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Boundary singularity for thermal transpiration problem of the linearized Boltzmann equation

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Abstract We study the boundary singularity of the fluid velocity for the thermal transpiration problem in the kinetic theory. Logarithmic singularity has been demonstrated through the asymptotic and computational analysis. The goal of this paper is to confirm this logarithmic singularity through exact analysis. We use an iterated scheme, with the "gain" part of the collision operator as a source. The iterated scheme is appropriate for large Knudsen number considered here and yields an explicit leading term.

Keywords Boltzmann equation · Boundary singularity · Thermal transpiration · Maxwellian bound

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1 Introduction

Consider the stationary linearized Boltzmann equation

$$\sum_{i=1}^3 \zeta_i \partial_{x_i} f = \frac{1}{\kappa} L(f), \quad (1)$$

where κ is the Knudsen number and L is the linearized collision operator. Here the linearization is around the normalized Maxwellian E :

$$E(\zeta) = \pi^{-\frac{3}{2}} e^{-|\zeta|^2}. \quad (2)$$

For the hard sphere model, the collision operator is the sum of an integral operator K with kernel $k(\zeta, \zeta_*)$ and a multiplicative operator v :

$$\begin{cases} L(\phi) = -v(|\zeta|)\phi + K(\phi), \\ K(\phi)(\zeta) = \int k(\zeta, \zeta_*) \phi(\zeta_*) d\zeta_*, \\ v(\eta) = 2^{-3/2} [e^{-\eta^2} + (2\eta + \frac{1}{\eta}) \int_0^\eta e^{-s^2} ds], \\ k(\zeta, \zeta_*) = k_1(\zeta, \zeta_*) + k_2(\zeta, \zeta_*), \\ k_1(\zeta, \zeta_*) = 2^{-\frac{1}{2}} \pi^{-1} |\zeta - \zeta_*|^{-1} \exp\left(\frac{|\zeta \times \zeta_*|^2}{|\zeta - \zeta_*|^2} - \frac{|\zeta|^2}{2} - \frac{|\zeta_*|^2}{2}\right) \\ \quad = 2^{-\frac{1}{2}} \pi^{-1} |\zeta - \zeta_*|^{-1} \exp\left(-\frac{1}{4} \frac{(|\zeta|^2 - |\zeta_*|^2)^2}{|\zeta - \zeta_*|^2} - \frac{1}{4} |\zeta - \zeta_*|^2\right), \\ k_2(\zeta, \zeta_*) = -2^{-\frac{3}{2}} \pi^{-1} |\zeta - \zeta_*| \exp\left(-\frac{|\zeta|^2}{2} - \frac{|\zeta_*|^2}{2}\right). \end{cases} \quad (3)$$

For $v(|\zeta|)$, it is clear that there exist positive constants $v_1 > v_0 > 0$ such that

$$v_0(1 + |\zeta|) < v(|\zeta|) < v_1(1 + |\zeta|). \quad (4)$$

The kernel of the collision operator L , the collision invariants, is spanned by the following five orthonormal functions

$$\Phi_0(\zeta) = E^{\frac{1}{2}}(\zeta), \quad \Phi_j(\zeta) = \zeta_j E^{\frac{1}{2}}(\zeta), \quad \Phi_4(\zeta) = (|\zeta|^2 - \frac{3}{2}) E^{\frac{1}{2}}(\zeta), \quad j = 1, 2, 3. \quad (5)$$

