

Cluster algebra and q -Painlevé equations: higher order generalization and degeneration structure

By

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Abstract

In this article we give a birational representation of an extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)}$ with the aid of a cluster mutation. This group provides several higher order generalizations of the q -Painlevé VI equation (q - P_{VI}) as translations. We also discuss a confluence of vertices of a quiver which can be applied to a degeneration structure of the q -Painlevé equations.

§ 1. Introduction

The cluster algebra was introduced by Fomin and Zelevinsky in [2, 3]. It is a variety of commutative ring described in terms of cluster variables and coefficients. Its generating set is defined by an operation called a mutation which transforms a seed consisting of cluster variables, coefficients and a quiver. Then new cluster variables (reps. coefficients) are rational in original cluster variables and coefficients (reps. coefficients). Hence we can obtain various discrete integrable systems from mutation-periodic quivers ([13]) as relations satisfied by cluster variables and coefficients.

In the previous work [15] the q -Painlevé VI equation ([6]) was derived from the mutation-periodic quiver with 8 vertices; see Figure 4 in Section 3. As its extension, we consider the quiver $Q_{mn}(A_{m-1}^{(1)})^{*1}$; see Figure 1. This quiver is invariant under some

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*¹This quiver is also investigated in a recent work [12]. We follow the notation of [4] in which the quiver $Q_m(\mathfrak{g})$ is proposed for a Kac-Moody Lie algebra $\mathfrak{g}$. Here the symbol $A_{m-1}^{(1)}$ stands for the affine Lie algebra of type $A_{m-1}^{(1)}$.

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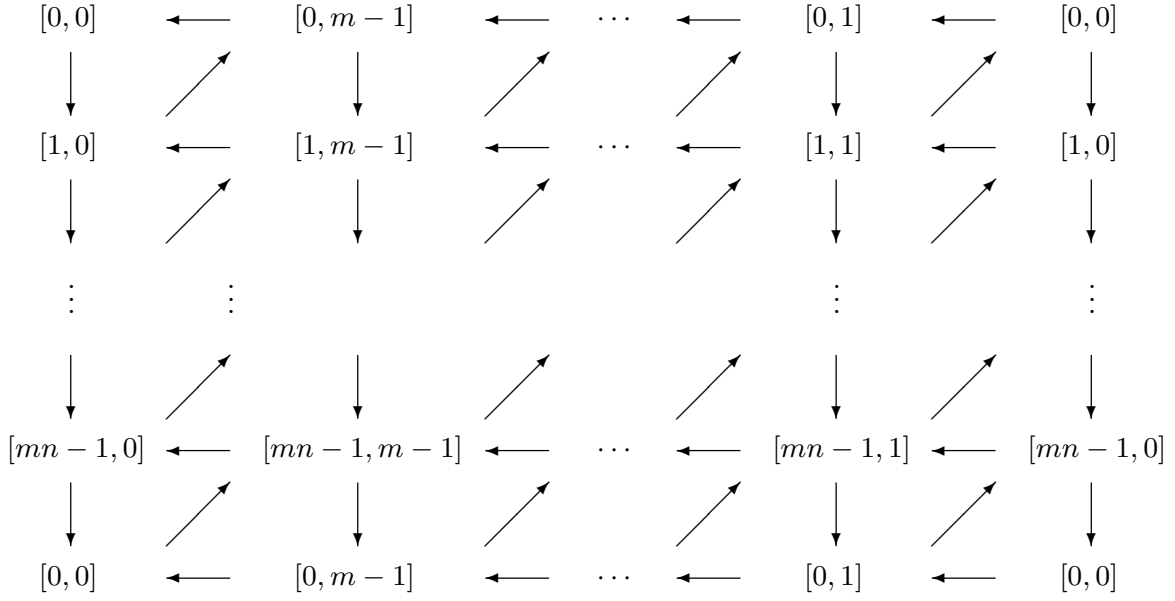
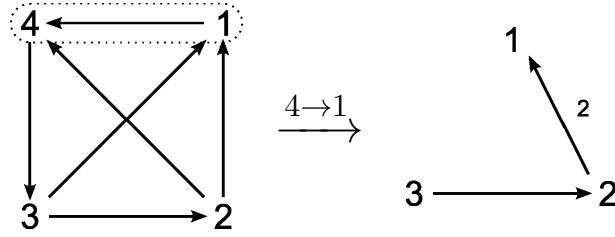
Figure 1. The quiver $Q_{mn}(A_{m-1}^{(1)})$ 

Figure 2. Confluence of quiver (example)

compositions of mutations and permutations of vertices of quivers. Those operations generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)*2}$. And this group provides several generalized q - P_{VI} 's as translations. In the previous work [16] we investigated the case $m = 2$ and derived three types of systems. In this article we review it quickly and give an additional explanation about the q -hypergeometric solution of the generalized q - P_{VI} ; see Section 2.

Our another aim is to propose a confluence of vertices of a quiver; see Figure 2. We define a confluence of vertices of a quiver $i \rightarrow j$ by replacing two vertices i, j and an arrow between them with one vertex j . In this article we only consider a confluence of two vertices which are connected by arrows directly. At the level of the corresponding skew-symmetric matrix, the confluence can be interpreted as the following operation.

^{*2}This fact is first obtained in [12] as an extension of the previous work [9, 10] in which the binational representation of the affine Weyl group of type $(A_{m-1} + A_{n-1})^{(1)}$ is formulated.

1. Add the i -th row to the j -th row.
2. Add the i -th column to the j -th column.
3. Delete the i -th row and the i -th column.

The confluence of skew-symmetric matrix corresponding to the one of Figure 2 is described as follows.

$$\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{4 \rightarrow 1} \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It can be applied to the degeneration structure of the q -Painlevé equations ([18]), which is described as follows.

$$\begin{array}{ccccccccccc} & & & & & & & & \frac{A_1^{(1)}}{|\alpha|^2=8} & & \\ & & & & & & & & \frac{A_7^{(1)}}{A_7^{(1)}} & & \\ & & & & & & & \nearrow & & \searrow & \\ \frac{E_8^{(1)}}{A_0^{(1)}} & \rightarrow & \frac{E_7^{(1)}}{A_1^{(1)}} & \rightarrow & \frac{E_6^{(1)}}{A_2^{(1)}} & \rightarrow & \frac{D_5^{(1)}}{A_3^{(1)}} & \rightarrow & \frac{A_4^{(1)}}{A_4^{(1)}} & \rightarrow & \frac{E_3^{(1)}}{A_5^{(1)}} & \rightarrow & \frac{E_2^{(1)}}{A_6^{(1)}} & \rightarrow & \frac{A_1^{(1)}}{A_7^{(1)}} & \rightarrow & \frac{E_0^{(1)}}{A_8^{(1)}} \end{array}$$

In this table numerators and denominators stand for symmetry and surface types respectively. Following [11], we use the symbols $E_3^{(1)} = (A_2 + A_1)^{(1)}$ and $E_2^{(1)} = (A_1 + \frac{A_1}{|\alpha|^2=14})^{(1)}$

for the sake of simplicity. The type $D_5^{(1)}/A_3^{(1)}$ corresponds to q - P_{VI} . In the previous work [15] we derived 4 types of q -Painlevé equations containing q - P_{VI} from some mutation-periodic quivers. At that time we gave the quivers by using the operation called a flattening which corresponds to the reduction of the discrete integrable system. Afterward, 9 equations below the one of type $E_6^{(1)}/A_2^{(1)}$ were derived in [1] by using a correspondence between quivers and Newton polygons. We obtain those quivers again with the aid of a confluence of vertices of a quiver; see Section 3.

