

## Motivations

### Bridgeland stability conditions on ruled surfaces $S$

- geometric stability conditions
  - existence? - Yes!!
  - potentially stable objects - all point sheaves
- gluing stability conditions** ..... from Orlov's SOD
  - existence? - **Unknown**.....
  - potentially stable objects - **Unknown**.....

Questions (→ Main Result)

- Declare all gluing stability conditions.
- Find potentially stable objects of gluing stability conditions.
- Positional relation between gluing and geometric on  $\text{Stab } S$ .

## Gluing stability conditions

$p: S \rightarrow C$ , ruled surface  $S$  with base curve  $C$

- $\exists s: C \rightarrow S$  s.t.  $p \circ s = \text{id}_C$  (projection and section)
- $C_0 := s(C)$ ,  $f := \text{fiber of } p$
- Orlov's semiorthogonal decomposition:
- $D^b(S) = \langle D_1, D_2 \rangle$ ,
- $D_1 := p^* D^b(C) \otimes \mathcal{O}_S(-C_0)$  and  $D_2 := p^* D^b(C)$

$\sigma_i \in \text{Stab } D_i$  for  $i = 1, 2$

- $D_1, D_2 \simeq D^b(C)$
- $\exists$  standard stability condition  $\sigma_{st}$  on  $\text{Stab } C$
- $g(C) \geq 1 \Rightarrow \sigma_{st} \cdot \text{GL}^+(2, \mathbb{R}) = \text{Stab } C$  ([Bri], [Mac])

Gluing  $\sigma_1 = (Z_1, \mathcal{A}_1) \in \text{Stab } D_1$  and  $\sigma_2 = (Z_2, \mathcal{A}_2) \in \text{Stab } D_2$

- $Z := Z_1 \circ \lambda_1 + Z_2 \circ \rho_2$ 
  - $\lambda_1$ : left adjoint functor of the inclusion  $D_1 \hookrightarrow D^b(S)$
  - $\rho_2$ : right adjoint functor of the inclusion  $D_2 \hookrightarrow D^b(S)$
- $\mathcal{A} := \{E \mid \lambda_1(E) \in \mathcal{A}_1 \text{ and } \rho_2(E) \in \mathcal{A}_2\}$  (gluing heart)
- $\rightarrow \sigma = (Z, \mathcal{A})$ , gluing **pre-stability** condition

Difficulties

- Confirm **gluing property** between  $\sigma_1$  and  $\sigma_2$ :  
 $\text{Hom}(\mathcal{A}_1, \mathcal{A}_2[i]) = 0$  for any  $i \leq 0$
- $\rightarrow \sigma = (Z, \mathcal{A})$ , gluing stability condition

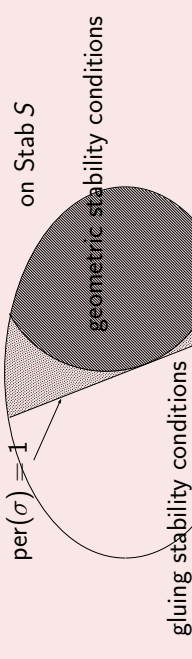
## Main Result [U]

$\sigma$ : gluing pre-stability condition of  $\sigma_1$  and  $\sigma_2$

- $\sigma_1 = (Z_1, \mathcal{P}_1)$ :  $\mathcal{P}_1(0) = p^* \mathcal{P}_{st}(\phi_1) \otimes \mathcal{O}_S(-C_0)$
- $\sigma_2 = (Z_2, \mathcal{P}_2)$ :  $\mathcal{P}_2(0) = p^* \mathcal{P}_{st}(\phi_2)$
- $\rightarrow \text{per}(\sigma) := \phi_1 - \phi_2$ , **gluing perversity**.

Theorem

- $\sigma$ : gluing pre-stability condition
- $\sigma$  gluing stability condition  $\Leftrightarrow \text{per}(\sigma) \geq 1$
- Potentially stable objects:  $\mathcal{O}_f$  and  $\mathcal{O}_f(-C_0)$
- Positional relation between gluing and geometric on  $\text{Stab } S$ :  
**Please see the right figure!!**



All point sheaves are stable of the same phase.

