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On 3-dimensional terminal singularities

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Introduction. Canonical and terminal singularities are introduced by M. Reid [4], [5]. He proved that 3-dimensional terminal singularities are cyclic quotient of smooth points or cDV points [5].

Let (X,p) be a 3-dimensional terminal singularity of index m with the associated $\mathbf{Z}_m$-cover $(\tilde{X},\tilde{p}) \to (X,p)$. If $(\tilde{X},\tilde{p})$ is smooth, then it is known as Terminal Lemma (Danilov [2], D. Morrison-G. Stevens [3]) that there exist an integer a prime to m and coordinates x, y, z of $(\tilde{X},\tilde{p})$ which are $\mathbf{Z}_m$-semi-invariants such that $\sigma(x) = \zeta x$, $\sigma(y) = \zeta^{-1}y$, $\sigma(x) = \zeta^a z$ for the standard generator σ of $\mathbf{Z}_m$, where ζ is a primitive m-th root of 1. In this paper, we consider the case where $(\tilde{X},\tilde{p})$ is a singular point and $m > 1$. The main results are Theorems 12, 23, 25 and Remarks 12.2, 23.1, 25.1. These, together with the Terminal Lamma above, almost classifies 3-dimensional terminal singularities.

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§1. Criteria for terminal and canonical singularities.

Let $\mathbf{C}$ be the field of complex numbers, and $\mathbf{C}\{x\}$ denotes the ring of convergent power series in variables x.

Lemma 1. Let (X,p) be the germ of an n-dimensional terminal (resp. canonical) singularity of index m. Let (X',p') be the germ of an n-dimensional reduced Gorenstein variety and $f : (X',p') \to (X,p)$ a morphism such that f factors as

$$X' \xrightarrow{g} Y \xrightarrow{h} X,$$

where h is a blow-up of X and g is quasi-finite. Let ω be a generator of $\omega_X{}^{(m)}$ at p. Then $f^*\omega$, as a meromorphic section of $\omega_{X'}{}^{\otimes m}$, vanishes (resp. is regular) along an arbitrary irreducible divisor $D \ni p$ such that $\dim f(D) < n-1$.

Proof. Let D be a divisor as in the lemma. Let $\pi : \tilde{X}' \to X'$ be the normalization, and $X'' \subset \tilde{X}'$ the complement of the singular locus of $\tilde{X}'$. Since $\operatorname{codim}_{\tilde{X}'}(\tilde{X}'-X'') \geqq 2$ and since $(\pi^*\omega_{X'})|_{X''} \supset \Omega^n{}_{X''}$, we may replace (X',p') by (X'',p'') for some smooth p'' such that $\pi(p'') \in D$. In other words, we may assume that X' is smooth. Hence in the factorization $X' \to Y \to X$, we may assume that Y is normal and $q = g(p)$ is a smooth point of Y (by moving p in D if necessary). Then $h^*\omega$ vanishes (resp. is regular) along $g(D)$. Since $g : (X',p') \to (Y,q)$ is a morphism of manifolds, $f^*\omega = g^*h^*\omega$ vanishes (resp. is regular) along D. q.e.d.

Let (X,p) be a 3-dimensional canonical singularity of index m such that, for the associated $\mathbf{Z}_m$-cover $\pi : (X',p') \to (X,p)$ [4,5], (X',p') is a hypersurface singularity. Then there exist $\mathbf{Z}_m$-semi-invariants $x_1, \ldots, x_4$ in the analytic local ring $\mathcal{O}_{X',p'}{}^h$ such that

(1.1) $\qquad \rho(x_i) = \zeta^{c_i} x_i \quad (i = 1, \ldots, 4)$

(1.2) $\qquad \mathcal{O}^h{}_{X',p'} \cong \mathbf{C}\{x_1, \ldots, x_4\}/(\varphi),$

where ρ (resp. ζ) is a generator of $\mathbf{Z}_m$ (resp. μ_m), $c = (c_1, \ldots, c_4) \in \mathbf{Z}^4$ be such that $\gcd(c,m) = \text{g.c.d.}\{c_1, \ldots, c_4, m\} = 1$, and φ is a semi-invariant. Since π is unramified outside p', one has