The macroscopic variables, the perturbation $\sigma(x)$ from the reference normalized density, the fluid velocity $q(x) = (q_1, q_2, q_3)$, and the perturbation $\tau(x)$ from the reference normalized temperature, are given as inner products with the collision invariants:

$$\sigma(x) = \langle f, \Phi_0 \rangle, \quad q_i(x) = \langle f, \Phi_i \rangle, \quad \tau(x) = \frac{2}{3} \langle f, \Phi_4 \rangle, \quad (6)$$

where the inner product is with respect to the microscopic variable ζ :

$$\langle f, h \rangle := \int_{\mathbb{R}^3} f(\zeta) h(\zeta) d\zeta.$$

We consider a gas between two parallel plates:

$$\Omega = \{(x_1, x_2, x_3) | -\frac{1}{2} < x_1 < \frac{1}{2}\}. \quad (7)$$

The perturbation τ_w of the wall temperature from unity has a positive constant gradient in the x_2 direction

$$\tau_w(\frac{1}{2}, x_2, x_3) = \tau_w(-\frac{1}{2}, x_2, x_3) = \varepsilon x_2, \quad \varepsilon > 0. \quad (8)$$

The thermal transpiration phenomenon is that the boundary temperature gradient on $\partial\Omega$, without the fluid pressure, can induce a gas flow in Ω . This phenomenon can be understood only with the physically realistic modeling of the boundary condition using the kinetic theory. We consider the diffuse boundary condition, which prescribes that the distribution leaving the boundary is in thermal equilibrium with the boundary and that there is no mass flux across the boundary, [7],

$$\begin{aligned} f(\pm\frac{1}{2}, x_2, x_3, \zeta) &= \sigma_{\pm}\Phi_0 + \tau_w\Phi_4, \text{ for } \zeta_1 \leq 0, \\ \sigma_{\pm} &= -\frac{1}{2}\tau_w + 2\sqrt{\pi} \int_{\zeta_1 \geq 0} |\zeta_1| E^{\frac{1}{2}}(\zeta) f(\pm\frac{1}{2}, x_2, x_3, \zeta) d\zeta. \end{aligned} \quad (9)$$

The linearity of the equation (1) and the boundary data (8) allow us to consider solutions in the following form, [5], [1]:

$$f(x, \zeta) = \varepsilon \left(\tilde{\phi}(x_1, \zeta) + x_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}}(\zeta) \right). \quad (10)$$

The second term on the right hand side of (10) absorbs τ_w occurring in the boundary condition (9), and consequently the symmetric part

$$\phi := \frac{1}{2} \left(\tilde{\phi}(x_1, \zeta_1, \zeta_2, \zeta_3) + \tilde{\phi}(-x_1, -\zeta_1, \zeta_2, \zeta_3) - (\sigma_+ + \sigma_-)\Phi_0 \right)$$

satisfies the zero boundary condition:

$$\begin{cases} \zeta_1 \partial_{x_1} \phi(x_1, \zeta) = \frac{1}{\kappa} L(\phi) - \zeta_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}}(\zeta), & \text{when } -\frac{1}{2} < x_1 < \frac{1}{2}, \\ \phi(\frac{1}{2}, \zeta) = 0, & \text{for } \zeta_1 < 0, \\ \phi(-\frac{1}{2}, \zeta) = 0, & \text{for } \zeta_1 > 0. \end{cases} \quad (11)$$

From $L = -v + K$, (3), we can rewrite (11) as an integral equation:

$$\phi(x_1, \zeta) = \begin{cases} \int_{-\frac{1}{2}}^{x_1} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|x_1-s|} \left(\frac{1}{\kappa} K(\phi)(\zeta) - \zeta_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}}(\zeta) \right) ds, & \text{for } \zeta_1 > 0, \\ \int_{x_1}^{\frac{1}{2}} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|x_1-s|} \left(\frac{1}{\kappa} K(\phi)(\zeta) - \zeta_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}}(\zeta) \right) ds, & \text{for } \zeta_1 < 0. \end{cases} \quad (12)$$