§ 2. Extended affine Weyl group and Generalized q - P_{VI}

§ 2.1. Cluster mutation

Let $Q = Q_{mn-1}(A_{m-1}^{(1)})$ ($m, n \geq 2$) be the quiver given in Figure 1. We define a mutation $\mu_{[j,i]}$ at the vertex $[j, i]$ as follows.

1. If there are k_1 arrows from $[j_1, i_1]$ to $[j, i]$ and k_2 arrows from $[j, i]$ to $[j_2, i_2]$, then we add $k_1 k_2$ arrows from $[j_1, i_1]$ to $[j_2, i_2]$.

2. If 2-cycles appear via the operation 1, then we remove all of them.

3. We reverse the direction of all arrows touching $[j, i]$.

Let $I = \{[j, i] \mid 0 \leq i \leq m-1, 0 \leq j \leq mn-1\}$ be a vertex set. We define a skew-symmetric matrix $\Lambda = (\lambda_{[j_1, i_1], [j_2, i_2]})_{[j_1, i_1], [j_2, i_2] \in I}$ corresponding to a quiver Q as follows.

- If there are k arrows from $[j_1, i_1]$ to $[j_2, i_2]$, then we set $\lambda_{[j_1, i_1], [j_2, i_2]} = k$ and $\lambda_{[j_2, i_2], [j_1, i_1]} = -k$.
- If there is no arrow connecting two vertices $[j_1, i_1]$ and $[j_2, i_2]$ directly, then we set $\lambda_{[j_1, i_1], [j_2, i_2]} = \lambda_{[j_2, i_2], [j_1, i_1]} = 0$.

Also let $\mathbf{y} = (y_{[j, i]})_{[j, i] \in I}$ be a m^2n -tuple of coefficients. Then the mutation $\mu_{[j, i]} : (\Lambda, \mathbf{y}) \mapsto (\Lambda', \mathbf{y}')$ is given by

$$\lambda'_{[j_1, i_1], [j_2, i_2]} = \begin{cases} -\lambda_{[j_1, i_1], [j_2, i_2]} & ([j_1, i_1] = [j, i] \vee [j_2, i_2] = [j, i]) \\ \lambda_{[j_1, i_1], [j_2, i_2]} + \lambda_{[j_1, i_1], [j, i]} \lambda_{[j, i], [j_2, i_2]} (\lambda_{[j_1, i_1], [j, i]} > 0 \wedge \lambda_{[j, i], [j_2, i_2]} > 0) \\ \lambda_{[j_1, i_1], [j_2, i_2]} - \lambda_{[j_1, i_1], [j, i]} \lambda_{[j, i], [j_2, i_2]} (\lambda_{[j_1, i_1], [j, i]} < 0 \wedge \lambda_{[j, i], [j_2, i_2]} < 0) & \\ \lambda_{[j_1, i_1], [j_2, i_2]} & (\text{otherwise}) \end{cases},$$

and

$$y'_{[j_1, i_1]} = \begin{cases} y_{[j, i]}^{-1} & ([j_1, i_1] = [j, i]) \\ y_{[j_1, i_1]} (1 + y_{[j, i]}^{-1})^{\lambda_{[j_1, i_1], [j, i]}} (\lambda_{[j_1, i_1], [j, i]} < 0) \\ y_{[j_1, i_1]} (1 + y_{[j, i]})^{\lambda_{[j_1, i_1], [j, i]}} (\lambda_{[j_1, i_1], [j, i]} > 0) & \\ y_{[j_1, i_1]} & (\text{otherwise}) \end{cases}.$$

Note that we don't consider cluster variables in this article. We denote a transposition of vertices $[j_1, i_1]$ and $[j_2, i_2]$ by $([j_1, i_1], [j_2, i_2])$. It act on coefficients as

$$([j_1, i_1], [j_2, i_2])(y_{[j, i]}) = \begin{cases} y_{[j_2, i_2]} & ([j, i] = [j_1, i_1]) \\ y_{[j_1, i_1]} & ([j, i] = [j_2, i_2]) \\ y_{[j, i]} & (\text{otherwise}) \end{cases}.$$

In the following we use a notation of periodicity

$$y_{[j, i]} = y_{[j, i+m]} = y_{[j+mn, i]}.$$

§ 2.2. Birational representation of affine Weyl group

We introduce parameters corresponding to the simple roots of the affine root system as

$$\alpha_j = \prod_{i=0}^{m-1} y_{[j,i]} \quad (j = 0, \dots, mn-1),$$

$$\beta_i = \prod_{j=0}^{mn-1} y_{[j,i]}, \quad \beta'_i = \prod_{j=0}^{mn-1} y_{[j,i+j]} \quad (i = 0, \dots, m-1),$$

with

$$\prod_{j=0}^{mn-1} \alpha_j = \prod_{i=0}^{m-1} \beta_i = \prod_{i=0}^{m-1} \beta'_i = \prod_{i=0}^{m-1} \prod_{j=0}^{mn-1} y_{[j,i]} = q.$$

Note that

$$\alpha_j = \alpha_{j+mn}, \quad \beta_i = \beta_{i+m}, \quad \beta'_i = \beta'_{i+m}.$$

Definition 2.1. We define birational transformations π, π', ρ , called Dynkin diagram automorphisms, by

$$\begin{aligned} \pi &= ([0, 0], [1, 1], \dots, [m-1, m-1], [m, 0], \dots, [mn-1, m-1]) \\ &\quad \times ([0, 1], [1, 2], \dots, [m-1, 0], [m, 1], \dots, [mn-1, 0]) \\ &\quad \times \dots \\ &\quad \times ([0, m-1], [1, 0], \dots, [m-1, m-2], [m, m-1], \dots, [mn-1, m-2]), \\ \pi' &= ([0, 0], [1, 0], \dots, [m-1, 0], [m, 0], \dots, [mn-1, 0]) \\ &\quad \times ([0, 1], [1, 1], \dots, [m-1, 1], [m, 1], \dots, [mn-1, 1]) \\ &\quad \times \dots \\ &\quad \times ([0, m-1], [1, m-1], \dots, [m-1, m-1], [m, m-1], \dots, [mn-1, m-1]), \\ \rho &= ([1, 0], [mn-1, m-1])([1, 1], [mn-1, 0]) \dots ([1, m-1], [mn-1, m-2]) \\ &\quad \times ([2, 0], [mn-2, m-2])([2, 1], [mn-2, m-1]) \dots ([2, m-1], [mn-2, m-3]) \\ &\quad \times \dots \\ &\quad \times ([m-1, 0], [mn-m+1, 1])([m-1, 1], [mn-m+1, 2]) \dots ([m-1, m-1], [mn-m+1, 0]) \\ &\quad \times ([m, 0], [mn-m, 0])([m, 1], [mn-m, 1]) \dots ([m, m-1], [mn-m, m-1]) \\ &\quad \times \dots \\ &\quad \times ([N, 0], [mn-N, 0])([N, 1], [mn-N, 1]) \dots ([N, m-1], [mn-N, m-1]), \end{aligned}$$

where

$$([j_1, i_1], [j_2, i_2], \dots, [j_k, i_k]) = ([j_1, i_1], [j_2, i_2])([j_2, i_2], [j_3, i_3]) \dots ([j_{k-1}, i_{k-1}], [j_k, i_k])$$

stands for a cyclic permutation and $N = \lfloor \frac{mn}{2} \rfloor$. Here we define a composition of two operations μ_1, μ_2 by $(\mu_1 \mu_2)(\mathbf{y}) = \mu_1(\mu_2(\mathbf{y}))$.