$\{q \in X' \mid x_i(q) = 0 \text{ if } c_i \not\equiv 0 \pmod{d}\} = \{p'\}$

for any divisor $d > 1$ of m. Hence, for any divisor $d > 1$ of m, one has

(1.3) $\qquad \gcd(c,m) = 1, \quad \#\{i \in [1,4] \mid c_i \equiv 0 \pmod{d}\} \leq 1.$

Now we reverse the process:

Notation (2.0). Let $\mathbf{Z}_m$ act on $\mathbf{C}\{x_1, \ldots, x_4\}$ by (1.1) with ρ (resp. ζ) a generator of $\mathbf{Z}_m$ (resp. μ_m), where $c \in \mathbf{Z}^4$ satisfies (1.3) for an arbitrary divisor $d > 1$ of m. Let φ be a semi-invariant of $\mathbf{C}\{x_1, \ldots, x_4\}$ such that $\mathbf{C}\{x_1, \ldots, x_4\}/(\varphi)$ is normal, and let (X',p') be the germ of a hypersurface at 0 defined by (1.2), and $(X,p) = (X',p')/\mu_m$.

Then we have

Theorem 2. Under Notation (2.0), let σ be an arbitrary element of $\mathbf{Z}_m$, and $a = (a_1, \dots, a_4)$ a 4-ple of arbitrary integers ≥ 0 such that $\sigma(x_i) = \zeta^{a_i} x_i$ $(i = 1, \dots, 4)$ and that at least three of $a_1, \dots, a_4$ are positive. Let $e(a) = \max\{j \mid \varphi(x_1 t^{a_1}, \dots, x_4 t^{a_4}) \equiv 0 \pmod{t^j}\}$ and $|a| = a_1 + \dots + a_4$. Then if (X,p) is terminal (resp. canonical), then $|a| - m - e(a) > 0$ (resp. ≥ 0).

Proof. (2.1) Let X_0' be $\mathbf{C}^4$ with global coordinates $x_1, \dots, x_4$, and let $\mathbf{Z}_m$ act on X_0' by (1.1). Then $T' = (\mathbf{C}^*)^4 \subset \mathbf{C}^4 = X_0'$ is an affine torus embedding, and to the affine torus embedding $T' \subset X_0'$ correspond the group $\Gamma(T') \cong \mathbf{Z}^4$ of 1 parameter subgroups of T' and a cone $C(X_0')$ of $\Gamma(T') \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $\Gamma(T') = \mathbf{Z}^4$ and $C(X_0') = \mathbf{Q}_+^4$ in the standard way, where $\mathbf{Q}_+ = \{q \in \mathbf{Q} \mid q \geq 0\}$. Then, to $T = T'/\mathbf{Z}_m \to X_0 = X_0'/\mathbf{Z}_m$, correspond $\Gamma(T) = \mathbf{Z}^4 + \mathbf{Z}\, c/m$ and $C(X_0) = \mathbf{Q}_+^4$. By the definition of a, there exist integers β, γ such that $\beta c_i \equiv \gamma a_i$ mod. m, where γ is prime to m. Hence $a/m \in \Gamma(T)$.

(2.2) Let φ' be a convergent power series defined by

$$\varphi'(w, y_1, \dots, y_4) = w^{-e(a)} \cdot \varphi(y_1 w^{a_1}, \dots, y_4 w^{a_4}).$$

Then $\varphi'(0,y)$ is a non-zero weighted homogeneous polynomial of weight $e(a)$ in $y_1, \dots, y_4$ for weight $y_i = a_i$ (cf. (*) below). Since $y_1 y_2$ and $y_3 y_4$ are coprime, $\varphi'(0,y)$ has a prime factor which is prime to $y_1 y_2$ or $y_3 y_4$. By symmetry, we may assume that $\varphi'(0,y)$ has a prime factor prime to $y_1 y_2$ and $a_1 > 0$. Since

$$(*)\qquad \varphi'(0,y_1t^{a_1},\dots,y_4t^{a_4}) = t^{e(a)}\varphi'(0,y_1,\dots,y_4),$$

one can find r_2, r_3, $r_4 \in \mathbf{C}$ such that

$$\varphi'(0,1,r_2,r_3,r_4) = 0.$$

(2.3). Let $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1) \in \mathbf{Z}^4$, and let C be the cone spanned by a, e_2, e_3, e_4 in $\mathbf{Q}^4$ and $M = \mathbf{Z}\,e_2 \oplus \mathbf{Z}\,e_3 \oplus \mathbf{Z}\,e_4 \subset \mathbf{Q}^4$. Then the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}\,a \oplus M & \longrightarrow & \mathbf{Z}\,\frac{a}{m} \oplus M \\ \downarrow & & \downarrow \\ \mathbf{Z}^4 & \longrightarrow & \mathbf{Z}^4 + \mathbf{Z}\,\frac{c}{m} \end{array}$$

gives a commutative diagram of tori:

