

On theories having a small Galois group

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Abstract

Let L be a first order language. Let D be an infinite L -structure and F a definably closed subset of D . Then $\text{Th}(D, F)$ is called small if it has a model (D_1, F_1) such that $\text{Aut}(\text{acl}(F_1)/F_1)$ is small. In this note, we prove the following:

Theorem: Suppose that $\text{Th}(D, F)$ is a small theory with the definable irreducibility property. Then $\text{Aut}(\text{acl}(F)/F) \cong \text{Aut}(\text{acl}(F_1)/F_1)$ for every model (D_1, F_1) of $\text{Th}(D, F)$.

As a corollary we show the following:

Corollary: Let F be a perfect field. Then the absolute Galois group of F is small if and only if the absolute Galois group of F_1 is isomorphic to that of F for any F_1 elementarily equivalent to F .

0. Introduction

Let L be a first order language. Let D be an infinite L -structure and F a definably closed subset of D . By $\text{Aut}(\text{acl}(F)/F)$ we mean a set of permutations of $\text{acl}(F)$ induced by elementary maps which fix F pointwise.

Let us observe the case where F is a pseudo-finite field (see, e.g., [1]) and D is an algebraically closed extension of F . It is seen that F is a perfect field, and so it is definably closed. Then $\text{Aut}(\text{acl}(F)/F)$ coincides with the absolute Galois group of F . It is also known that the absolute Galois group is isomorphic to the profinite completion of the group of integers $\mathbf{Z}$. In this case, $\text{Th}(D, F)$ satisfies the following condition:

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(*) $\text{Aut}(\text{acl}(F)/F) \cong \text{Aut}(\text{acl}(F_1)/F_1)$ for every model (D_1, F_1) of $\text{Th}(D, F)$.

In this note we want to give a criterion for $\text{Th}(D, F)$ to satisfy (*). To state our results, we need some preparations.

In case D is an algebraically closed field we can consider $\text{Aut}(\text{acl}(F)/F)$ as a profinite group. In a general context we can do as well: Let A be a definably closed subset of D such that $F \subset A \subset \text{acl}(F)$. Then we say that A is *normal* over F if it is invariant under $\text{Aut}(\text{acl}(F)/F)$. And we say that A is *finitely generated* over F if $A = \text{dcl}(\bar{a}F)$ for some $\bar{a} \in \text{acl}(F)$. Let $\mathcal{A}$ be a family of the subsets of $\text{acl}(F)$ which are finitely generated and normal over F . In the obvious way, $\text{Aut}(\text{acl}(F)/F)$ can be identified with a profinite group:

$$\text{Aut}(\text{acl}(F)/F) \cong \text{projlim}_{A \in \mathcal{A}} \text{Aut}(A/F)$$

Through this isomorphism, the Krull topology is induced on $\text{Aut}(\text{acl}(F)/F)$. A profinite group G is said to be *small* if for any finite groups H there are only finitely many continuous homomorphisms of G into H (see [1, p.185]). In particular the profinite completion of $\mathbf{Z}$ is small.

Here we define $\text{Th}(D, F)$ to be *small* if it has a model (D_1, F_1) such that $\text{Aut}(\text{acl}(F_1)/F_1)$ is small. (Our definition is related to that of Hrushovski. See Remark 2.5). Does $\text{Th}(D, F)$ satisfy the condition (*) if it is small? The answer is No. In general $\text{Th}(D, F)$ does not necessarily satisfy (*), even if it has a model (D_1, F_1) such that $\text{Aut}(\text{acl}(F_1)/F_1)$ is finite. We introduce some property on $\text{Th}(D, F)$, the *definable irreducibility property*, and prove that a small theory with the definable irreducibility property satisfies the condition (*) (Theorem 3.1). As a corollary we give a characterization of a perfect field with a small absolute Galois group (Corollary 3.3).

Notation . We only assume basic knowledge of model theory. For the rest of the paper we fix an infinite L -structure D and a definably closed subset F of D . The type of a over A is denoted by $\text{tp}(a/A)$. We say that p is an algebraic type if it has a finite number of realizations. An element a is algebraic over A if the type $\text{tp}(a/A)$ is an algebraic type. The set of all algebraic elements over A is denoted by $\text{acl}(A)$. If a is the unique realization of $\text{tp}(a/A)$ then we say that a is definable over A . The set of all definable elements over A is denoted by $\text{dcl}(A)$. For any $A \subset B$, $\text{Aut}(B/A)$ means the set of permutations of B induced by elementary maps which fix A pointwise.