In [1], it is proved that there is a thermal transpiration flow, i.e. there exist positive constants C_1 and C_2 such that, for κ sufficiently large,

$$C_1 \ln \kappa < \frac{1}{\varepsilon} q_2 < C_2 \ln \kappa. \quad (13)$$

Our analysis is based on an iteration scheme, (50), (20), with the leading term

$$\phi_0(x_1, \zeta) = \begin{cases} -\int_{-\frac{1}{2}}^{x_1} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|x_1-s|} \zeta_2(|\zeta|^2 - \frac{5}{2}) E^{\frac{1}{2}}(\zeta) ds, & \text{for } \zeta_1 > 0, \\ -\int_{x_1}^{\frac{1}{2}} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|x_1-s|} \zeta_2(|\zeta|^2 - \frac{5}{2}) E^{\frac{1}{2}}(\zeta) ds, & \text{for } \zeta_1 < 0. \end{cases} \quad (14)$$

Takata and Funagane, [9], use similar iteration scheme for the design of a numerical scheme. Their numerical experiments gives strong evidence of the optimal convergent rate of $C \frac{\ln \kappa}{\kappa}$ for their numerical scheme. Their high quality numerics on the boundary behavior of the distribution function prompts the present analytical study of the boundary singularity. We first improve on the estimate (13) with the explicit leading term of the fluid velocity in the direction of the plate.

Theorem 1 *For the solution of (12), the gas flow velocity $q_2(x_1)$, (6), satisfies, for κ sufficiently large,*

$$\begin{aligned} \frac{1}{\varepsilon} q_2(x_1) &= \frac{1}{4\sqrt{\pi}} \left(\ln \kappa + \frac{1}{2} - \gamma - C_T - |x_1 + \frac{1}{2}| \ln |x_1 + \frac{1}{2}| - |x_1 - \frac{1}{2}| \ln |x_1 - \frac{1}{2}| \right) \\ &\quad + O\left(\frac{\ln^2 \kappa}{\kappa}\right), \end{aligned} \quad (15)$$

where γ is the Euler constant and

$$C_T := 4 \int_0^\infty \rho^3 \left(\frac{5}{2} - \rho^2 \right) e^{-\rho^2} \ln \frac{v(\rho)}{\rho} d\rho. \quad (16)$$

The derivative of the fluid velocity generated by the leading term ϕ_0 , (14),

$$\partial_{x_1} q_2^0(x_1) = \partial_{x_1} \varepsilon \int \zeta_2 E^{\frac{1}{2}} \phi_0(x_1, \zeta) d\zeta$$

is singular near the plate $x_1 = \pm \frac{1}{2}$. We will show that, for large Knudsen number, this singularity dominates those induced by the remaining terms in the iterations and we have:

Theorem 2 *For large Knudsen number, the derivative of the flow velocity q_2 tends to infinity logarithmically near the boundary:*

$$\frac{1}{\varepsilon} \partial_{x_1} q_2 = \left(\frac{1}{4\sqrt{\pi}} + O\left(\frac{\ln \kappa}{\kappa}\right) \right) \left(\ln \left| \frac{1}{2} - x_1 \right| - \ln \left| \frac{1}{2} + x_1 \right| \right) + O\left(\frac{(\ln \kappa)^2}{\kappa}\right). \quad (17)$$

The logarithmic singularity near the boundary has been derived through the asymptotic methods for Boltzmann-Krook-Welande (BKW) models. The singular boundary behavior is one of the main features of the kinetic theory, see [7] and references therein. It would be interesting to establish this generic singular behavior for general boundary layers. This is, however, left for the future. In Section 2 and Section 3, we establish two key lemmas concerning the operators contained in the integral equation (12). This allows us to analyze the solution in the subsequent sections. Numerical demonstration of the singular behavior of the logarithmic type will be given in the last section.

The study of boundary singularity requires quantitative, pointwise analysis. There is another pointwise analysis making use of the Green's function, e.g. [3]. The present paper makes use of the deep understanding, through asymptotic analysis, of the boundary singularity by the Kyoto School.