These transformations act on the coefficients and the parameters as

$$\begin{aligned}\pi(y_{[j,i]}) &= y_{[j+1,i+1]}, \\ \pi'(y_{[j,i]}) &= y_{[j+1,i]}, \\ \rho(y_{[j_2 m + j_1, i]}) &= y_{[mn - j_2 m - j_1, i - j_1]} \quad (j_1 = 0, \dots, m-1; j_2 = 0, \dots, n-1).\end{aligned}$$

and

$$\begin{aligned}\pi(\alpha_j) &= \alpha_{j+1}, & \pi(\beta_i) &= \beta_{i+1}, & \pi(\beta'_i) &= \beta'_i, \\ \pi'(\alpha_j) &= \alpha_{j+1}, & \pi'(\beta_i) &= \beta_i, & \pi'(\beta'_i) &= \beta'_{i-1}, \\ \rho(\alpha_j) &= \alpha_{mn-j}, & \rho(\beta_i) &= \beta'_i, & \rho(\beta'_i) &= \beta_i,\end{aligned}$$

for $i = 0, \dots, m-1$ and $j = 0, \dots, mn-1$.

Definition 2.2. We define birational transformations r_0 , called a simple reflection, by

$$r_0 = \mu_{[0,0]} \mu_{[0,1]} \cdots \mu_{[0,m-2]} ([0, m-2], [0, m-1]) \mu_{[0,m-2]} \cdots \mu_{[0,1]} \mu_{[0,0]}.$$

We also define birational transformations $r_1, \dots, r_{mn-1}$ by using π, r_0 as

$$r_j = \pi^{-1} r_{j-1} \pi \quad (j = 1, \dots, mn-1).$$

The transformation r_0 acts on the coefficients and the parameters as

$$\begin{aligned}r_0(y_{[0,i]}) &= \frac{\sum_{k_1=0}^{m-1} \prod_{k_2=0}^{k_1-1} y_{[0,i+k_2]}}{\sum_{k_1=0}^{m-1} \prod_{k_2=0}^{k_1} y_{[0,i+k_2+1]}}, & r_0(y_{[1,i]}) &= \frac{\sum_{k_1=0}^{m-1} \prod_{k_2=0}^{k_1} y_{[0,i+k_2]}}{\sum_{k_1=0}^{m-1} \prod_{k_2=0}^{k_1-1} y_{[0,i+k_2]}} y_{[1,i]}, \\ r_0(y_{[mn-1,i]}) &= \frac{\sum_{k_1=0}^{m-1} \prod_{k_2=0}^{k_1} y_{[0,i+k_2+1]}}{\sum_{k_1=0}^{m-1} \prod_{k_2=0}^{k_1-1} y_{[0,i+k_2+1]}} y_{[mn-1,i]}, & r_0(y_{[j,i]}) &= y_{[j,i]} \quad (j \neq 0, 1, mn-1),\end{aligned}$$

for $i = 0, \dots, m-1$ and

$$\begin{aligned}r_0(\alpha_0) &= \frac{1}{\alpha_0}, & r_0(\alpha_1) &= \alpha_0 \alpha_1, & r_0(\alpha_{mn-1}) &= \alpha_0 \alpha_{mn-1}, & r_0(\alpha_k) &= \alpha_k \quad (k \neq 0, 1, mn-1), \\ r_0(\beta_i) &= \beta_i, & r_0(\beta'_i) &= \beta'_i \quad (i = 0, \dots, m-1),\end{aligned}$$

Definition 2.3. We define birational transformations s_0 , called a simple reflection, by

$$s_0 = \mu_{[0,0]} \mu_{[1,0]} \cdots \mu_{[mn-2,0]} ([mn-2, 0], [mn-1, 0]) \mu_{[mn-2,0]} \cdots \mu_{[1,0]} \mu_{[0,0]}.$$

We also define birational transformations $s_1, \dots, s_{m-1}$ and $s'_0, \dots, s'_{m-1}$ by using π, ρ, s_0 as

$$s_i = \pi^{-1} s_{i-1} \pi \quad (i = 1, \dots, m-1),$$

and

$$s'_i = \rho s_i \rho \quad (i = 0, \dots, m-1).$$

In the case $m = 2$, the transformation r_0 acts on the coefficients and the parameters as

$$s_0(y_{[j,0]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}, \quad s_0(y_{[j,1]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}} y_{[j,0]} y_{[j,1]},$$

for $j = 0, \dots, mn-1$ and

$$\begin{aligned} s_0(\alpha_j) &= \alpha_j \quad (j = 0, \dots, mn-1), \\ s_0(\beta_0) &= \frac{1}{\beta_0}, \quad s_0(\beta_1) = \beta_0^2 \beta_1, \quad s_0(\beta'_i) = \beta'_i \quad (i = 0, \dots, m-1). \end{aligned}$$

In the case $m \geq 3$, the action of r_0 is described as

$$\begin{aligned} s_0(y_{[j,0]}) &= \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}, \quad s_0(y_{[j,1]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}} y_{[j,1]}, \\ s_0(y_{[j,m-1]}) &= \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2+1,0]}} y_{[j,m-1]}, \quad s_0(y_{[j,i]}) = y_{[j,i]} \quad (i \neq 0, 1, m-1), \end{aligned}$$

for $j = 0, \dots, mn-1$ and

$$\begin{aligned} s_0(\alpha_j) &= \alpha_j \quad (j = 0, \dots, mn-1), \\ s_0(\beta_0) &= \frac{1}{\beta_0}, \quad s_0(\beta_1) = \beta_0 \beta_1, \quad s_0(\beta_{m-1}) = \beta_0 \beta_{m-1}, \quad s_0(\beta_k) = \beta_k \quad (k \neq 0, 1, m-1), \\ s_0(\beta'_i) &= \beta'_i \quad (i = 0, \dots, m-1). \end{aligned}$$

Fact 2.4 ([5, 12, 16]). The birational transformations defined in the above satisfy the fundamental relations of the extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)}$

$$\begin{aligned} r_j^2 &= s_i^2 = (s'_i)^2 = 1, \\ (r_j r_{j+1})^3 &= (s_i s_{i+1})^3 = (s'_i s'_{i+1})^3 = 1, \\ (r_{j_1} r_{j_2})^2 &= (s_{i_1} s_{i_2})^2 = (s'_{i_1} s'_{i_2})^2 = 1 \quad (i_1 \neq i_2, i_2 \pm 1; j_1 \neq j_2, j_2 \pm 1), \\ (r_j s_i)^2 &= (r_j s'_i)^2 = (s_{i_1} s'_{i_2})^2 = 1, \end{aligned}$$

and

$$\begin{aligned}\pi^{mn} &= 1, \quad (\pi')^{mn} = 1, \quad \rho^2 = 1, \quad \pi \pi' = \pi' \pi, \quad \pi^m = (\pi')^m, \quad \pi' \rho = \rho \pi^{-1}, \\ \pi r_j &= r_{j-1} \pi, \quad \pi' r_j = r_{j-1} \pi', \quad \rho r_j = r_{mn-j} \rho, \\ \pi s_i &= s_{i-1} \pi, \quad \pi s'_i = s'_i \pi, \quad \pi' s_i = s_i \pi', \quad \pi' s'_i = s'_{i+1} \pi', \quad \rho s_i = s'_i \rho,\end{aligned}$$

for $i, i_1, i_2 = 0, \dots, m-1$ and $j = 0, \dots, mn-1$, where

$$r_j = r_{j+mn}, \quad s_i = s_{i+m}, \quad s'_i = s'_{i+m}.$$

§ 2.3. Example (case $m = 2$)

This case has already been considered in our previous work.