## Idea of gluing perversity

Theorem:  $\sigma$  gluing pre-stability condition

$\sigma$  gluing stability condition  $\Leftrightarrow \text{per}(\sigma) \geq 1$

(Proof)

- assume:  $\mathcal{A}_1 = p^* \mathcal{P}_{st}(\phi, \phi + 1] \otimes \mathcal{O}_S(-C_0)$ ,  $\mathcal{A}_2 = p^* \text{Coh } C$ .

- $\text{per}(\sigma) < 1$
- $\forall q \in \frac{1}{\pi} \arctan \frac{1}{2} \exists L$  line bundle s.t.  $L \in \mathcal{P}_{st}(q)$ 
  - $\rightarrow \exists q \in (\phi - [\phi], 1)$  s.t.  $\exists L$  line bundle  $L \in \mathcal{P}_{st}(q)$
- $\phi < 1 \Rightarrow \text{Hom}(p^* L \otimes \mathcal{O}_S(-C_0)[[\phi]], p^* L[[\phi]]) \neq 0$ 
  - $\rightarrow \phi < 1 \Rightarrow \exists i \leq 0$  s.t.  $\text{Hom}(\mathcal{A}_1, \mathcal{A}_2[i]) \neq 0$
- $\text{per}(\sigma) \geq 1$
- $Rp_* \mathcal{O}_S(C_0)$  locally free sheaf
- $p^* F \otimes \mathcal{O}_S(-C_0) \in \mathcal{A}_1$ ,  $p^* G \in \mathcal{A}_2$ 
  - $\text{Hom}(p^* F \otimes \mathcal{O}_S(-C_0), p^* G[i]) = \text{Hom}(F, G \otimes Rp_* \mathcal{O}_S(C_0)[i])$
  - $\phi_{st}(F) \in (\phi, \phi + 1]$ ,  $\phi_{st}(G \otimes Rp_* \mathcal{O}_S(C_0)[i]) \in (i, i + 1]$
  - $\rightarrow \phi \geq 1 \Rightarrow \forall i \leq 0 \text{Hom}(F, G \otimes Rp_* \mathcal{O}_S(C_0)[i]) = 0$

## Further study

Stability space of ruled surfaces:

- topology
- simply connectedness? - **Unknown**.....
- ex) Stability space of  $\mathbb{P}^1$  is simply connected. ([Mac])
- of Hirzebruch surface, especially
- $\{S_k\}_{k \in \mathbb{Z}}$ : family of exceptional objects on  $D^b(\mathbb{P}^1)$
- $\Theta_k := \{\sigma \in \text{Stab } \mathbb{P}^1 \mid S_k \text{ and } S_{k+1} \text{ are stable.}\}$ 
  - $\text{Stab } \mathbb{P}^1 = \bigcup_{k \in \mathbb{Z}} \Theta_k$
  - $i \neq j \Rightarrow \Theta_i \cap \Theta_j = \sigma_{st} \cdot \text{GL}^+(2, \mathbb{R})$
  - $\rightarrow$  Gluing is more complicated.....

My dreams

- $S$ , ruled surface
- Understand gluing stability conditions on  $D^b(S)$ .
- Get a complete description of  $\text{Stab } S$ !
- positional relation, topology, .....

## References

- T. Bridgeland, Stability conditions on triangulated categories, *Annals of Math.* **166** (2007) no.2, 317-345.
- J. Collins and A. Polishchuck, Gluing stability conditions, *Adv. Theor. Math. Phys.* Volume **14** (2010) Number 2, 563-608.
- E. Macri, Stability conditions on curves, *Math. Res. Lett.* **14** (2007), 657-672.
- T. Uchiba, Gluing stability conditions on ruled surfaces with base curve of positive genus, preprint (2015).