2 Pointwise estimate

Pointwise estimate of the solution is needed for our analysis. First we rewrite the integral equation (12) as:

$$\phi = H\left(\frac{1}{\kappa}K(\phi) + h(\zeta)\right), \quad (18)$$

where

$$h(\zeta) = \pi^{-3/4} \zeta_2 \left(\frac{5}{2} - |\zeta|^2\right) e^{-\frac{1}{2}|\zeta|^2}; \quad (19)$$

$$H(\phi)(x_1, \zeta) := \begin{cases} \int_{-\frac{1}{2}}^{x_1} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|s-x_1|} \phi(s, \zeta) ds, & \zeta_1 > 0, \\ \int_{x_1}^{\frac{1}{2}} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|s-x_1|} \phi(s, \zeta) ds, & \zeta_1 < 0. \end{cases} \quad (20)$$

Note that in (20) the integration is only with respect to s and so

$$|H(\phi)| \leq g(x_1, \zeta) \|\phi(\cdot, \zeta)\|_{L_{x_1}^\infty}, \quad (21)$$

where

$$g(x_1, \zeta) := \begin{cases} \int_{-\frac{1}{2}}^{x_1} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|s-x_1|} ds, & \text{for } \zeta_1 > 0, \\ \int_{x_1}^{\frac{1}{2}} \frac{1}{|\zeta_1|} e^{-\frac{v(|\zeta|)}{\kappa|\zeta_1|}|s-x_1|} ds, & \text{for } \zeta_1 < 0. \end{cases} \quad (22)$$

Lemma 1 *The function $g(x_1, \zeta)$ of (22) has the following bounds in terms of the spatial location x and the microscopic velocity ζ_1 along the plates:*

$$g(x_1, \zeta) \leq \begin{cases} \frac{x_1 + \frac{1}{2}}{|\zeta_1|}, & \text{for } \zeta_1 \geq \frac{v(|\zeta|)|x_1 + \frac{1}{2}|}{\kappa}, \\ \frac{\kappa}{v(|\zeta|)}, & \text{for } -\frac{v(|\zeta|)|x_1 - \frac{1}{2}|}{\kappa} < \zeta_1 < \frac{v(|\zeta|)|x_1 + \frac{1}{2}|}{\kappa}, \\ \frac{|x_1 - \frac{1}{2}|}{|\zeta_1|}, & \text{for } \zeta_1 \leq -\frac{v(|\zeta|)|x_1 - \frac{1}{2}|}{\kappa}. \end{cases} \quad (23)$$

Thus, there exists a positive constant C such that

$$|g(x_1, \zeta)| \leq C(\kappa^{-1} + |\zeta_1|)^{-1}. \quad (24)$$

Proof We only consider the case $\zeta_1 > 0$; the case $\zeta_1 < 0$ is similar. We can either take the supremum of the integrand over s to obtain

$$g(x_1, \zeta) \leq \frac{x_1 + \frac{1}{2}}{|\zeta_1|}, \quad (25)$$

or integrate directly to obtain

$$g(x_1, \zeta) = \frac{\kappa}{v(|\zeta|)} \left(1 - e^{-\frac{v(|\zeta|)|x_1 + \frac{1}{2}|}{\kappa|\zeta_1|}}\right) \leq \frac{\kappa}{v(|\zeta|)}. \quad (26)$$

(23) follows from the above two estimates, while (24) follows from (26) because of (4). $\square$

Lemma 1 and (21) yield the following coarser but simpler in form estimate, which we will use later.

Corollary 1

$$|H(\phi)| \leq C(\kappa^{-1} + |\zeta_1|)^{-1} \|\phi(\cdot, \zeta)\|_{L_{x_1}^\infty}. \quad (27)$$

3 Maxwellian Bounds

We study the properties of the operator K , (3).