Fact 2.5 ([16]). Let

$$\begin{aligned}T_1 &= s'_1 s_1 \pi' \pi^{-1}, \\ T_2 &= (r_0 r_1 \dots r_{n-2} r_n r_{n+1} \dots r_{2n-2} \pi')^2, \\ T_3 &= r_1 r_2 \dots r_{2n-1} s'_1 \pi', \\ T_4 &= (r_0 r_2 \dots r_{2n-2} \pi')^2.\end{aligned}$$

Then they provides three types of q -Painlevé systems as follows^{*3}.

- T_1 provides the q -Painlevé system q - $P_{(n,n)}$ arising from the q -DS hierarchy given in [20, 21].
- T_2 provides the q -Garnier system given in [19].
- T_4 provides the q -Painlevé system arising from the q -LUC hierarchy given in §3.4 of [23].

In this section we focus on the translation T_1 and investigate a particular solution in terms of the basic hypergeometric function ${}_n\phi_{n-1}$. The actions of T_1 on the parameters are described as

$$\begin{aligned}T_1(\alpha_j) &= \alpha_j \quad (j = 0, \dots, 2n-1), \quad T_1(\beta_0) = q \beta_0, \quad T_1(\beta_1) = \frac{\beta_1}{q}, \\ T_1(\beta'_0) &= q \beta'_0, \quad T_1(\beta'_1) = \frac{\beta'_1}{q}.\end{aligned}$$

^{*3}We conjecture that T_3 provides the variation of the q -Garnier system $T_{a_{N+1}}^{-1} T_{c_1}^{-1}$ given in §3.2.4 of [14].

Recall that

$$\alpha_j = y_{[j,0]} y_{[j,1]} \quad (j = 0, \dots, 2n-1),$$

$$\beta_i = \prod_{j=0}^{2n-1} y_{[j,i]}, \quad \beta'_i = \prod_{j=0}^{n-1} y_{[2j,i]} y_{[2j+1,i+1]} \quad (i = 0, 1),$$

and

$$y_{[j,i]} = y_{[j,i+2]} = y_{[j+2n,i]}, \quad \alpha_j = \alpha_{j+2n}, \quad \beta_i = \beta_{i+2}, \quad \beta'_i = \beta'_{i+2}.$$

The actions of T_1 on the coefficients $y_{[0,0]}, \dots, y_{[2n-1,0]}$ are described as

$$(2.1) \quad T_1(y_{[2j,0]}) = \alpha_{2j} \alpha_{2j+1} y_{[2j,0]} \frac{S'_j S_{j+1}}{S_j S'_{j+1}},$$

$$T_1(y_{[2j+1,0]}) = \frac{1 + y_{[2j+2,0]}}{1 + y_{[2j,0]}} y_{[2j+3,0]} \frac{S_j S'_{j+1} + \alpha_{2j+1} y_{[2j,0]} S'_j S_{j+1}}{S_{j+1} S'_{j+2} + \alpha_{2j+3} y_{[2j+2,0]} S'_{j+1} S_{j+2}},$$

for $j = 0, \dots, n-1$, where

$$(2.2) \quad S_j = \sum_{k=j}^{n-1} (1 + y_{[2k,0]}) \prod_{l=j}^{k-1} y_{[2l,0]} y_{[2l+1,0]} + \sum_{k=0}^{j-1} (1 + y_{[2k,0]}) \prod_{l=0}^{k+n-1} y_{[2l,0]} y_{[2l+1,0]},$$

$$S'_j = \sum_{k=j}^{n-1} \left(1 + y_{[2k,0]}^{-1}\right) \prod_{l=j}^{k-1} y_{[2l,0]}^{-1} y_{[2l+1,1]}^{-1} + \sum_{k=0}^{j-1} \left(1 + y_{[2k,0]}^{-1}\right) \prod_{l=0}^{k+n-1} y_{[2l,0]}^{-1} y_{[2l+1,1]}^{-1}.$$

Lemma 2.6. If, in system (2.1), we assume that

$$y_{[2j+1,1]} = -1 \quad (j = 0, \dots, n-1),$$

then the coefficients $y_{[0,0]}, y_{[2,0]}, \dots, y_{[2n-2,0]}$ satisfy

$$(2.3) \quad T_1(y_{[2j,0]}) = \alpha_{2j} \alpha_{2j+1} y_{[2j,0]} \frac{S_{j+1}}{S_j} \quad (j = 0, \dots, n-1),$$

where

$$S_j = 1 + \sum_{k=j}^{j+n-2} (-1)^{k-j} (1 - \alpha_{2k+1}) \prod_{l=j}^{k-1} \alpha_{2l+1} \prod_{l=j}^k y_{[2l,0]} + (-1)^{n-1} \frac{\prod_{k=0}^{n-1} \alpha_{2k+1}}{\alpha_{2j-1}} \prod_{k=0}^{n-1} y_{[2k,0]}.$$

Proof. Substituting

$$y_{[2j+1,0]} = -\alpha_{2j+1}, \quad y_{[2j+1,1]} = -1 \quad (j = 0, \dots, n-1),$$

into (2.2), we obtain

$$S_j = 1 + \sum_{k=j}^{j+n-2} (-1)^{k-j} (1 - \alpha_{2k+1}) \prod_{l=j}^{k-1} \alpha_{2l+1} \prod_{l=j}^k y_{[2l,0]} + (-1)^{n-1} \frac{\prod_{k=0}^{n-1} \alpha_{2k+1}}{\alpha_{2j-1}} \prod_{k=0}^{n-1} y_{[2k,0]},$$

$$S'_j = 1 + \frac{(-1)^{n-1}}{\prod_{k=0}^{n-1} y_{[2k,0]}},$$

and

$$S_j + \alpha_{2j+1} y_{[2j,0]} S_{j+1} = (1 + y_{[2j,0]}) \left(1 + (-1)^{n-1} \prod_{k=0}^{n-1} \alpha_{2k+1} y_{[2k,0]} \right),$$

for $j = 0, \dots, n-1$. Then system (2.1) implies (2.3). We also obtain

$$T_1(y_{[2j+1,0]}) = y_{[2j+3,0]} \quad (j = 0, \dots, n-1),$$

namely the assumption of this lemma is consistent with system (2.1). $\square$

Thanks to this lemma, we can show the following theorem easily by a direct calculation.

