

On James and Schäffer constants for Banach spaces

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We introduce James and Schaffer type constants for Banach spaces X , and investigate the relation between these constants and some geometrical properties of Banach spaces.

Let X be a Banach space with $\dim X \geq 2$. Then, geometrical properties of X are determined by its unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ or its unit sphere $S_X = \{x \in X : \|x\| = 1\}$. The modulus of convexity of X is a function $\delta_X : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|x+y\|/2 : x, y \in S_X, \|x-y\| = \varepsilon\}$$

In the above definition, it is well-known that S_X may be replaced by B_X . The space X is called uniformly convex (Clarkson [1]) if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon < 2$, and called uniform non-square (James [5]) if $\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$.

James and Schäffer constants:

James constant of X is defined by

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

and Schäffer constant of X is defined by

$$S(X) = \inf\{\max(\|x+y\|, \|x-y\|) : x, y \in S_X\}.$$

Known Facts (cf. [3], [4], [7]):

- (1) In the definition of $J(X)$, S_X may be replaced by B_X .
- (2) $J(X)S(X) = 2$
- (3) X : unif. non-square $\Leftrightarrow J(X) < 2 \Leftrightarrow S(X) > 1$
- (4) Let $1 \leq p \leq \infty$, $1/p + 1/p' = 1$, $t = \min\{p, p'\}$ and $s = \max\{p, p'\}$.
Then, $J(L_p) = 2^{1/t}$ and $S(L_p) = 2^{1/s}$.
- (5) $\sqrt{2} \leq J(X) \leq 2$ and $1 \leq S(X) \leq \sqrt{2}$ for any Banach space X .
- (6) If X is a Hilbert space, then $J(X) = \sqrt{2}$, but the converse is not true.

(7) There is a Banach space X such that $J(X) \neq J(X^*)$ ($S(X) \neq S(X^*)$), where X^* is a dual space of X .

(8) $2J(X) - 2 \leq J(X^*) \leq J(X)/2 + 1$ for any Banach space X .

New constants of James and Schäffer type:

We denote by $M_t(a, b)$ the power means of order t of the positive real numbers a and b , that is,

$$M_t(a, b) = \{(a^t + b^t)/2\}^{1/t} \quad (t \neq 0) \quad \text{and} \quad M_0(a, b) = (ab)^{1/2}$$

Remark. (1) $M_t(a, b)$ is defined for $a, b \geq 0$ ($M_t(a, b) = 0$ if $t < 0$, $ab = 0$).

(2) If $t \rightarrow -\infty$ ($t \rightarrow +\infty$), then $M_t(a, b) \rightarrow \min\{a, b\}$ ($M_t(a, b) \rightarrow \max\{a, b\}$).

James type constants:

$$J_t(X) = \sup\{M_t(\|x+y\|, \|x-y\|) : x, y \in S_X\}, \quad -\infty < t < +\infty$$

Schäffer type constants:

$$S_t(X) = \inf\{M_t(\|x+y\|, \|x-y\|) : x, y \in S_X\}, \quad -\infty < t < +\infty$$

Remark. In the definition of $J_t(X)$, S_X may be replaced by B_X .

Proposition 1. (1) $\sqrt{2} \leq J(X) \leq J_t(X) \leq 2$ for all $t \in (-\infty, +\infty)$, and if $t \geq 2$, then $J_t(X) \geq 2^{1-1/t}$.

(2) $J_t(X)$ is non-decreasing on $(-\infty, +\infty)$, $J_t(X) \rightarrow 2$ if $t \rightarrow +\infty$, and $J_t(X) \rightarrow J(X)$ if $t \rightarrow -\infty$.

(3) $S_t(X) = 0$ if $t \leq 0$, $S_t(X) = 2^{1-1/t}$ if $0 < t \leq 1$, $S_t(X) \leq 2^{1-1/t}$ for all $t < \infty$, and $1 \leq S_t(X) \leq S(X) \leq \sqrt{2}$ for all $t \in (1, +\infty)$.

(4) $S_t(X)$ is non-decreasing on $(-\infty, +\infty)$, $S_t(X) \rightarrow 1$ if $t \rightarrow 1+0$, and $S_t(X) \rightarrow S(X)$ if $t \rightarrow +\infty$.

Theorem 2. The following assertions are equivalent:

(1) X is uniformly non-square.

(2) $J_t(X) < 2$ for all t (some t).

(3) $J(X) < J_t(X)$ for some t .

(4) There exists t_0 such that $J_t(X)$ is strictly increasing on $[t_0, +\infty)$.

(5) $S_t(X) > 1$ for all $t > 1$ (some $t > 1$).

(6) $S(X) > S_t(X)$ for some $t > 1$.

Let $1 \leq p \leq 2$ and $1/p + 1/p' = 1$. We say that the (p, p') -Clarkson inequality holds in a Banach space X if for any $x, y \in X$, the inequality

$$(CI_p) \quad (\|x + y\|^{p'} + \|x - y\|^{p'})^{1/p'} \leq 2^{1/p'} (\|x\|^p + \|y\|^p)^{1/p}$$

holds.

