

Product Possibility Space with Finitely Many Independent Fuzzy Vectors

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Abstract

This paper discusses about product possibility space and independent fuzzy variables.

Keywords: uncertainty theory, possibility measure, product possibility space, fuzzy variable, fuzzy vector, independence

1 Introduction

Fuzzy set was first introduced by Zadeh [24] in 1965. This notion has been very useful in human decision making under uncertainty. We can see lots of papers which use this fuzzy set theory in K.Iwamura and B.Liu [3] [4], B.Liu and K.Iwamura [18] [19] [20], X.Ji and K.Iwamura [9], X.Gao and K.Iwamura [2], G.Wang and K.Iwamura [22], M.Wen and K.Iwamura [23] and others. We also have some books on fuzzy decision making under fuzzy environments such as D.Dubois and H.Prade [1], H-J.Zimmermann [26], M.Sakawa [27], J.Kacprzyk [10], B.Liu and A.O.Esogbue [17].

Recently B.Liu has founded a frequentist fuzzy theory with huge amount of applications in fuzzy mathematical programming. We see it in books such as B.Liu [12] [16] [13]. In his book [13] published in 2004, B.Liu [13] has succeeded in establishing an axiomatic foundation for uncertainty theory, where they proposed a notion of independent fuzzy variables.

In this paper, we discuss on possibility axioms, construct product possibility space and show that we not only have finitely many independent fuzzy variables[6] but also we have finitely many independent fuzzy vectors in a wide sense.

The rest of the paper is organized as follows. The next section provides a brief review on the results of possibility measure axioms with definitions of fuzzy variables, fuzzy vectors, independence of the two. Section 3 presents how we can get finitely many independent fuzzy vectors under possibility measures.

2 Possibility Measure and Product Possibility Space

We start with the axiomatic definition of possibility measure given by B. Liu [13] in 2004. Let Θ be an arbitrary nonempty set, and let $\mathcal{P}(\Theta)$ be the power set of Θ .

The three axioms for possibility measure are listed as follows:

Axiom 1. $\text{Pos}\{\Theta\} = 1$.

Axiom 2. $\text{Pos}\{\emptyset\} = 0$.

Axiom 3. $\text{Pos}\{\cup_i A_i\} = \sup_i \text{Pos}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$.

We call Pos a possibility measure over Θ if it satisfies these three axioms. We call the triplet $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ a possibility space.

Theorem 2.1 (B. Liu, 2004). Let Θ_i be nonempty sets on which $\text{Pos}_i\{\cdot\}$ satisfy the three axioms, $i = 1, 2, \dots, n$, respectively, and let $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n$. Define $\text{Pos}\{\cdot\}$ by

$$\text{Pos}\{A\} = \sup_{(\theta_1, \theta_2, \dots, \theta_n) \in A} \text{Pos}_1\{\theta_1\} \wedge \text{Pos}_2\{\theta_2\} \wedge \dots \wedge \text{Pos}_n\{\theta_n\} \quad (1)$$

for each $A \in \mathcal{P}(\Theta)$. Then $\text{Pos}\{\cdot\}$ satisfies the three axioms for possibility measure.

Therefore, a newly defined triplet $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ is a possibility space. $\text{Pos}\{\cdot\}$ is a possibility measure on $\mathcal{P}(\Theta)$. We call it product possibility measure derived from $(\Theta_i, \text{Pos}_i\{\cdot\}, \mathcal{P}(\Theta_i))$, $i = 1, \dots, n$. We call $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ the product possibility space derived from $(\Theta_i, \text{Pos}_i\{\cdot\}, \mathcal{P}(\Theta_i))$, $i = 1, \dots, n$.

Lemma 2.1 Let $0 \leq a_i \leq 1$ and let $\epsilon > 0$, for $1 \leq i \leq n$. Then we get

$$(\epsilon + a_1) \wedge (\epsilon + a_2) \wedge \dots \wedge (\epsilon + a_n) \leq \epsilon + a_1 \wedge a_2 \wedge \dots \wedge a_n \quad (2)$$

Lemma 2.2 Let $A_i \in \mathcal{P}(\Theta_i)$ for $1 \leq i \leq n$. Then we get $A_1 \times \dots \times A_n \in \mathcal{P}(\Theta)$ and

$$\text{Pos}\{A_1 \times \dots \times A_n\} = \text{Pos}_1\{A_1\} \wedge \text{Pos}_2\{A_2\} \wedge \dots \wedge \text{Pos}_n\{A_n\}. \quad (3)$$