Theorem 2.7. Let n -tuple $(x_0, \dots, x_{n-1})$ be a solution of a system of linear q -difference equations

$$(2.4) \quad T_1(x_j) = \left(\prod_{l=0}^{2j-1} \alpha_l \right) x_j + \sum_{k=j+1}^{n-1} \left((-1)^{k-j-1} (1 - \alpha_{2k-1}) \prod_{l=0}^{2k-2} \alpha_l \right) x_k$$

$$+ \sum_{k=0}^{j-1} \left((-1)^{k-j-1+n} (1 - \alpha_{2k-1}) \prod_{l=0}^{2k-2} \alpha_l \right) q \beta'_0 x_k + \left((-1)^{n-1} \prod_{l=0}^{2j-2} \alpha_l \right) q \beta'_0 x_j,$$

for $j = 0, \dots, n-1$. We also set

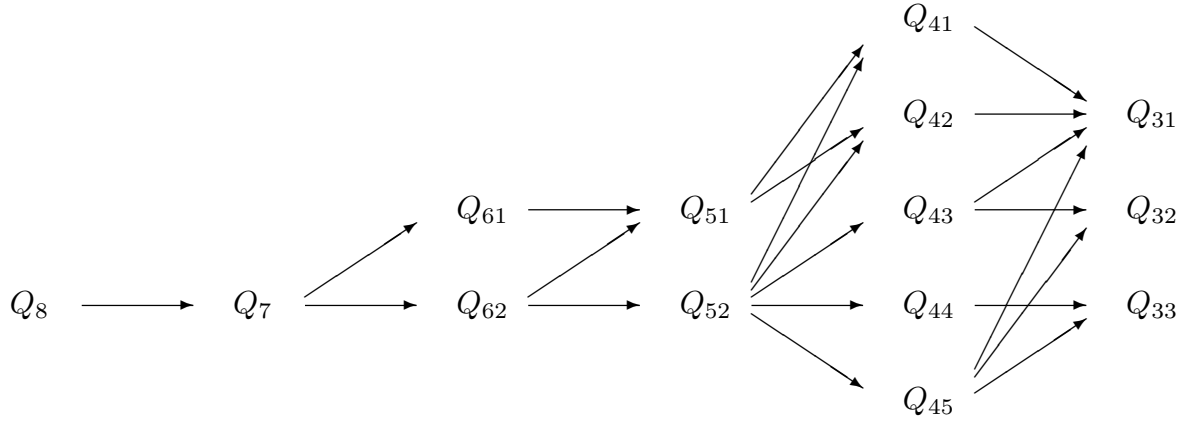
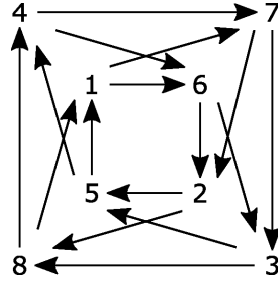
$$y_{[2j,0]} = \alpha_{2j} \frac{x_{j+1}}{x_j} \quad (j = 0, \dots, n-2), \quad y_{[2n-2,0]} = \alpha_{2n-2} \beta'_0 \frac{x_0}{x_{n-1}}.$$

Then $y_{[0,0]}, y_{[2,0]}, \dots, y_{[2n-2,0]}$ satisfy system (2.3).

System (2.4) is equivalent to the one given in [20] whose solution is described in terms of the basic hypergeometric function ${}_n\phi_{n-1}$. Here the parameter β'_0 plays the role of the independent variable.

§ 3. Degeneration structure of q -Painlevé equations

In this section we start with the quiver $Q_8 = Q_4(A_1^{(1)})$ and consider confluences of vertices of quivers. These procedures give the degeneration scheme of q -Painlevé

Figure 3. Confluence of the quiver $Q_4(A_1^{(1)})$ Figure 4. Q_8

equations below the one of type $D_5^{(1)}/A_3^{(1)}$. We list a correspondence between the quivers in Figure 3 and the q -Painlevé equations below^{*4}.

Q_8	Q_7	Q_{62}	Q_{52}	Q_{44}	Q_{45}	Q_{33}
$D_5^{(1)}/A_3^{(1)}$	$A_4^{(1)}/A_4^{(1)}$	$E_3^{(1)}/A_5^{(1)}$	$E_2^{(1)}/A_6^{(1)}$	$A_1^{(1)}/A_7^{(1)}$	$A_1^{(1)}/A_7^{(1)}$ $ \alpha ^2=8$	$E_0^{(1)}/A_8^{(1)}$

§ 3.1. Quiver Q_8

For the sake of simplicity, we rename the vertices of $Q_4(A_1^{(1)})$ as

$$\begin{aligned}
 [0, 0] &= 1, & [0, 1] &= 2, & [1, 0] &= 6, & [1, 1] &= 5, \\
 [2, 0] &= 3, & [2, 1] &= 4, & [3, 0] &= 8, & [3, 1] &= 7.
 \end{aligned}$$

^{*4}Among the other 7 quivers, Q_{41} and Q_{31} are ones of finite type. We expect that the rest 5 quivers correspond to the q -hypergeometric functions for the following reasons. The assumption of Lemma 2.6 turns into $y_5 = y_7 = -1$ in the quiver Q_8 . On the other hand, if we remove two vertices 5, 7 and all arrows touching 5, 7 from the quiver Q_8 , then we obtain the one Q_{61} . Besides, the degeneration scheme below Q_{61} is similar to the one of q -hypergeometric functions.

Then we obtain the quiver Q_8 ; see Figure 4. The skew-symmetric matrix Λ_8 corresponding to Q_8 is given by

$$\Lambda_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the following we denote a mutation at the vertex i by μ_i and a transposition of vertices i_1, i_2 by (i_1, i_2) .

The quiver Q_8 is invariant under compositions of mutations and permutations of vertices of quivers

$$\begin{aligned} r_0 &= (1, 4), \quad r_1 = (2, 3), \quad r_2 = \mu_1(1, 2)\mu_1, \quad r_3 = \mu_5(5, 6)\mu_5, \quad r_4 = (5, 8), \quad r_5 = (6, 7), \\ \pi_1 &= (1, 5, 2, 6)(4, 8, 3, 7), \quad \pi_2 = (1, 2)(3, 4)(5, 6)(7, 8). \end{aligned}$$

Their actions on the coefficients $\mathbf{y} = (y_1, \dots, y_8)$ generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $D_5^{(1)*5}$. The parameters corresponding to the simple roots of the affine root system are given by

$$\alpha_0 = \frac{y_4}{y_1}, \quad \alpha_1 = \frac{y_3}{y_2}, \quad \alpha_2 = y_1 y_2, \quad \alpha_3 = y_5 y_6, \quad \alpha_4 = \frac{y_8}{y_5}, \quad \alpha_5 = \frac{y_7}{y_6}.$$

The transformations π_1, π_2 act on the parameters as

$$\begin{aligned} \pi_1((\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)) &= (\alpha_4, \alpha_5, \alpha_3, \alpha_2, \alpha_1, \alpha_0), \\ \pi_2((\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)) &= (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_5, \alpha_4). \end{aligned}$$

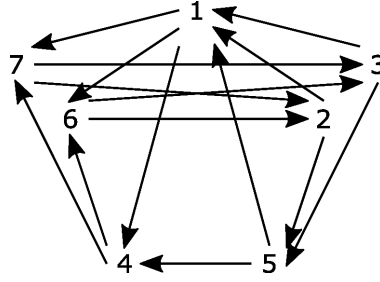
A translation of this Weyl group provides q - P_{VI} ; see [24].

§ 3.2. Quiver Q_7

Thanks to a symmetry of the quiver Q_8 , it is enough to investigate the following confluences.

$$8 \rightarrow 1, \quad 8 \rightarrow 2,$$

^{*5}The formulations of the extended affine Weyl groups in this section were given systematically in the previous work [1].

Figure 5. Q_7

from which we obtain the quiver Q_7 . To be precise, we have to take a permutation $(1, 2, 6)(3, 7, 4, 5)$ after the confluence $8 \rightarrow 2$. In the following we omit permutations after confluence procedures.