Lemma 2 (Maxwellian Bound)

(i) Suppose that a function ϕ satisfies the following Gaussian estimate for some constant a , $0 \leq a \leq \frac{1}{2}$,

$$|\phi(\zeta)| \leq e^{-a|\zeta|^2}. \quad (28)$$

Then there exist positive constants C_a and C such that, for $0 \leq a < \frac{1}{2}$,

$$|K(\phi)(\zeta)| \leq C_a e^{-a|\zeta|^2}, \quad (29)$$

where C_a is a constant depending on a :

$$C_a = C \left(1 + |a - \frac{1}{2}|^{-1}\right); \quad (30)$$

and, for $a = \frac{1}{2}$,

$$|K(\phi)(\zeta)| \leq C(1 + |\zeta|)e^{-\frac{1}{2}|\zeta|^2}. \quad (31)$$

(ii) Suppose that a function ϕ satisfies

$$|\phi(\zeta)| \leq (\kappa^{-1} + |\zeta_1|)^{-1} e^{-a|\zeta|^2}, \text{ for } 0 \leq a \leq \frac{1}{2}. \quad (32)$$

Then, for $\kappa > 2$,

$$\begin{aligned} |K(\phi)(\zeta)| &\leq C_a \ln(\kappa) e^{-a|\zeta|^2}, & \text{for } 0 \leq a < \frac{1}{2}, \\ |K(\phi)(\zeta)| &\leq C \ln(\kappa) (1 + |\zeta|) e^{-\frac{1}{2}|\zeta|^2}, & \text{for } a = \frac{1}{2}. \end{aligned} \quad (33)$$

Proof

(i) Consider first the $k_1(\zeta, \zeta_*)$ part of K :

$$K_1(\phi) := \int k_1(\zeta, \zeta_*) \phi(\zeta_*) d\zeta_*.$$

From the form of $k_1(\zeta, \zeta_*)$, (3), and the assumption (28) for ϕ ,

$$|K_1(\phi)(\zeta)| \leq \int \frac{c}{|\zeta - \zeta_*|} e^{\frac{|\zeta \times \zeta_*|^2}{|\zeta - \zeta_*|^2} - \frac{|\zeta|^2}{2} - (a + \frac{1}{2})|\zeta_*|^2} d\zeta_*. \quad (34)$$

We choose an orthogonal basis $\{e_1, e_2, e_3\}$ centered at ζ with $e_3 = -\frac{\zeta}{|\zeta|}$ as the north pole and (ρ, θ, φ) its spherical coordinates:

$$\zeta_* = \zeta + \rho \cos \theta e_3 + \rho \sin \theta \cos \varphi e_1 + \rho \sin \theta \sin \varphi e_2. \quad (35)$$

Then we have the expressions

$$|\zeta - \zeta_*| = \rho, \quad |\zeta \times \zeta_*| = \rho |\zeta| \sin \theta, \quad |\zeta_*|^2 = |\zeta|^2 + \rho^2 - 2|\zeta|\rho \cos \theta. \quad (36)$$

From (36), the inequality (34) yields

$$\begin{aligned}
& |K_1(\phi)(\zeta)| \\
& \leq \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{c}{\rho} e^{|\zeta|^2 \sin^2 \theta - \frac{|\zeta|^2}{2} - (a+\frac{1}{2})(|\zeta|^2 + \rho^2 - 2|\zeta|\rho \cos \theta)} \rho^2 \sin \theta d\varphi d\theta d\rho \\
& \leq 2\pi c e^{-a|\zeta|^2} \int_0^\infty \int_0^\pi e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2 + (a-\frac{1}{2})|\zeta|^2 \cos^2 \theta} \rho \sin \theta d\theta d\rho.
\end{aligned}$$

The last integral is dominated by the following two terms

$$\begin{aligned}
I + II &:= \int_0^\infty \int_0^\pi e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2 + (a-\frac{1}{2})|\zeta|^2 \cos^2 \theta} |\rho - |\zeta| \cos \theta| \sin \theta d\theta d\rho \\
&+ \int_0^\infty \int_0^\pi e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2 + (a-\frac{1}{2})|\zeta|^2 \cos^2 \theta} |\zeta| |\cos \theta| \sin \theta d\theta d\rho.
\end{aligned}$$

Clearly,

$$|I| < \int_0^\infty \int_0^\pi e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2} |\rho - |\zeta| \cos \theta| \sin \theta d\theta d\rho \leq C. \quad (37)$$