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

and $C \subset \Gamma(R') \otimes \mathbf{Q}$ gives a torus embedding

$$R' \subset Z_0' = \mathrm{Spec}\,\mathbf{C}[w,\bar{x}_2,\bar{x}_3,\bar{x}_4],$$

where a, e_2, e_3, e_4 of $\Gamma(R')$ correspond to w, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$. Then

$$R \subset Z_0 = \mathrm{Spec}\,\mathbf{C}[\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4],$$

with $\bar{x}_1 = w^m$, is the torus embedding corresponding to

$C \subset \Gamma(R) \otimes \mathbf{Q}$, and commutative diagrams

$$\begin{array}{ccccc} \Gamma(R') \otimes \mathbf{Q} & \longrightarrow & \Gamma(R) \otimes \mathbf{Q} & \longrightarrow & \Gamma(T) \otimes \mathbf{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & C & \longrightarrow & \mathbf{Q}_+^{\,4} \end{array}$$

and

$$\begin{array}{ccccc} \Gamma(R') \otimes \mathbf{Q} & \longrightarrow & \Gamma(T') \otimes \mathbf{Q} & \longrightarrow & \Gamma(T) \otimes \mathbf{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & \mathbf{Q}_+^{\,4} & \longrightarrow & \mathbf{Q}_+^{\,4} \end{array}$$

give a commutative diagram

$$\begin{array}{ccc} Z_0' & \longrightarrow & Z_0 \\ f' \downarrow & & f \downarrow \\ X_0' & \longrightarrow & X_0, \end{array}$$

where f' is given by

$$x_1 = w^{a_1},\ x_2 = \bar{x}_2 w^{a_2},\ x_3 = \bar{x}_3 w^{a_3},\ x_4 = \bar{x}_4 w^{a_4},$$

If $T \subset Y_0$ is the torus embedding corresponding to $C \subset \Gamma(T) \otimes \mathbf{Q}$, then $Z_0 \to Y_0$ is finite and $Y_0 \to X_0$ is a blow-up.

(2.4). Let $s' = V(w, \bar{x}_2 - r_2, \bar{x}_3 - r_3, \bar{x}_4 - r_4) \in Z_0'$ (resp. s = the image of s' in Z_0), and let

$$\psi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = w^{-e(a)} \cdot \varphi(w^{a_1}, \bar{x}_2 w^{a_2}, \bar{x}_3 w^{a_3}, \bar{x}_4 w^{a_4}),$$

(note that the right hand side is a holomorphic function in

$\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ defined near s.) Let $(Z,s) \subset (Z_0,s)$ (resp. $(Z',s') \subset (Z_0',s')$) be defined by $\psi = 0$. Then f (resp. f') induces $f : (Z,s) \to (X,p)$ (resp. $f' : (Z',s') \to (X',p')$), which satisfies the conditions of Lemma 1 by (2.3). We have the following commutative diagram of natural morphisms:

$$\begin{array}{ccc} Z' & \xrightarrow{\tau} & Z \\ f' \downarrow & & f \downarrow \\ X' & \xrightarrow{\pi} & X. \end{array}$$

By Poincaré residue formula,

$$\omega_{X'} = \mathrm{Res}\, \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\varphi} = -\frac{dx_1 \wedge dx_2 \wedge dx_3}{\varphi_4},$$

$$f'^*\omega_{X'} = -\frac{d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3})}{f'^*\varphi_4},$$

where $\varphi_4 = \partial\varphi/\partial x_4$. By calculation, one sees

$$d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3})$$

$$= a_1 \cdot w^{a_1+a_2+a_3-1} \cdot dw \wedge d\bar{x}_2 \wedge d\bar{x}_3.$$

Since $f'^*\varphi = w^{e(a)}\psi$, it follows from the chain rule that

$$(f'^* \varphi_4) \cdot w^{a_4} = w^{e(a)} \psi_4,$$

where $\psi_4 = \partial\psi/\partial\bar{x}_4$. Thus

$$f'^*\omega_{X'} = -a_1 \cdot w^{|a|-e(a)-1} \cdot \frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\psi_4}.$$

