3 it follows that $\pi_1(S^4 - K_0^2) \cong \mathbb{Z}$, where K_0^2 is an equatorial knot of K^3 . By Proposition 4.2, we have $H_i(\tilde{X}_0) = 0$ for all $i \geq 1$. This implies that

$H_j(\tilde{X}) = 0$ for $j \geq 1$. Therefore $S^5 - K^3$ is homotopy equivalent to S^1 , hence by [7], [13] and [16], K^3 is unknotted. Similarly, for $n \geq 4$, it is easy to see the assertion is true by induction on the dimension n . This completes the proof.

Recently A.Kawauchi and T.Matamoto have obtained independently the same result as Theorem 4.3 [6].

The following is obtained by Proposition 3.3 and Theorem 4.3 :

4.5. Corollary. For a ribbon n -knot K^n with $n \geq 4$, any equatorial knot of K^n is unknotted if K^n is unknotted.

For $n = 2$, Corollary 4.5 is false. For example, Kinoshita-Terasaka knot is an equatorial knot of the unknot [9]. The case $n = 3$ still remains open.

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