Let $\text{Th}(D, F)$ be a theory in a language $L \cup \{P\}$, where P is a new unary predicate whose interpretation in (D, F) is F . $L \cup \{P\}$ is denoted by L^* and $\text{Th}(D, F)$ by T^* .

1. The Definable Irreducibility Property

- 1.1. Definition .** (i) Let A be a set. Then we say that $\phi(\bar{x}\bar{a}) \in L(A)$ is *A-irreducible* if it is algebraic and isolated over A .
(ii) Let (D_1, F_1) be a model of T^* and $\phi(\bar{x}\bar{a})$ an F_1 -irreducible formula with exactly n realizations. Then we say that $\phi(\bar{x}\bar{a})$ has the *definable irreducibility property* (DIP) if there is an L^* -formula $\theta(\bar{y})$ such that for any model (D_2, F_2) of T^* and any realization $\bar{b}$ of θ in F_2 , $\phi(\bar{x}\bar{b})$ is an F_2 -irreducible formula with n realizations. We denote such a $\theta(\bar{y})$ by $\theta_\phi^n(\bar{y})$.
(iii) We say that T^* has the DIP if any F_1 -irreducible formula has the DIP for any model (D_1, F_1) of T^* .

1.2 Example . Let A be a countable set and E an equivalence relation on A with infinitely many four element classes. Let $\{A_n : n < \omega\}$ be an enumeration of the classes of E . For each $n < \omega$, let U_n and V_n be subsets of A such that $A_n = U_n \cup V_n$ and $|U_n| = |V_n| = 2$. Let $\pi : A \rightarrow A/E$ be a projection and $F = \{\pi(a) : a \in A\}$. Let $D = (A \cup F, E, \pi, \{U_n\}_{n < \omega}, \{V_n\}_{n < \omega})$. Clearly $\text{dcl}(F) = F$. Then $T^* = \text{Th}(D, F)$ does not have the DIP: Let $\phi(xy) = " \pi(x) = y "$. For every $b \in F$, $\phi(xb)$ is not F -irreducible. On the other hand, for a saturated model (D_1, F_1) of T^* , we can take an element $c \in F_1$ such that $\phi(xc)$ is F_1 -irreducible. Then $\phi(xc)$ does not have the DIP.

1.3. Lemma . If D is an algebraically closed field, then T^* has the DIP.

Proof . Take any model (D_1, F_1) of T^* . Note that F_1 is a perfect field since it is definably closed. And take an F_1 -irreducible formula $\phi(\bar{x}\bar{a})$. Let $\phi(\bar{x}\bar{a})$ have n realizations. Pick a realization $\bar{e}$ of $\phi(\bar{x}\bar{a})$. By the primitive element theorem we can get an element d such that $\text{dcl}(dF_1) = \text{dcl}(\bar{e}F_1)$. Take an F_1 -irreducible formula $\psi(x\bar{b}) \in \text{tp}(d/F_1)$. Let $\psi(x\bar{b})$ have exactly m realizations.

First we show that $\psi(x\bar{b})$ has the DIP. By elimination of quantifiers, $\psi(x\bar{b})$ may be identified with a polynomial equation " $p(x, \bar{b}) = 0$ " of degree m . Let

X be the set of the general polynomials of degree $< m$. Then we can define $\theta_\psi^m(\bar{z})$ by

$$\bigwedge_{p_1, p_2 \in X} \neg \exists \bar{z}_1, \bar{z}_2 \in P \forall x [p(x, \bar{z}) = p_1(x, \bar{z}_1) p_2(x, \bar{z}_2)]$$

Hence $\psi(x\bar{b})$ has the DIP.

We must show that $\phi(x\bar{a})$ has the DIP. Now $\bar{e}$ and d are inter-definable over F_1 , and so we can take an L^* -formula $\alpha(\bar{x}x)$ which satisfies

1. $(D_1, F_1) \models \alpha(\bar{e}d)$;
2. For any model (D_2, F_2) of T^* , $(D_2, F_2) \models \alpha(\bar{e}'d')$ implies $\text{dcl}(\bar{e}'F_2) = \text{dcl}(d'F_2)$.