One also has

$$\omega_{Z'} = \mathrm{Res}\ \frac{dw\wedge d\bar{x}_2\wedge d\bar{x}_3\wedge d\bar{x}_4}{\psi} = -\frac{dw\wedge d\bar{x}_2\wedge d\bar{x}_3}{\psi_4}.$$

One has

$$f'^{*}\omega_{X'} = a_1 \cdot w^{|a|-e(a)-1}\ \omega_{Z'}.$$

By construction, we have

$$\tau^{*}\omega_{Z} = m \cdot w^{m-1}\ \omega_{Z'}.$$

Hence we have

$$f'^{*}\omega_{X'} = (a_1/m) \cdot w^{|a|-e(a)-m}\ \tau^{*}\omega_{Z}.$$

Since $\pi : X' \to X$ is unramified in codimension 1 by (1.3), one has $\pi^{*}\ \omega_X{}^{(m)} = (\mathrm{unit}) \cdot \omega_{X'}{}^{\otimes m}$ near p' (by abuse of language, $\omega_X{}^{(m)}$ denotes one of its generators at p), and

$$f^{*}\ \omega_X{}^{(m)} = (\mathrm{unit})\cdot\bar{x}_1{}^{|a|-e(a)-m}\ \omega_Z{}^{\otimes m} \quad \text{near} \quad s.$$

Since $\{\bar{x}_1 = 0\}$ is a divisor through s collapsed by f, one has $|a| - e - m > 0$ (resp. ≥ 0). q.e.d.

Under Notation (2.0), let $\chi_1, \ldots, \chi_r \in \mathbf{C}\{x_1, \ldots, x_4\}$ $(r \geq 2)$ be $\mathbf{Z}_m$-semi-invariants with the same character such that $\chi_i \mathbf{C}\{x_1, \ldots, x_4\} = u_i \mathbf{C}\{x_1, \ldots, x_4\}$ for some monomial u_i in $x_1, \ldots, x_4$ $(i = 1, \ldots, r)$ and that $u_1, \ldots, u_r$ are linearly independent over $\mathbf{C}$ and the locus defined by $\chi_1 = \cdots = \chi_r = 0$ is of dimension ≤ 1. Let Φ be the linear system generated by $\chi_1, \ldots, \chi_r$, and assume that our φ is written as

$$\varphi = \sum_{i=1}^{r} \lambda_i \chi_i$$

for some $\lambda = (\lambda_1, \ldots, \lambda_r) \in (\mathbf{C}^*)^r$. By Bertini's theorem, $\varphi = 0$ defines a normal variety for general λ since the base locus of Φ is of dimension ≤ 1. Let σ, a, ζ be as in Theorem 2, then the value of e(a) given in Theorem 2 does not depend on the choice of $\lambda \in (\mathbf{C}^*)^r$. Then under the notation of Theorem 2, the following is the corollary to the proof of Theorem 2.

Corollary 2.1. If $|a| - m - e(a) > 0$ (resp. ≥ 0) for arbitrary σ, a, ζ as in Theorem 2, then (X,p) is terminal (resp. canonical) for general λ. If (X,p) has an isolated singular point at p (resp. canonical singularity outside p) for general λ, then one can add the extra conditions $a_1, \ldots, a_4 > 0$ on a in the statement above.

Proof. Let $X_0 = \mathbf{C}^4/\mathbf{Z}_m$ with respect to the action given above. Then Φ induces a linear system Φ_0 (of Weil divisors) in a neighborhood of 0 in X_0 which is free from fixed components. By the conditions on χ_i's, there exists a toric resolution