Remark. Let $1 \leq p \leq 2$.

- (1) (CI_p) holds in L_p and $L_{p'}$ (Clarkson [1]).
- (2) (CI_p) holds in X if and only if it holds in X^* ; if (CI_p) holds in X , then (CI_t) holds in X for any $t \in [1, p]$; and if (CI_p) holds in X , then (CI_t) holds in $L_r(X)$, where $1 \leq r \leq \infty$ and $t = \min\{p, r, r'\}$ (Takahashi and Kato [9]).

A Banach space Y is said to be finitely representable (f.r.) in a Banach space X if for any $\lambda > 1$ and for any finite dimensional subspace F of Y there is a finite dimensional subspace E of X with $\dim E = \dim F$ such that the Banach-Mazur distance $d(E, F) \leq \lambda$.

Proposition 3. If Y is f.r. in X , then $J_t(Y) \leq J_t(X)$ and $S_t(Y) \geq S_t(X)$ for any t .

Theorem 4. Let $1 < p \leq 2$ and suppose that the (p, p') -Clarkson inequality holds in X .

- (1) $J_t(X) = 2^{1-1/t}$ for $t \geq p'$, and $S_t(X) = 2^{1-1/t}$ for $0 < t \leq p$.
- (2) If $\mathcal{O}_p$ (or $\mathcal{O}_{p'}$) is finitely representable (f.r.) in X , then $J_t(X) = 2^{1/p}$ for $t \leq p'$, and $S_t(X) = 2^{1/p'}$ for $t \geq p$.

Corollary 1. (1) $J_t(H) = \sqrt{2}$ if $t \leq 2$, $J_t(H) = 2^{1-1/t}$ if $t \geq 2$, $S_t(X) = 2^{1-1/t}$ if $0 < t \leq 2$, and $S_t(X) = \sqrt{2}$ if $t \geq 2$, where H is a Hilbert space.

(2) $J_t(L_p) = 2^{1/r}$ if $t \leq r'$, $J_t(L_p) = 2^{1-1/t}$ if $t \geq r'$, $S_t(L_p) = 2^{1-1/t}$ if $0 < t \leq r$, and $S_t(L_p) = 2^{1/r'}$ if $t \geq r$, where $r = \min\{p, p'\}$.

(3) Let $X = L_p(L_q)$, and $r = \min\{p, p', q, q'\}$. Then $J_t(X) = 2^{1/r}$ if $t \leq r'$, $J_t(X) = 2^{1-1/t}$ if $t > r'$, $S_t(X) = 2^{1-1/t}$, and $S_t(X) = 2^{1/r'}$ if $t \geq r$.

Corollary 2. Let $X = L_p(L_q)$, $1 < p, q < \infty$. Then, $J(X) = 2^{1/r}$ and $S(X) = 2^{1/r'}$, where $r = \min\{p, p', q, q'\}$.

Remark. As already mentioned, for any Banach space X , it holds $J(X)S(X) = 2$, $J_t(X) \rightarrow J(X)$ if $t \rightarrow -\infty$, and $S_t(X) \rightarrow S(X)$ if $t \rightarrow +\infty$. By Corollary 1, we know that for various Banach spaces X , $J_t(X)S_{t'}(X) = 2$, where $1 < t < \infty$ and $1/t + 1/t' = 1$. Note that for any t ($1 < t < \infty$), there is a Banach space X such that $J_t(X)S_t(X) \neq 2$.

Now we give a characterization of a Hilbert space. As mentioned before, if X is a Hilbert space, then $J(X) = \sqrt{2}$; but the converse is not true.

Theorem 5. A Banach space X is isometric to a Hilbert space if and only if $J_2(X) = \sqrt{2}$.

Remark. Let $C_{NJ}(X)$ denote the von Neumann-Jordan constant of X (Clarkson [2]). Then it is easy to see that $\sqrt{2} \leq J_2(X) \leq \sqrt{2C_{NJ}(X)}$ for any Banach space X . Hence, Theorem 5 generalizes a result of Jordan and von Neumann [6], which asserts that X is a Hilbert space if and only if $C_{NJ}(X) = 1$.

Proposition 6. Let X be a Banach space. If there is $t \in [2, \infty)$ such that $J_t(X) = 2^{1-1/t}$, then X is uniformly convex.

Remark. For any Banach space X , we have $J_t(X) \geq 2^{1-1/t}$ for all $t \geq 2$ (see, Proposition 1). It can be shown that for any $\varepsilon > 0$, there is a Banach space X which is not uniformly convex such that $J_t(X) < 2^{1-1/t} + \varepsilon$.

Theorem 7. (1) For any Banach space X , $J_1(X) = J_1(X^*)$.

(2) For any $t < 1$, there is a Banach space X such that $J_t(X) \neq J_t(X^*)$.

Corollary 3. X : uniformly non-square $\Leftrightarrow X^*$: uniformly non-square

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