The quiver Q_7 is invariant under compositions of mutations and permutations

$$r_0 = \mu_1 \mu_2 (2, 6) \mu_2 \mu_1, \quad r_1 = (2, 3), \quad r_2 = \mu_2 (2, 4) \mu_2, \quad r_3 = \mu_5 (5, 6) \mu_5, \quad r_4 = (6, 7), \\ \pi_1 = (1, 5, 3, 7, 4)(2, 6) \mu_2.$$

Their actions on the coefficients generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $A_4^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1 y_2 y_6, \quad \alpha_1 = \frac{y_3}{y_2}, \quad \alpha_2 = y_2 y_4, \quad \alpha_3 = y_5 y_6, \quad \alpha_4 = \frac{y_7}{y_6}.$$

The transformation π_1 acts on the parameters as

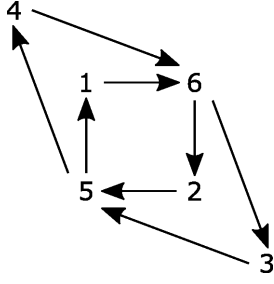
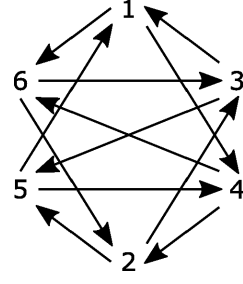
$$\pi_1((\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)) = (\alpha_3, \alpha_4, \alpha_0, \alpha_1, \alpha_2).$$

A translation of this Weyl group provides q -Painlevé V equation; see [17, 22].

In the confluence $8 \rightarrow 1$ a degeneration of the coefficients is given by a replacement $y_1 \rightarrow y_1/\varepsilon$, $y_8 \rightarrow \varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_8	$r_2 r_4 r_3 r_4 r_2 = \mu_1 \mu_8 \mu_2 (2, 6) \mu_2 \mu_8 \mu_1$	r_1	$r_0 r_2 r_0$	r_3	r_5
Q_7	r_0	r_1	r_2	r_3	r_4

For example, the action $r_2 r_4 r_3 r_4 r_2(y_1 y_8)$ in Q_8 is reduced to the one $r_0(y_1)$ in Q_7 as

Figure 6. Q_{61} Figure 7. Q_{62}

follows.

$$\begin{aligned}
 r_2 r_4 r_3 r_4 r_2 (y_1 y_8) &= \frac{(1 + y_1 + y_1 y_6 + y_1 y_6 y_2)(1 + y_8 + y_8 y_1 + y_8 y_1 y_6)}{y_1 y_6 (1 + y_2 + y_2 y_8 + y_2 y_8 y_1)(1 + y_6 + y_6 y_2 + y_6 y_2 y_8)} \\
 &\xrightarrow[y_8 \rightarrow \varepsilon]{y_1 \rightarrow y_1 / \varepsilon} \frac{(\varepsilon + y_1 + y_1 y_6 + y_1 y_6 y_2)(1 + \varepsilon + y_1 + y_1 y_6)}{y_1 y_6 (1 + y_2 + y_2 \varepsilon + y_2 y_1)(1 + y_6 + y_6 y_2 + y_6 y_2 \varepsilon)} \\
 &\xrightarrow{\varepsilon \rightarrow 0} \frac{1 + y_1 + y_1 y_6}{y_6 (1 + y_2 + y_2 y_1)} \\
 &= r_0(y_1).
 \end{aligned}$$

Note that, throughout this section, we haven't clarified degenerations of mutations or transformations denoted by π_1, π_2 yet. It is a future problem.

§ 3.3. Quiver with 6 vertices

For the quiver Q_7 , it is enough to investigate the following confluences.

$$7 \rightarrow 1, \quad 7 \rightarrow 2, \quad 7 \rightarrow 4, \quad 5 \rightarrow 1, \quad 5 \rightarrow 2, \quad 5 \rightarrow 4, \quad 4 \rightarrow 1, \quad 3 \rightarrow 1.$$

Then we obtain the quivers Q_{61} and Q_{62} .

3.3.1. Q_{61}

The quiver Q_{61} is obtained via the following confluences.

$$Q_7 \rightarrow Q_{61} : \quad 7 \rightarrow 1, \quad 5 \rightarrow 4, \quad 3 \rightarrow 1.$$

To be precise, the quiver obtained after the confluence $5 \rightarrow 4$ is different from the one Q_{61} . We have to take a mutation μ_1 after the confluence procedure in order to obtain Q_{61} .

The quiver Q_{61} is invariant under compositions of mutations and permutations

$$\begin{aligned}
 r_1 &= \mu_5 (5, 6) \mu_5, \quad r_2 = \mu_1 (1, 2) \mu_1, \quad r_3 = (1, 4), \quad r_4 = (2, 3), \\
 \pi_1 &= (1, 5, 4, 6) \mu_1 \mu_5, \quad \pi_2 = (1, 2)(3, 4)(5, 6).
 \end{aligned}$$

The actions of simple reflections $r_1, \dots, r_4$ on the coefficients generate a group of birational transformations which is isomorphic to the Weyl group of type D_4 . The parameters corresponding to the simple roots are given by

$$\alpha_1 = y_5 y_6, \quad \alpha_2 = y_1 y_2, \quad \alpha_3 = \frac{y_4}{y_1}, \quad \alpha_4 = \frac{y_3}{y_2}.$$

The transformations π_1, π_2 act on the parameters as

$$\pi_1((\alpha_1, \alpha_2, \alpha_3, \alpha_4)) = (\alpha_3, \alpha_2, \alpha_1, \alpha_4), \quad \pi_2((\alpha_1, \alpha_2, \alpha_3, \alpha_4)) = (\alpha_1, \alpha_2, \alpha_4, \alpha_3).$$

Note that π_1 is not an involution unlike π_2 because $\pi_1^2(y_i) \neq y_i$ for any i .

In this case the simple reflections aren't obtained via a limiting procedure. They are derived from the simple reflections of the quiver Q_8 . A set of the coefficients $\{y_1, \dots, y_6\}$ is closed under actions of $r_0, \dots, r_3$ in Q_8 . Moreover, the quiver Q_{61} is obtained by removing two vertices 7, 8 and all arrows touching 7, 8 from Q_8 . These facts induce the following degeneration.

Q_8	r_3	r_2	r_0	r_1
Q_{61}	r_1	r_2	r_3	r_4

3.3.2. Q_{62}

The quiver Q_{62} is obtained via the following confluences.

$$Q_7 \rightarrow Q_{62} : \quad 7 \rightarrow 2, \quad 7 \rightarrow 4, \quad 5 \rightarrow 1, \quad 5 \rightarrow 2, \quad 4 \rightarrow 1.$$

Similarly as Q_{61} , we have to take mutations μ_2 and μ_7 after the confluences $7 \rightarrow 4$ and $5 \rightarrow 2$ respectively. We also have to take μ_2 and μ_7 in this order after $5 \rightarrow 1$ or $4 \rightarrow 1$.

The quiver Q_{62} is invariant under compositions of mutations and permutations

$$\begin{aligned} r_0 &= \mu_1(1, 2) \mu_1, & r_1 &= \mu_3(3, 4) \mu_3, & r_2 &= \mu_5(5, 6) \mu_5, & \pi_1 &= (1, 3, 6)(2, 4, 5), \\ s_0 &= \mu_1 \mu_3(3, 6) \mu_3 \mu_1, & s_1 &= \mu_2 \mu_4(4, 5) \mu_4 \mu_2, & \pi_2 &= (1, 2)(3, 4)(5, 6). \end{aligned}$$

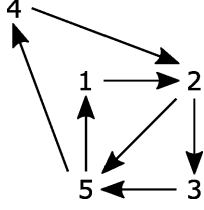
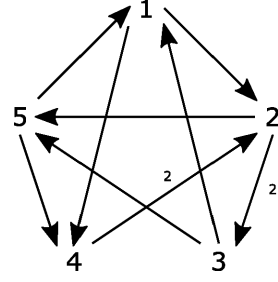
Their actions on the coefficients generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $(A_2 + A_1)^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1 y_2, \quad \alpha_1 = y_3 y_4, \quad \alpha_2 = y_5 y_6, \quad \beta_0 = y_1 y_3 y_6, \quad \beta_1 = y_2 y_4 y_5.$$

The transformations π_1, π_2 act on the parameters as

$$\pi_1((\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)) = (\alpha_1, \alpha_2, \alpha_0, \beta_0, \beta_1), \quad \pi_2((\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)) = (\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_0).$$

Translations of this Weyl group provide q -Painlevé III and IV equations; see [7, 8, 22].