For the second term II , by changing the order of integrations, we obtain

$$|II| \leq C \int_0^\pi |\zeta| |\cos \theta| e^{(a-\frac{1}{2})|\zeta|^2 \cos^2 \theta} \sin \theta d\theta. \quad (38)$$

Clearly, for the case $a \leq \frac{1}{2}$, we have $|II| \leq C|\zeta|$, which yields, in particular, the K_1 part of the estimate (31). To complete the proof of (i) for K_1 part, it remains to consider the case $a < \frac{1}{2}$ for $|\zeta| > 1$. For this, we change the variable $\beta = \sqrt{\frac{1}{2} - a} |\zeta| \cos \theta$ in the integral in (38) to obtain

$$|II| \leq \frac{C}{|\frac{1}{2} - a| |\zeta|} \int_{-\sqrt{\frac{1}{2}-a}|\zeta|}^{\sqrt{\frac{1}{2}-a}|\zeta|} |\beta| e^{-\beta^2} d\beta \leq C \left| \frac{1}{2} - a \right|^{-1}. \quad (39)$$

This proves (i) for the K_1 part.

Next, consider the other part $k_2(\zeta, \zeta_*)$ of K :

$$K_2(\phi) = \int k_2(\zeta, \zeta_*) \phi(\zeta_*) d\zeta_*.$$

Again from (3) and the assumption (28) for ϕ , we have

$$\begin{aligned}
|K_2(\phi)(\zeta)| &\leq c e^{-\frac{1}{2}|\zeta|^2} \int |\zeta - \zeta_*| e^{-(a+\frac{1}{2})|\zeta_*|^2} d\zeta_* \\
&\leq c e^{-\frac{1}{2}|\zeta|^2} |\zeta| \int e^{-(a+\frac{1}{2})|\zeta_*|^2} d\zeta_* + c e^{-\frac{1}{2}|\zeta|^2} \int |\zeta_*| e^{-(a+\frac{1}{2})|\zeta_*|^2} d\zeta_* \\
&\leq C(1 + |\zeta|) e^{-\frac{1}{2}|\zeta|^2}.
\end{aligned} \quad (40)$$

For $0 \leq a < \frac{1}{2}$, we have

$$|\zeta| e^{-\frac{1}{2}|\zeta|^2} \leq C \left| \frac{1}{2} - a \right|^{-1} e^{-a|\zeta|^2}. \quad (41)$$

This completes the proof for (i).

(ii) For the proof of (ii), we need to treat the cases $|\zeta_{*1}|$ small and $|\zeta_{*1}|$ large separately. We define a bump function ψ with the properties: $\psi(s) \in C^\infty(\mathbb{R})$, $0 \leq \psi(s) \leq 1$, $|\psi'(s)| \leq 2$, and

$$\psi(s) = \begin{cases} 1 & \text{when } |s| \leq 1, \\ 0 & \text{when } |s| > 2. \end{cases} \quad (42)$$

Here we consider the K_1 part only, because the same analysis applies to the K_2 part. We have

$$\begin{aligned} |K_1(\phi)(\zeta)| &\leq \int k_1(\zeta, \zeta_*) (1 - \psi(\zeta_{*1})) (\kappa^{-1} + |\zeta_{*1}|)^{-1} e^{-a|\zeta_*|^2} d\zeta_* \\ &\quad + \int k_1(\zeta, \zeta_*) \psi(\zeta_{*1}) (\kappa^{-1} + |\zeta_{*1}|)^{-1} e^{-a|\zeta_*|^2} d\zeta_* \\ &=: A + B. \end{aligned}$$

