

On the evaluation of the multiplicity of the ground states for quantum field models

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[Abstract] We consider an abstract interaction system of Bose fields. The Hilbert space for the system is defined by a tensor product of a Hilbert space and boson Fock space. It is proven that if the total Hamiltonian has a ground state, its multiplicity is finite.

1 Main Result

In this article, we consider an abstract interaction system of Bose fields. The Hilbert space for the system is given by

$$\mathcal{H} = \mathcal{K} \otimes \mathcal{F}_b(L^2(\mathbb{R}^d)),$$

where $\mathcal{F}_b(L^2(\mathbb{R}^d))$ denotes the boson Fock space over $L^2(\mathbb{R}^d)$. The free Hamiltonian is given by

$$H_0 = K \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma_b(\omega),$$

where $d\Gamma_b(\omega)$ is the second quantization of $\omega = \omega(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^d$. We assume that K and ω satisfy the condition below, respectively.

(A.1) K is self-adjoint and bounded from below.

(A.2) $\omega \in C(\mathbb{R}^d)$ and $\inf_{\mathbf{k} \in \mathbb{R}^d} \omega(\mathbf{k}) \geq 0$.

We denote the annihilation operator by $a(f)$ and the creation operator by $a^\dagger(f)$. The creation operators and annihilation operators satisfy the canonical commutation relation

$$\begin{aligned} [a(f), a^\dagger(g)] &= (f, g), \\ [a(f), a(g)] &= [a^\dagger(f), a^\dagger(g)] = 0. \end{aligned}$$

The total Hamiltonian is defined by

$$H_\kappa = H_0 + \kappa H_I,$$

where $\kappa \in \mathbb{R}$ is called coupling constant.

We suppose the condition below.

(A.3) H_I is symmetric.

(A.4) For all $\kappa \in \mathbb{R}$, H_κ is self-adjoint.

For a self-adjoint operator H , we denote the lowest spectrum by $E_0(H) = \inf(\sigma(H))$, and we say that H has a ground state if $E_0(H)$ is an eigenvalue of H . We are interested in the multiplicity of the ground state of H_κ . The difficulty of the analysis is that the ground state is an eigenvalue which is embedded in continuous spectrum (refer to e.g., [1, 5]).

Suppose the condition below.

(A.5) K has a compact resolvent.

(A.6) There exists $Q_I : \mathbb{R}^d \rightarrow \mathcal{L}(\mathcal{H})$ such that for all $\Phi, \Psi \in \mathcal{D}(H_0)$,

$$[H_I, \mathbf{1} \otimes a(f)]^0(\Phi, \Psi) = \int_{\mathbb{R}^d} f(\mathbf{k})^*(\Phi, Q_I(\mathbf{k})) d\mathbf{k}.$$

where $[X, Y]^0(\Phi, \Psi)$ is the weak commutator and $a(\mathbf{k})$ is the operator kernel of the annihilation operator. In addition, there exists $\zeta(\kappa) \geq 0$ and $\tau(\kappa) \geq 0$ such that for all $\Phi \in \mathcal{D}(H_0)$,

$$\int_{\mathbb{R}^d} \frac{1}{\omega(\mathbf{k})^2} \|Q_I(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq \zeta(\kappa) \|H_\kappa \Phi\|^2 + \tau(\kappa) \|\Phi\|^2.$$

(A.7) $Q_I(\mathbf{k})$ is strongly differentiable and there exists $\mu_l(\kappa) \geq 0$ and $\nu_l(\kappa) \geq 0$, for each $l = 0, \dots, d$, such that for all $\Phi \in \mathcal{D}(H_0)$,

$$\int_{\mathbb{R}^d} \frac{1}{\omega(\mathbf{k})^4} \|Q_I(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq \mu_0(\kappa) \|H_\kappa \Phi\|^2 + \nu_0(\kappa) \|\Phi\|^2,$$

and

$$\int_{\mathbb{R}^d} \frac{1}{\omega(\mathbf{k})^2} \|\partial_{k_j} Q_I(\mathbf{k})\Phi\|^2 d\mathbf{k} \leq \mu_j(\kappa) \|H_\kappa \Phi\|^2 + \nu_j(\kappa) \|\Phi\|^2.$$

The following is the main result.

Suppose (A.1)-(A.7). Then if H_κ has a ground state, its multiplicity is finite.

[Remark]

If we suppose $|\kappa|$ is sufficiently small, then it follows that $\dim \ker (H_\kappa - E_0(H_\kappa)) \leq \dim \ker (K - E_0(K))$ from the main result in [6].

[Outline of Proof]

The strategy is to use a method considered in [7], which is a combined technics for showing the the existence of the ground state for all valued coupling constants in [3, 4] and it is proven that the multiplicity of the ground state for the quantum electrodynamics is finite for all values of coupling constants. Let us suppose that H_κ has a ground state with $\dim \ker (H_\kappa - E_0(H_\kappa)) = \infty$. Then there exists a ground state $\Omega_{\kappa,j}$, $j \in \mathbb{N}$, such that $H_\kappa \Omega_{\kappa,j} = E_0(H_\kappa) \Omega_{\kappa,j}$ and $\|\Omega_{\kappa,j}\| = 1$. We see that $\text{w-}\lim_{j \rightarrow \infty} \Omega_{\kappa,j} = 0$. On the other hand, using the combined method above, we can prove $\text{w-}\lim_{j \rightarrow \infty} \Omega_{\kappa,j} \neq 0$, and it yields a contradiction. Hence $\dim \ker (H_\kappa - E_0(H_\kappa)) < \infty$.

[Application]

We shortly introduce the generalized the spin-boson model [2]. The Hilbert space for the system is given by $\mathcal{H}_{\text{GSB}} = \mathcal{K} \otimes \mathcal{F}_b(L^2(\mathbb{R}^d))$ and the total Hamiltonian by

$$H_{\text{GSB}}(\kappa) = A \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma_b(\omega) + \kappa \sum_{j=1} B_j \otimes \phi(f_j).$$

The precise conditions are not explained here, but the main result can be applied to this model.

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