Figure 8. Q_{51} Figure 9. Q_{52}

In the confluence $7 \rightarrow 2$ a degeneration of the coefficients is given by a replacement $y_2 \rightarrow y_2/\varepsilon$, $y_7 \rightarrow \varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_7	$r_4 r_0 r_4 = \mu_7 \mu_2 (1, 2) \mu_2 \mu_7$	$r_1 r_2 r_1$	r_3	$r_1 r_0 r_1$	$r_2 r_4 r_3 r_4 r_2 = \mu_2 \mu_7 \mu_4 (4, 5) \mu_4 \mu_7 \mu_2$
Q_{62}	r_0	r_1	r_2	s_0	s_1

§ 3.4. Quiver with 5 vertices

For the quivers Q_{61} and Q_{62} , it is enough to investigate the following confluences.

$$Q_{61} : \quad 6 \rightarrow 1, \quad 6 \rightarrow 2.$$

$$Q_{62} : \quad 6 \rightarrow 1, \quad 6 \rightarrow 2, \quad 6 \rightarrow 3, \quad 6 \rightarrow 4, \quad 4 \rightarrow 1, \quad 4 \rightarrow 2.$$

Then we obtain the quivers Q_{51} and Q_{52} .

3.4.1. Q_{51}

The quiver Q_{51} is obtained via the following confluences.

$$Q_{61} \rightarrow Q_{51} : \quad 6 \rightarrow 1, \quad 6 \rightarrow 2.$$

$$Q_{62} \rightarrow Q_{51} : \quad 6 \rightarrow 1, \quad 6 \rightarrow 3, \quad 4 \rightarrow 2.$$

It is invariant under compositions of mutations and permutations

$$r_1 = \mu_1 \mu_2 (2, 5) \mu_2 \mu_1, \quad r_2 = (1, 4), \quad r_3 = \mu_1 (1, 3) \mu_1, \quad \pi_1 = (2, 3, 5) \mu_2.$$

The actions of simple reflections r_1, r_2, r_3 on the coefficients generate a group of birational transformations which is isomorphic to the Weyl group of type A_3 . The parameters corresponding to the simple roots are given by

$$\alpha_1 = y_1 y_2 y_5, \quad \alpha_2 = \frac{y_4}{y_1}, \quad \alpha_3 = y_1 y_3.$$

Note that the transformation π_1 is not an involution due to the same reason as Q_{61} , although it acts on the parameters as

$$\pi_1((\alpha_1, \alpha_2, \alpha_3)) = (\alpha_3, \alpha_2, \alpha_1).$$

In the confluence $6 \rightarrow 2$ ($Q_{61} \rightarrow Q_{51}$) a degeneration of the coefficients is given by a replacement $y_2 \rightarrow y_2/\varepsilon$, $y_6 \rightarrow \varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{61}	$r_2 r_1 r_2 = \mu_1 \mu_6 \mu_2 (2, 5) \mu_2 \mu_6 \mu_1$	r_3	$r_4 r_2 r_4$
Q_{51}	r_1	r_2	r_3

3.4.2. Q_{52}

The quiver Q_{52} is obtained via the following conflucences.

$$Q_{62} \rightarrow Q_{52} : \quad 6 \rightarrow 2, \quad 6 \rightarrow 4, \quad 4 \rightarrow 1.$$

It is invariant under compositions of mutations and permutations

$$r_0 = \mu_1 \mu_2 (2, 5) \mu_2 \mu_1, \quad r_1 = \mu_3 (3, 4) \mu_3, \quad \pi_1 = (1, 3, 2, 4, 5) \mu_1.$$

The actions of simple reflections r_0, r_1 on the coefficients generate a group of birational transformations which is isomorphic to the affine Weyl group of type $A_1^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1 y_2 y_5, \quad \alpha_1 = y_3 y_4.$$

The transformation π_1 is not an involution due to the same reason as Q_{61} , although it acts on the parameters as

$$\pi_1((\alpha_0, \alpha_1)) = (\alpha_1, \alpha_0).$$

Therefore the group $\langle r_0, r_1, \pi_1 \rangle$ is not an extended affine Weyl group of type $A_1^{(1)}$. Nevertheless those transformations provide the q -Painlevé II equation and another q -Painlevé equation; see [1, 15].

In the confluence $6 \rightarrow 2$ a degeneration of the coefficients is given by a replacement $y_2 \rightarrow y_2/\varepsilon$, $y_6 \rightarrow \varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{62}	$r_0 r_2 r_0 = \mu_1 \mu_6 \mu_2 (2, 5) \mu_2 \mu_6 \mu_1$	r_1
Q_{52}	r_0	r_1

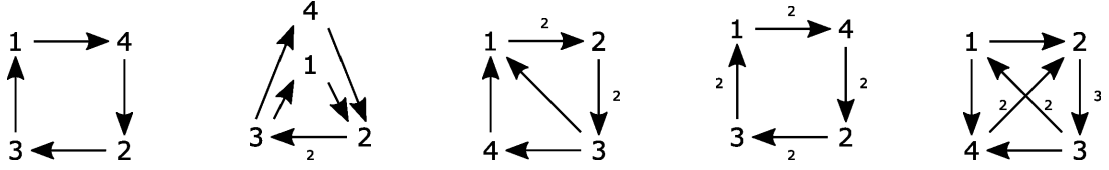


Figure 10. Q_{41} Figure 11. Q_{42} Figure 12. Q_{43} Figure 13. Q_{44} Figure 14. Q_{45}

§ 3.5. Quiver with 4 vertices

For the quivers Q_{51} and Q_{52} , it is enough to investigate the following confluences.

$$Q_{51} : \quad 5 \rightarrow 1, \quad 5 \rightarrow 3, \quad 4 \rightarrow 2, \quad 3 \rightarrow 2.$$

$$Q_{52} : \quad 5 \rightarrow 1, \quad 5 \rightarrow 2, \quad 5 \rightarrow 3, \quad 5 \rightarrow 4, \quad 4 \rightarrow 1, \quad 4 \rightarrow 2, \quad 3 \rightarrow 1, \quad 3 \rightarrow 2, \quad 2 \rightarrow 1.$$

Then we obtain the quivers Q_{41} , Q_{42} , Q_{43} , Q_{44} and Q_{45} . Note that, in the quiver Q_{51} , all of arrows are removed via the confluence $5 \rightarrow 2$.