For A , since $|(1 - \psi(\zeta_{*1}))(\kappa^{-1} + |\zeta_{*1}|)^{-1}| < 1$, the case reduces to the one in (i). For B , although the integrand is not differentiable at $\zeta_{*1} = 0$ and is singular at $\zeta_* = \zeta$, we can perform the integration by parts on upper half and lower half spaces in the sense of improper integral to obtain

$$B = - \int (\ln(\kappa^{-1} + |\zeta_{*1}|) + \ln \kappa) \text{sign}(\zeta_{*1}) \partial_{\zeta_{*1}} [k_1(\zeta, \zeta_*) e^{-a|\zeta_*|^2} \psi(\zeta_{*1})] d\zeta_*.$$

Here the constant $\ln \kappa$ is chosen so as to eliminate the boundary terms in the process of the integration by parts. Since ψ is supported on $[-2, 2]$, $\kappa > 2$, and $|\ln(\kappa^{-1} + |\zeta_{*1}|)| \leq 2 \ln \kappa$, we have

$$|B| \leq 3 |\ln \kappa| \int |\partial_{\zeta_{*1}} (k_1(\zeta, \zeta_*) e^{-a|\zeta_*|^2} \psi(\zeta_{*1}))| d\zeta_*. \quad (43)$$

When the derivative hits ψ , we use $|\psi'(\zeta_{*1})| < 2$ and reduce the situation as in (i) to obtain the estimates. Thus, the remaining task is to estimate the following integral B' that is generated when the derivative hits $k_1(\zeta, \zeta_*) e^{-a|\zeta_*|^2}$:

$$\begin{aligned} B' &= \int |\psi(\zeta_{*1}) \partial_{\zeta_{*1}} (k_1(\zeta, \zeta_*) e^{-a|\zeta_*|^2})| d\zeta_* \\ &= \int |\psi(\zeta_{*1}) (\frac{\partial \rho}{\partial \zeta_{*1}} \frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial \zeta_{*1}} \frac{\partial}{\partial \theta}) F(\rho, \theta, |\zeta|)| d\zeta_*, \end{aligned}$$

where

$$F(\rho, \theta, |\zeta|) = \frac{c}{\rho} e^{-a|\zeta|^2} e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2 + (a-\frac{1}{2})(|\zeta|^2 \cos^2 \theta)}, \quad (44)$$

and ρ and θ are the ones introduced in (35). Direct calculations yield

$$\begin{aligned} \frac{\partial}{\partial \rho} F &= -(\frac{1}{\rho} + 2(a + \frac{1}{2})(\rho - |\zeta| \cos \theta)) F, \\ \frac{\partial}{\partial \theta} F &= -(2(a + \frac{1}{2})(\rho - |\zeta| \cos \theta) |\zeta| \sin \theta + 2(a - \frac{1}{2}) |\zeta|^2 \cos \theta \sin \theta) F, \\ \frac{\partial \rho}{\partial \zeta_{*1}} &= a_1 \sin \theta \cos \varphi + a_2 \sin \theta \sin \varphi + a_3 \cos \theta, \\ \frac{\partial \theta}{\partial \zeta_{*1}} &= a_1 \frac{\cos \theta \cos \varphi}{\rho} + a_2 \frac{\cos \theta \sin \varphi}{\rho} - a_3 \frac{\sin \theta}{\rho}, \end{aligned} \quad (45)$$

where a_1, a_2, a_3 are the components of the unit vector $(1, 0, 0)$ in the orthonormal basis $\{e_1, e_2, e_3\}$, namely $(1, 0, 0) = a_1 e_1 + a_2 e_2 + a_3 e_3$ and $(a_1)^2 + (a_2)^2 + (a_3)^2 = 1$. Thus from (45), we have

$$\begin{aligned} B' &\leq C e^{-a|\zeta|^2} \int_0^\infty \int_0^\pi \left((a + \frac{1}{2})(\rho - |\zeta| \cos \theta)^2 + (a + \frac{1}{2})|\zeta| |(\rho - |\zeta| \cos \theta)| \right. \\ &\quad \left. + 1 + |a - \frac{1}{2}| |\zeta|^2 |\cos \theta| \right) \cdot e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2 + (a-\frac{1}{2})|\zeta|^2 \cos^2 \theta} \sin \theta d\theta d\rho \\ &=: C e^{-a|\zeta|^2} (B'_1 + B'_2 + B'_3 + B'_4). \end{aligned}$$