Then set $\theta(\bar{y})$ by

$$\exists^n \bar{x} \phi(\bar{x}\bar{y}) \wedge \exists \bar{z} \in P \forall \bar{x} [\phi(\bar{x}\bar{y}) \wedge \theta_\psi^m(\bar{z}) \rightarrow \exists x (\psi(x\bar{z}) \wedge \alpha(\bar{x}x))].$$

The formula $\theta(\bar{y})$ is consistent since it is realized by $\bar{a}$. It is easily seen that $\theta(\bar{y}) = \theta_\phi^n(\bar{y})$. Hence $\phi(x\bar{a})$ has the DIP. This completes the proof of the lemma.

2. Theories with a Small Galois Group

2.1. Definition . We say that T^* has a *small Galois group* (or for short, T^* is small) if it has a model (D_1, F_1) such that $\text{Aut}(\text{acl}(F_1)/F_1)$ is small.

2.2. Example . Let A be an infinite set and E an equivalence relation on A with infinitely many two element classes. Let $\pi : A \rightarrow A/E$ be a projection and $F = \{\pi(a) : a \in A\}$. Let $D = (A \cup F, E, \pi)$. It is clear that $\text{dcl}(F) = F$. Take any model (D_1, F_1) of T^* and let $\kappa = |D_1|$. Then it can be seen that $\text{Aut}(\text{acl}(F_1)/F_1)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^\kappa$. Hence $\text{Th}(D, F)$ is not small.

2.3. Lemma . Let T^* be a theory with the DIP. Let (D_1, F_1) and (D_2, F_2) be models of T^* . Suppose that $A \subset \text{acl}(F_1)$ is finitely generated and normal over F_1 . Then there is a $B \subset \text{acl}(F_2)$ which is finitely generated and normal over F_2 such that $\text{Aut}(A/F_1) \cong \text{Aut}(B/F_2)$.

Proof . Since A is finitely generated over F_1 , there is a tuple $\bar{a}$ with $A = \text{dcl}(\bar{a}F_1)$. Let $\{\bar{a}_1, \dots, \bar{a}_n\}$ be a set of all conjugates of $\bar{a}$ over F_1 . Note that $\bar{a}_i$'s are inter-definable over F_1 , since A is normal over F_1 . Take an irreducible formula $\phi(\bar{x}_1 \dots \bar{x}_n \bar{c}) \in \text{tp}(\bar{a}_1 \dots \bar{a}_n / F_1)$. Let $\phi(\bar{x}_1 \dots \bar{x}_n \bar{c})$ have m realizations. By the DIP we can get an L^* -formula $\theta_\phi^m(\bar{y})$. For each $g \in \text{Aut}(\text{acl}(F_1)/F_1)$ define $\sigma_g \in S_n$ by

$$\sigma_g(i) = j \Leftrightarrow g(\bar{a}_i) = \bar{a}_j (\forall i, j \leq n)$$

Let $X = \{\sigma_g \in S_n : g \in \text{Aut}(A/F_1)\}$. Clearly $X \cong \text{Aut}(A/F_1)$. Set a formula $\Sigma(\bar{y})$ by

1. $\theta_\phi^m(\bar{y}) \wedge P(\bar{y})$ and
2. $\bigwedge_{\sigma \in X} \forall \bar{x}_1 \dots \forall \bar{x}_n [\phi(\bar{x}_1 \dots \bar{x}_n \bar{y}) \rightarrow \phi(\bar{x}_{\sigma(1)} \dots \bar{x}_{\sigma(n)} \bar{y})]$.