Lemma 2.3 $\text{Pos}\{\cdot\}$ on $\mathcal{P}(\Theta)$ at (1) is monotone with respect to set inclusion, i.e., for any sets $A, B \in \mathcal{P}(\Theta)$ with $A \subset B$, we get

$$\text{Pos}\{A\} \leq \text{Pos}\{B\}. \quad (4)$$

A fuzzy variable is defined as a function from Θ of a possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ to the set of reals $\mathcal{R}$. An n -dimensional fuzzy vector is defined as a function from Θ of a possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ to an n -dimensional Euclidian space $\mathcal{R}^n$. Let ξ be a fuzzy variable defined on the possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$.

Then its membership function is derived through the possibility measure Pos by

$$\mu(x) = \text{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\}, \quad x \in \mathcal{R}. \quad (5)$$

Theorem 2.2 (B. Liu, 2004). Let $\mu : \mathcal{R} \rightarrow [0, 1]$ be a function with $\sup \mu(x) = 1$. Then there is a fuzzy variable whose membership function is μ .

The fuzzy variables $\xi_1, \xi_2, \dots, \xi_m$ are said to be independent if and only if

$$\text{Pos}\{\xi_i \in B_i, i = 1, 2, \dots, m\} = \min_{1 \leq i \leq m} \text{Pos}\{\xi_i \in B_i\}$$

for any subsets $B_1, B_2, \dots, B_m$ of the set of reals $\mathcal{R}$. The fuzzy vectors $\xi_i (1 \leq i \leq m)$ are said to be independent if and only if

$$\text{Pos}\{\xi_i \in B_i, i = 1, 2, \dots, m\} = \min_{1 \leq i \leq m} \text{Pos}\{\xi_i \in B_i\}$$

for any subsets $B_i \in \mathcal{R}^n (1 \leq i \leq m)$. Here after we use $a \wedge b$ in place of $\min\{a, b\}$.

Note 2.1 : We have fuzzy variables ξ_1, ξ_2 which are not independent (Liu[13]). Let $\Theta = \{\theta_1, \theta_2\}$, $\text{Pos}\{\theta_1\} = 1$, $\text{Pos}\{\theta_2\} = 0.8$ and define ξ_1, ξ_2 by

$$\xi_1(\theta) = \begin{cases} 0, & \text{if } \theta = \theta_1 \\ 1, & \text{if } \theta = \theta_2, \end{cases} \quad \xi_2(\theta) = \begin{cases} 1, & \text{if } \theta = \theta_1 \\ 0, & \text{if } \theta = \theta_2 \end{cases}$$

Then we have $\text{Pos}\{\xi_1 = 1, \xi_2 = 1\} = \text{Pos}\{\emptyset\} = 0 \neq 0.8 \wedge 1 = \text{Pos}\{\xi_1 = 1\} \wedge \text{Pos}\{\xi_2 = 1\}$.

3 Finitely Many Independent Fuzzy Vectors

Let ξ_i be a fuzzy vector from a possibility space $(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i)$ to the l_i -th dimensional Euclidian space $\mathcal{R}^{l_i}$, for $i = 1, 2, \dots, n$. Define Θ by

$$\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n \quad (6)$$

and $\tilde{\xi}_i$ on Θ by

$$\tilde{\xi}_i(\theta) = \xi_i(\theta_i) \quad \text{for any } \theta = (\theta_1, \theta_2, \dots, \theta_n) \in \Theta, 1 \leq i \leq n. \quad (7)$$

For any subset B_i of $\mathcal{R}^{l_i}$, we get ([8])

Theorem 3.1 The fuzzy vectors $\tilde{\xi}_i (1 \leq i \leq n)$ given above are independent fuzzy vectors, i.e.,

$$\text{Pos}\{\tilde{\xi}_i(\theta) \in B_i (1 \leq i \leq n)\} = \text{Pos}\{\tilde{\xi}_1 \in B_1\} \wedge \text{Pos}\{\tilde{\xi}_2 \in B_2\} \wedge \cdots \wedge \text{Pos}\{\tilde{\xi}_n \in B_n\} \quad (8)$$

4 Conclusion

We have shown that Axiom. 4 in B.Liu([13]) can be proved through Axiom. 1, 2 and 3. We have proved the fact that there exists finitely many independent fuzzy vectors. Through these proofs we have shown that for possibility measures existence of finitely many independent fuzzy vectors depends on product possibility space. Although the notion of independence was discovered by L.A.Zadeh and others([25]) under the term of "noninteractiveness" or "unrelatedness", our notion of independence through product possibility space have brought about a grand world of fuzzy process, hybrid process and uncertain process of B.Liu([15]).