3.5.1. Q_{41}

The quiver Q_{41} is obtained via the following confluences.

$$Q_{51} \rightarrow Q_{41} : \quad 5 \rightarrow 1, \quad 4 \rightarrow 2.$$

$$Q_{52} \rightarrow Q_{41} : \quad 5 \rightarrow 2, \quad 2 \rightarrow 1.$$

It is invariant under compositions of mutations and permutations

$$r_1 = \mu_1 (1, 2) \mu_1, \quad r_2 = \mu_3 (3, 4) \mu_3, \quad \pi_1 = (1, 4, 2, 3).$$

Since the fundamental relation $(r_1 r_2)^3 = 1$ is satisfied, the actions of simple reflections r_1, r_2 on the coefficients generate a group of birational transformations which is isomorphic to the Weyl group of type A_2 . The parameters corresponding to the simple roots are given by

$$\alpha_1 = y_1 y_2, \quad \alpha_2 = y_3 y_4.$$

The transformation π_1 acts on the parameters as

$$\pi_1((\alpha_1, \alpha_2)) = (\alpha_2, \alpha_1).$$

In the confluence $5 \rightarrow 1$ ($Q_{51} \rightarrow Q_{41}$) a degeneration of the coefficients is given by a replacement $y_1 \rightarrow y_1/\varepsilon$, $y_5 \rightarrow \varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{51}	$r_1 = \mu_5 \mu_1 (1, 2) \mu_1 \mu_5$	$r_2 r_3 r_2$
Q_{41}	r_1	r_2

3.5.2. Q_{42}

The quiver Q_{42} is obtained via the following confluences.

$$\begin{aligned} Q_{51} \rightarrow Q_{42} : \quad & 5 \rightarrow 3, \quad 3 \rightarrow 2. \\ Q_{52} \rightarrow Q_{42} : \quad & 5 \rightarrow 4, \quad 3 \rightarrow 1. \end{aligned}$$

It is invariant under compositions of mutations and permutations

$$r_1 = (1, 4), \quad \pi_1 = (2, 3) \mu_2.$$

The action of a simple reflection r_1 on the coefficients generates a group of birational transformations which is isomorphic to the Weyl group of type A_1 . The parameter corresponding to the simple root is given by

$$\alpha_1 = \frac{y_4}{y_1},$$

which is invariant under the action of π_1 .

In the confluence $5 \rightarrow 3$ ($Q_{51} \rightarrow Q_{42}$) a degeneration of the coefficients is given by a replacement $y_3 \rightarrow y_3/\varepsilon$, $y_5 \rightarrow \varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{51}	r_2
Q_{42}	r_1

3.5.3. Q_{43}

The quiver Q_{43} is obtained via the following confluences.

$$Q_{52} \rightarrow Q_{43} : \quad 4 \rightarrow 2, \quad 3 \rightarrow 2.$$

It is invariant under a composition of a mutation and a permutation

$$\pi_1 = (1, 4, 3) \mu_1.$$

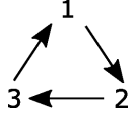
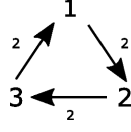
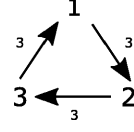
3.5.4. Q_{44}

The quiver Q_{44} is obtained via the following confluence.

$$Q_{52} \rightarrow Q_{44} : \quad 5 \rightarrow 1.$$

It is invariant under compositions of mutations and permutations

$$r_0 = \mu_1 (1, 2) \mu_1, \quad r_1 = \mu_3 (3, 4) \mu_3, \quad \pi_1 = (1, 4, 2, 3).$$

Figure 15. Q_{31} Figure 16. Q_{32} Figure 17. Q_{33}

Their actions on the coefficients generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $A_1^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1 y_2, \quad \alpha_1 = y_3 y_4.$$

The transformation π_1 acts on the parameters as

$$\pi_1((\alpha_0, \alpha_1)) = (\alpha_1, \alpha_0).$$

A translation of this Weyl group provides a q -Painlevé equation; see [1].

In the confluence $5 \rightarrow 1$ a degeneration of the coefficients is given by a replacement $y_1 \rightarrow y_1/\varepsilon$, $y_5 \rightarrow \varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{52}	$r_0 = \mu_5 \mu_1 (1, 2) \mu_1 \mu_5$	r_1
Q_{44}	r_0	r_1

3.5.5. Q_{45}

The quiver Q_{45} is obtained via the following confluences.

$$Q_{52} \rightarrow Q_{45} : \quad 5 \rightarrow 3, \quad 4 \rightarrow 1.$$

It is invariant under a composition of a mutation and a permutation

$$\pi_1 = (1, 2, 3, 4) \mu_1,$$

which provides the q -Painlevé I equation; see [15].

§ 3.6. Quiver with 3 vertices

For the quiver Q_{41} , Q_{42} , Q_{43} , Q_{44} and Q_{45} , it is enough to investigate the following confluences.

$$Q_{41} : 4 \rightarrow 1.$$

$$Q_{42} : 4 \rightarrow 2, \quad 4 \rightarrow 3.$$

$$Q_{43} : 4 \rightarrow 1, \quad 4 \rightarrow 3, \quad 3 \rightarrow 2, \quad 2 \rightarrow 1.$$

$$Q_{44} : 4 \rightarrow 1.$$

$$Q_{45} : 4 \rightarrow 1, \quad 4 \rightarrow 2, \quad 4 \rightarrow 3, \quad 3 \rightarrow 1, \quad 3 \rightarrow 2, \quad 2 \rightarrow 1.$$

Then we obtain the quivers Q_{31} , Q_{32} and Q_{33} . Note that, in the quiver Q_{42} and Q_{43} , all of arrows are removed via the confluence $3 \rightarrow 2$ and $3 \rightarrow 1$ respectively.

3.6.1. Q_{31}

The quiver Q_{31} is obtained via the following confluences.

$$Q_{41} \rightarrow Q_{31} : 4 \rightarrow 1.$$

$$Q_{42} \rightarrow Q_{31} : 4 \rightarrow 2, \quad 4 \rightarrow 3.$$

$$Q_{43} \rightarrow Q_{31} : 3 \rightarrow 2, \quad 2 \rightarrow 1.$$

$$Q_{45} \rightarrow Q_{31} : 4 \rightarrow 3, \quad 3 \rightarrow 2, \quad 2 \rightarrow 1.$$

It is invariant under compositions of mutations and permutations

$$r_1 = \mu_1 \mu_2 (2, 3) \mu_2 \mu_1, \quad \pi_1 = (1, 2, 3).$$

The action of r_1 on the coefficients generates a group of birational transformations which is isomorphic to the Weyl group of type A_1 . The parameter corresponding to the simple root is given by

$$\alpha_1 = y_1 y_2 y_3.$$

which is invariant under the action of π_1 .

In the confluence $4 \rightarrow 1$ ($Q_{41} \rightarrow Q_{31}$) a degeneration of the coefficients is given by a replacement $y_1 \rightarrow \varepsilon$, $y_4 \rightarrow y_1/\varepsilon$ and taking a limit $\varepsilon \rightarrow 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{41}	$r_1 r_2 r_1 = \mu_1 \mu_4 \mu_2 (2, 3) \mu_2 \mu_4 \mu_1$
Q_{31}	r_1

3.6.2. Q_{32}

The quiver Q_{32} is obtained via the following confluences.

$$Q_{43} \rightarrow Q_{32} : \quad 4 \rightarrow 1, \quad 4 \rightarrow 3.$$

$$Q_{44} \rightarrow Q_{32} : \quad 4 \rightarrow 1.$$

$$Q_{45} \rightarrow Q_{32} : \quad 4 \rightarrow 2, \quad 3 \rightarrow 1.$$

It is invariant under compositions of mutations and permutations

$$\pi_1 = (2, 3) \mu_1, \quad \pi_2 = (1, 2, 3).$$

3.6.3. Q_{33}

The quiver Q_{33} is obtained via the following confluence.

$$Q_{45} \rightarrow Q_{33} : \quad 4 \rightarrow 1.$$

It is invariant under a permutation

$$\pi_1 = (1, 2, 3).$$

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