Then $\Sigma(\bar{y})$ is consistent since it is realized by $\bar{c}$. So we can pick a realization $\bar{d}$ of $\Sigma(\bar{y})$ in (D_2, F_2) . By 1, $\phi(\bar{x}_1 \dots \bar{x}_n \bar{d})$ is an F_2 -irreducible formula with m realizations. Take a realization $\bar{b}_1 \dots \bar{b}_n$ of $\phi(\bar{x}_1 \dots \bar{x}_n \bar{d})$. Let $B = \text{dcl}(\bar{b}_1 F_2)$. Clearly B is finitely generated over F_2 . Since $\phi(\bar{x}_1 \dots \bar{x}_n \bar{d})$ is F_2 -irreducible, $\bar{b}_i$'s are inter-definable over F_2 , and so B is normal over F_2 . By 2 we have $X \cong \text{Aut}(B/F_2)$. Hence $\text{Aut}(A/F_1) \cong \text{Aut}(B/F_2)$.

2.4. Lemma . If T^* is a small theory with the DIP, then $\text{Aut}(\text{acl}(F_1)/F_1)$ is small for every model (D_1, F_1) of T^* .

Proof . Assume otherwise. Then there is a model (D_1, F_1) of T^* such that $\text{Aut}(\text{acl}(F_1)/F_1)$ is not small. So, for some $n < \omega$ there are infinitely many A_i 's such that $|\text{Aut}(A_i/F_1)| \leq n$. On the other hand, by our assumption, we can take a model (D_2, F_2) of T^* such that $\text{Aut}(\text{acl}(F_2)/F_2)$ is small. However, using 2.3 we get infinitely many B_i 's such that $|\text{Aut}(B_i/F_2)| \leq n$. This contradicts the smallness of $\text{Aut}(\text{acl}(F_2)/F_2)$.

2.5. Remark . In [2] and [3], Hrushovski has defined T^* to be *bounded* if $\text{Aut}(\text{acl}(F_1)/F_1)$ is small for every model (D_1, F_1) of T^* (under the stronger assumption than ours). So 2.4 states that every small theory with the DIP is bounded. On the other hand there is a small, unbounded theory (and therefore it does not have the DIP): Let A be a countable set and E an equivalence relation on A with infinitely many two element classes. Let

$\{U_i\}_{i<\omega}$ be an enumeration of all classes of E . For each $i < j < \omega$, let $f_{ij} : U_i \rightarrow U_j$ be a bijection. Let $\pi : A \rightarrow A/E$ be a projection and $F = \{\pi(a) : a \in A\}$. Set $D = (A \cup F, E, \pi, \{U_i\}_{i<\omega}, \{f_{ij}\}_{i<j<\omega})$. Then, for each model (D_1, F_1) of $\text{Th}(D, F)$ we have

$$\text{Aut}(\text{acl}(F_1)/F_1) \cong (\mathbf{Z}/2\mathbf{Z})^{\kappa+1},$$

where $\kappa = |F_1 - F|$. So $\text{Aut}(\text{acl}(F_1)/F_1)$ is small if κ is finite. Otherwise it is not small.

3. Theorem and Corollary

In this section, we prove that a small theory with the DIP satisfies the condition (*) in the introduction. As a corollary we give a characterization of a perfect field with a small absolute Galois group.

3.1. Theorem . *Let D be an infinite structure and F a definably closed subset of D . Suppose that $\text{Th}(D, F)$ is a small theory with the definable irreducibility property. Then $\text{Aut}(\text{acl}(F_1)/F_1) \cong \text{Aut}(\text{acl}(F)/F)$ for every model (D_1, F_1) of $\text{Th}(D, F)$.*

Proof . Take any models (D_1, F_1) and (D_2, F_2) of $\text{Th}(D, F)$. Let $G = \text{Aut}(\text{acl}(F_1)/F_1)$ and $H = \text{Aut}(\text{acl}(F_2)/F_2)$. We will show that $G \cong H$. Let $\mathcal{A}, \mathcal{B}$ be families of all subsets which are finitely generated and normal over F_1, F_2 respectively. For each $n < \omega$ let

$$A_n = \text{dcl}(\bigcup \{A \in \mathcal{A} : |\text{Aut}(A/F_1)| \leq n\});$$

$$B_n = \text{dcl}(\bigcup \{B \in \mathcal{B} : |\text{Aut}(B/F_2)| \leq n\});$$

$$G_n = \text{Aut}(A_n/F_1); H_n = \text{Aut}(B_n/F_2).$$

By 2.4, G and H are small. So, for each $n < \omega$, A_n and B_n are finitely generated and hence G_n and H_n are finite groups.