$h : U_0 \to X_0$ such that the proper transform Ψ_0 of Φ_0 to U_0 is free from base points (principalizer of the coherent sheaf associated to Φ_0). Then a general member X of Φ_0 is normal at 0 and its proper transform $U = h^{-1}[X]$ is smooth *in a neighborhood of* $h^{-1}(0)$. Thus it is enough to show that $h^*\omega_X{}^{(m)}$, as a meromorphic section of $\omega_U{}^{\otimes m}$, vanishes (resp. is regular) along $D_0 \cap U$, where $D_0 \subset U_0$ is an arbitrary exceptional divisor of h. We now use the notation of the *proof of Theorem 2*. Let L_+ be the 1-simplex $\subset \mathbf{Q}_+{}^4 \subset \Gamma(T) \otimes \mathbf{Q}$ corresponding to D_0. Let $a = (a_1, \ldots, a_4)$ be a 4-ple of integers ≥ 0 such that $\mathbf{Z}a/m = \mathbf{Q}L_+ \cap \Gamma(T)$. By (1.3), the singular locus of X_0 is of dimension ≤ 1, and the base locus of Φ_0 *is of dimension* ≤ 1. Thus *one may assume* that the image of D_0 to X_0 is of dimension ≤ 1, whence at least 3 of $a_1, \ldots, a_4$ are positive. Then the notation of (2.3) can be used, and the torus embedding $T \subset V_0 = T \cup D_0$ corresponds to $L_+ \subset \Gamma(T) \otimes \mathbf{Q}$. $L_+ \subset \Gamma(R) \otimes \mathbf{Q}$ *corresponds to the open* subset W_0 of Z_0 defined by $\bar{x}_3\bar{x}_4 \neq 0$. Since $\mathbf{Z}a/m = \mathbf{Q}a/m \cap \Gamma(T)$, $\{\bar{x}_1 = 0\}$ is not in the branch locus of $W_0 \to V_0$. Let V, W be the proper transforms of X to V_0 and W_0, respectively. Then, since Ψ_0 is free from base points, $g : W \to V$ is unramified over general points of arbitrary irreducible components of $V \cap D_0$. Thus by

$$g^*h^*\omega_X{}^{(m)} = (\text{unit})\cdot \bar{x}_1{}^{|a|-e(a)-m}\, \omega_W{}^{\otimes m} \quad \text{along} \quad W \cap D_0,$$

(2.4), we have corollary 2.1. q.e.d.

Corollary 2.2. Under the assumptions of Theorem 2, assume that m is odd, and

$$\varphi = x_1^2 + f(x_2, x_3, x_4) \quad (f \in \mathbf{C}\{x_2, x_3, x_4\}).$$

Let $n = \max\{j \mid \varphi(x_2 t^{a_2}, x_3 t^{a_3}, x_4 t^{a_4}) \equiv 0 \ (t^j)\}$. Then

$$a_2+a_3+a_4 > (\text{resp.} \geq) \begin{cases} m + n/2 & \text{if } n \text{ is even} \\ m/2 + n/2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. One has $2\cdot\mathrm{wt}\, x_1 \equiv n \pmod{m}$. If n is even, then we choose $a_1 = n/2$, keeping a_2, a_3, a_4 the same. Then $n/2 + a_2 + a_3 + a_4 > (\text{resp.} \geq)\ m + n$. If n is odd, then we choose $a_1 = (m+n)/2$, keeping a_2, a_3, a_4 the same. Then $(n + m)/2 + a_2 + a_3 + a_4 > (\text{resp.} \geq)\ m + n$. q.e.d.

For the approximation of φ, we need the following which easily follows by the argument of [1]:

Theorem 3. Let $\varphi \in \mathbf{C}\{x,y,z,u\}$. Assume that φ has an isolated singular point at (O), and that $\mathbf{Z}_m$ acts on $\mathbf{C}\{x,y,z,u\}$ in such a way that x, y, z, u, and φ are semi-invariants. Then for an arbitrary integer $b > 0$, there exists an integer $n > 0$ such that, for an arbitrary semi-invariant $\psi \in \mathbf{C}\{x,y,z,u\}$ with the property $\psi \equiv \varphi \ (x,y,z,u)^n$, there exists an analytic $\mathbf{C}$-automorphism σ of $\mathbf{C}\{x,y,z,u\}$ commuting with $\mathbf{Z}_m$-action (will be called a $\mathbf{Z}_m$-automorphism, for short) such that $\sigma(\varphi)\mathbf{C}\{x,y,z,u\} = \psi\mathbf{C}\{x,y,z,u\}$, $\sigma \equiv \mathrm{id.}$ modulo $(x,y,z,u)^b$.

Corollary 4. Under the notation and the assumptions of Theorem 3, if $\varphi \in (x,y,z,u)^2$ and if xy appears in φ, then there exist a $\mathbf{Z}_m$-automorphism σ of $\mathbf{C}\{x,y,z,u\}$ such that

$$\sigma(x) - x \in (y,z,u) + (x,y,z,u)^2,$$
$$\sigma(y) - y \in (x,z,u) + (x,y,z,u)^2,$$
$$\sigma(z) - z,\ \sigma(u) - u \in (x,y,z,u)^2,$$
$$(\sigma(\varphi)) = (xy + f(z,u)) \text{ as ideals for some } f \in \mathbf{C}\{z,u\}.$$

§2. Notation and classification.