References

- [1] Dubois, D., and Prade, H., *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.
- [2] X.Gao and K.Iwamura, Fuzzy Multi-Criteria Minimum Spanning Tree Problem, *handwriting*, 2005.
- [3] K.Iwamura and B.Liu, A genetic algorithm for chance constrained programming, *Journal of Information & Optimization Sciences*, Vol. 17, 409-422, 1996.
- [4] K.Iwamura and B.Liu, Dependent-chance integer programming applied to capital budgeting, *Journal of the Operations Research Society of Japan*, Vol. 42, 117-127, 1999.
- [5] K.Iwamura and M.Horiike, λ Credibility, *Proceedings of the Fifth International Conference on Information and Management Sciences*, Chengdu, China, July 1-8, 2006, 310-315.
- [6] K.Iwamura and M.Wen, Creating Finitely Many Independent Fuzzy Variables, *handwriting*, 2005.
- [7] K.Iwamura and M.Kageyama, From finitely many independent fuzzy sets to possibility-based fuzzy linear programming problems, *Journal Interdisciplinary Mathematics*, Vol. 10, 757-766, 2007.
- [8] K.Iwamura, M.Kageyama, M.Horiike and S.Kitakubo, Existence proof of finitely many independent fuzzy vectors, submitted to *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*.
- [9] X.Ji and K.Iwamura, New-Models for Shortest Path Problem, 2005.
- [10] J.Kacprzyk, *Multistage Fuzzy Control*, John Wiley & Sons, 1997.

- [11] R.Liang , and J.Gao, Dependent-Chance Programming Models for Capital Budgeting in Fuzzy Environments, to appear in *Tsinghua Science and Technology*, Beijing.
- [12] B.Liu, *Uncertain Programming*, Wiley, New York, 1999.
- [13] B.Liu, *Uncertainty Theory: An Introduction to its Axiomatic Foundations*, Springer-Verlag, Berlin, 2004.
- [14] B.Liu, *Uncertainty Theory Second Edition*, Springer-Verlag, Berlin, 2007.
- [15] B.Liu, Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, Vol. 2, 3-16, 2008.
- [16] B.Liu, *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.
- [17] B.Liu, and A.O. Esogbue, *Decision Criteria and Optimal Inventory Processes*, Kluwer, Boston, 1999.
- [18] B.Liu, and K.Iwamura, Chance constrained programming with fuzzy parameters, *Fuzzy Sets and Systems*, Vol. 94, No. 2, 227-237, 1998.
- [19] B.Liu, and K.Iwamura, A note on chance constrained programming with fuzzy coefficients, *Fuzzy Sets and Systems*, Vol. 100, Nos. 1-3, 229-233, 1998.
- [20] B.Liu, and K.Iwamura, Fuzzy programming with fuzzy decisions and fuzzy simulation-based genetic algorithm, *Fuzzy Sets and Systems*, Vol. 122, No. 2, 253-262, 2001.
- [21] B.Liu, and Y.-K.Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, Vol. 10, No. 4, 445-450, 2002.
- [22] G.Wang and K.Iwamura, Yield Management with Random Fuzzy Demand, *Asian Information-Science-Life*, Vol. 1, No. 3, 279-284, 2002.
- [23] M.Wen and K.Iwamura, Fuzzy Facility Location-Allocation Problem under the Hurwicz Criterion, to appear in *European Journal of Operational Research*.
- [24] L.A.Zadeh, Fuzzy sets, *Information and Control*, Vol. 8, 338-353, 1965.
- [25] L.A.Zadeh, Fuzzy Sets as a Basis for a Theory of Possibility, *Fuzzy Sets and Systems*, Vol.1, 3-28, 1978.
- [26] H.-J.Zimmermann, *Fuzzy Set Theory and its Applications*, Kluwer Academic Publishers, 1991.
- [27] M.Sakawa, *Fuzzy Sets and Interactive Multiobjective Optimization*, Plenum Press, 1993
- [28] T.Murofushi and M.Sugeno, An Interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, *Fuzzy sets and Systems*, Vol. 29, 201-227, 1989.