First we see that $G_n \cong H_n$ for each $n < \omega$. Fix any $n < \omega$. By 2.3 we can take an element $B \in \mathcal{B}$ such that $\text{Aut}(B/F_2) \cong G_n$. Then we have $B \subset B_n$. Hence there is a homomorphism of G_n onto H_n . By the similar argument, we obtain a homomorphism of H_n onto G_n . Hence G_n and H_n are isomorphic.

Next find isomorphisms $\Phi_n : G_n \rightarrow H_n$ ($n < \omega$) satisfying

$$n < m < \omega \Rightarrow \pi_{mn}^H \circ \Phi_m = \Phi_n \circ \pi_{mn}^G,$$

where $\pi_{mn}^G : G_m \rightarrow G_n$ and $\pi_{mn}^H : H_m \rightarrow H_n$ are canonical projections. In fact we can get such isomorphisms, since the number of isomorphisms of G_n with H_n is at most finite. Using the sequence $(\Phi_n)_{n < \omega}$, we can define an isomorphism of G with H in a natural way.

3.2. Remarks . (i) The theory of the example in 2.5 is small, but does not satisfy the condition (*). This shows that the DIP is necessary for the above theorem.

(ii) Let $\{U_i\}_{i < \omega}$ be a disjoint family of two element sets, and let $A = \bigcup_{i < \omega} U_i$. Let F be an arbitrary set which is distinct from A . Put $D = (A \cup F, \{U_i\}_{i < \omega})$. Then it is clear that $\text{Th}(D, F)$ has the DIP. For each model (D_1, F_1) of $\text{Th}(D, F)$, we have $\text{Aut}(\text{acl}(F_1)/F_1) \cong (\mathbf{Z}/2\mathbf{Z})^\omega$. Hence $\text{Th}(D, F)$ satisfies the condition (*), but is not small.

3.3. Corollary . *Let F be a perfect field. Then the following are equivalent:*

- (i) *The absolute Galois group of F is small;*
- (ii) *If F_1 is elementarily equivalent to F , then the absolute Galois group of F_1 is isomorphic to that of F .*

Proof . Let D be the algebraic closure of F . Let $T^* = \text{Th}(D, F)$. By 1.3 T^* has the DIP. Note that if a field F_1 is elementarily equivalent to F then there is a structure D_1 such that (D_1, F_1) is a model of $\text{Th}(D, F)$. So, by 3.1 we obtain the implication (i) $\rightarrow$ (ii). We must show (ii) $\rightarrow$ (i). Suppose that $\text{Aut}(D/F)$ is not small. Then, for some $n < \omega$ there is an infinite set $\{A_i\}_{i < \kappa}$ of the finitely generated normal extensions of F such that $|\text{Aut}(A_i/F)| = n$ for each $i < \kappa$. For each $i < \kappa$ let d_i be a primitive element such that $A_i = \text{dcl}(d_i F)$. Let $p(x, \bar{y})$ be a general polynomial of degree n . Then there is a set $\{\bar{a}_i\}_{i < \kappa}$ of n -tuples from F such that d_i is a solution of $p(x, \bar{a}_i)$ for each $i < \kappa$. Take any $\lambda > \kappa$. By compactness there are a model (D_1, F_1) of T^* , a set $\{\bar{b}_i\}_{i < \lambda}$ of n -tuples from F_1 and a set $\{e_i\}_{i < \lambda}$ of elements of D_1 with the following property:

1. $p(x, \bar{b}_i)$ is an F_1 -irreducible polynomial with a solution e_i , for each $i < \lambda$;

2. The solutions of $p(x, \bar{b}_i)$ are inter-definable over F_1 , for each $i < \lambda$;
3. $\text{dcl}(e_i F_1) \neq \text{dcl}(e_j F_1)$, for each $i, j < \lambda$ with $i \neq j$.

For each $i < \lambda$ let $B_i = \text{dcl}(e_i F_1)$. By 1, 2 and 3, $\{B_i\}_{i < \lambda}$ is the family of the distinct finitely generated normal extensions of F_1 such that $|\text{Aut}(B_i/F_1)| = n$ for each $i < \lambda$. Hence the absolute Galois group of F_1 is not isomorphic to that of F . This completes the proof of the corollary.

Reference

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