Let φ be an element of $(x,y,z,u)^2\mathbf{C}\{x,y,z,y\}$ which has a $\mathbf{Z}_m$-action ($m > 1$) such that x,y,z,u,φ are semi-invariants. Assume that φ has an isolated cDV singularity at the origin (O), that the quotient of $\{\varphi = 0\}$ by $\mathbf{Z}_m$ has a terminal singularity at (O), and that the action of $\mathbf{Z}_m$ is free on $U - (O)$, where $U \ni (O)$ is an open set of $\{\varphi = 0\}$. By a $\mathbf{Z}_m$-automorphism, we mean an analytic $\mathbf{C}$-automorphism of $\mathbf{C}\{x,y,z,u\}$ commuting with $\mathbf{Z}_m$-action unless otherwise mentioned. We will keep these assumptions and notation, unless otherwise mentioned.

Fixing a primitive m-th root ζ of 1, and given the $\mathbf{Z}_m$-action above, we associate to each $\sigma \in \mathbf{Z}_m$ a weight modulo m (denoted by σ-wt mod. m); $\sigma\text{-wt}(x) \equiv a(\sigma), \ldots, \sigma\text{-wt}(u) \equiv d(\sigma) \pmod{m}$ are determined by $\sigma(x) = \zeta^{a(\sigma)}\cdot x, \ldots, \sigma(u) = \zeta^{d(\sigma)}\cdot u$. If σ is a generator of $\mathbf{Z}_m$, we may simply call σ-wt a wt., if there is no danger of confusion. By Theorem 2 and Corollary 2.2, one can almost classify such φ as above. The results are as follows.

Theorem 5. If xy appears in φ, then one of the following holds (after exchanging z,u if necessary):

(1) $\text{wt } x + \text{wt } y \equiv \text{wt } z \equiv 0$ mod. m, and wt u, wt x, wt y are prime to m.

(2) $m = 4$ and there exists a generator σ of $\mathbf{Z}_4$ such that σ-wts of x,y,z,u are 1,1,2,3 mod. 4.

By Theorem 5, one can easily prove

Lemma 6. If x^2 and y^2 appear in φ and $m = 2$, then

(1) if $\mathrm{wt}\, x \equiv \mathrm{wt}\, y$ mod. 2, then after exchanging z, u (if necessary), one has wts of x, y, z, u equal to 1, 1, 0, 1 mod. 2, and modulo $\mathbf{Z}_2$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u^2))$ for some $f \in \mathbf{C}\{z,u\}$,

(2) if $\mathrm{wt}\, x \not\equiv \mathrm{wt}\, y$ mod. 2, then after exchanging x, y (if necessary), one has wts of x, y, z, u equal to 1, 0, 1, 1 mod. 2, and modulo $\mathbf{Z}_2$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u))$ for some $f \in \mathbf{C}\{z,u\}$.

Thus one gets

Theorem 7. If the quadratic part φ_2 of φ has rank ≥ 2 as a quadratic form, then after permutation of x, y, z, u (if necessary), one of the following holds.

(1) $\mathrm{wt}\, x + \mathrm{wt}\, y \equiv 0$; $\mathrm{wt}\, z \equiv \mathrm{wt}\, \varphi \equiv 0$; $\mathrm{wt}\, x$, $\mathrm{wt}\, y$, $\mathrm{wt}\, u$ are prime to m, and modulo $\mathbf{Z}_m$-automorphism, $(\varphi) = (xy + f(z, u^m))$ for some $f \in \mathbf{C}\{z,u^m\}$.

(2) $m = 4$: wts of x, y, z, u, φ are 1, 3, 2, 1, 2 mod. 4 (after choosing a generator of $\mathbf{Z}_4$), and modulo $\mathbf{Z}_4$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u^2))$ for some $f \in \mathbf{C}\{z,u^2\}$.

(3) $m = 2$: wts of x, y, z, u, φ are 1, 0, 1, 1, 0 mod. 2, and modulo $\mathbf{Z}_2$-automorphism, $(\varphi) = (x^2 + y^2 + f(z, u))$ for some $f \in (z,u)^4\mathbf{C}\{z,u\}$.

Remark 7.1. In case (1) (resp. (2), (3)) of Theorem

7, if f is a general linear combination of a finite number of monomials in $(z^2, u^m)\mathbf{C}\{z, u^m\}$ (resp. $(z^3, u^2)\mathbf{C}\{z^2, zu^2, u^4\}$, $(z^4, z^3u, z^2u^2, zu^3, u^4)\mathbf{C}\{z^2, zu, u^2\}$) and if f has an isolated singular point at 0, then (X,p) is a terminal singularity. This follows from Corollary 2.1.

Lemma 8. If rk $\varphi_2 \leq 1$ and u^2 appears in φ_2, then one has

$$(\varphi) = (u^2 + f(x,y,z)) \text{ modulo } \mathbf{Z}_m\text{-automorphism},$$

where $f \in (x,y,z)^3\mathbf{C}\{x,y,z\}$ has non-zero cubic term f_3.

This follows from the definition of cDV points.

Theorem 9. Under the situation of Lemma 8, if f_3 is not a cube of a linear factor, then after permutation of x, y, z and choice of generator σ of $\mathbf{Z}_m$, one of the following holds.

(1) m = 2, wts of x, y, z, u, φ are 1, 1, 0, 1, 0 mod.2, and xyz or y^2z appears in f_3.

(2) m = 3, σ-wts of x, y, z, u, φ are 1, 2, 2, 0, 0 mod. 3, and $f_3 = x^3 + y^3 + z^3$, $x^3 + yz^2$, or $x^3 + y^3$ modulo $\mathbf{Z}_3$-automorphism of $\mathbf{C}\{x,y,z\}$.

Remark 9.1. In cases (1), (2) of Theorem 9, it is easy to see that modulo $\mathbf{Z}_m$-automorphism and units of $\mathbf{C}\{x, y, z, u\}$, one may put φ in one of the following forms by Theorem 3:

Case 1. m = 2, wts of x, y, z, u are 1, 1, 0, 1 mod.

2,

(9.1.1) $\varphi = u^2 + xyz + x^{2a} + y^{2b} + z^c$,

(9.1.2) $\varphi = u^2 + y^2z + \lambda yx^{2a-1} + g$,

$(a, b \geq 2, c \geq 3,\ \lambda \in k,\ g \in (x^4, x^2z^2, z^3)\mathbf{C}\{x^2, z\})$,

Case 2. $m = 3$, wts of x, y, z, u are 1, 2, 2, 0 mod. 3,

(9.2.1) $\varphi = u^2 + x^3 + y^3 + z^3$,

(9.2.2) $\varphi = u^2 + x^3 + yz^2 + \lambda y^{3r+1}x + \mu y^{3s}$,

(9.2.3) $\varphi = u^2 + x^3 + y^3 + xyz^3\cdot\alpha + xz^4\cdot\beta + yz^5\cdot\gamma + z^6\cdot\delta$,

$(r \geq 1,\ s \geq 2,\ \lambda, \mu \in k,\ \alpha, \beta, \gamma, \delta \in \mathbf{C}\{z^3\})$.

For (9.1.1) and (9.2.1), (X, p) is terminal. If φ is a general linear combination of a finite number of monomials as in (9.1.2), (9.2.2), or (9.2.3), and if φ has an isolated singularity at 0, then (X, p) is terminal.

Lemma 10. Under the situation of Lemma 8, if $f_3 = x^3$, then modulo $\mathbf{Z}_m$-automorphism,

$$(\varphi) = (u^2 + x^3 + g(y,z)x + h(y,z)),$$

where $g, h \in \mathbf{C}\{y, z\}$, $g \in (y,z)^3$, $h \in (y, z)^4$.

Theorem 11. Under the situation of Lemma 10, one has $m = 2$ and wts of x, y, z, u, φ are 0, 1, 1, 1, 0 mod. 2, and $h \notin (y,z)^5$.

Remark 11.1. If $m = 2$ and wts of x, y, z, u are 0, 1, 1, 1 mod. 2 and if an even polynomial g (resp. h) in y, z is a general linear combination of a finite number of monomials $\in (y, z)^4$ and $h \notin (y,z)^5$, and if

$$\varphi = u^2 + x^3 + g(y, z)x + h(y, z)$$

has an isolated singularity at O, then (X,p) is terminal.

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