For the first two terms $B'_1 + B'_2$, we change the order of integration to obtain

$$\begin{aligned} B'_1 + B'_2 &= \int_0^\pi \left(\int_0^\infty (a + \frac{1}{2}) \left(|\zeta| |(\rho - |\zeta| \cos \theta)| + (\rho - |\zeta| \cos \theta)^2 \right) \right. \\ &\quad \left. \cdot e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2} d\rho \right) e^{-|a-\frac{1}{2}||\zeta|^2 \cos^2 \theta} \sin \theta d\theta \\ &\leq (|\zeta| + C) \int_0^\pi e^{-|a-\frac{1}{2}||\zeta|^2 \cos^2 \theta} \sin \theta d\theta \\ &\leq C + \int_0^\pi |\zeta| e^{-|a-\frac{1}{2}||\zeta|^2 \cos^2 \theta} \sin \theta d\theta. \end{aligned}$$

In the case $a = \frac{1}{2}$, this yields the desired estimate $B'_1 + B'_2 \leq C(1 + |\zeta|)$. In the case $0 \leq a < \frac{1}{2}$, we change the variables as in (39) to obtain

$$B'_1 + B'_2 \leq C + \frac{1}{\sqrt{\frac{1}{2} - a}} \int_{-\sqrt{\frac{1}{2}-a}|\zeta|}^{\sqrt{\frac{1}{2}-a}|\zeta|} e^{-\beta^2} d\beta \leq C \left(1 + \left| \frac{1}{2} - a \right|^{-1} \right).$$

Similarly, for $0 \leq a \leq \frac{1}{2}$,

$$\begin{aligned} B'_3 + B'_4 &= \int_0^\pi \left(\int_0^\infty e^{-(a+\frac{1}{2})(\rho - |\zeta| \cos \theta)^2} d\rho \right) \\ &\quad \cdot \left(1 + \left(\frac{1}{2} - a \right) |\zeta|^2 |\cos \theta| \right) e^{-(\frac{1}{2}-a)|\zeta|^2 \cos^2 \theta} \sin \theta d\theta \\ &\leq C \int_0^\pi \left(1 + \left(\frac{1}{2} - a \right) |\zeta|^2 |\cos \theta| \right) e^{-(\frac{1}{2}-a)|\zeta|^2 \cos^2 \theta} \sin \theta d\theta \\ &\leq C \left(2 + 2 \int_0^{\sqrt{\frac{1}{2}-a}|\zeta|} \beta e^{-\beta^2} d\beta \right) \leq C. \end{aligned}$$

This finishes the estimate for B' and thus for B . The desired estimate for A and B yields the estimate for $K_1(\phi)$. The same analysis applies to $K_2(\phi)$, and we complete the proof for (ii). $\square$

Remark 1 (ii) of the lemma is what we need to develop our existence theory in the next section. For the case $a = \frac{1}{2}$, Lemma 2 is optimal. In fact, by considering $\phi = \Phi_0 = E^{\frac{1}{2}}$ in the kernel of L , we have $K(\Phi_0) = v(|\zeta|)\Phi_0$. Because of (4),

$$v_0(1 + |\zeta|)e^{-\frac{1}{2}|\zeta|^2} < |K(\Phi_0)| < v_1(1 + |\zeta|)e^{-\frac{1}{2}|\zeta|^2}, \quad (46)$$

which shows that the inequality (31) is sharp.

Remark 2 There are some related estimates that can lead to a similar but different characterization. For instance, as an application of Lemma 5 in [2] or by a modification of the proof above, we have

Lemma 3 Suppose that

$$|\phi(\zeta)| \leq e^{-a|\zeta|^2}, \text{ for some } a > \frac{1}{2}. \quad (47)$$

Then,

$$|K_1(\phi)(\zeta)| \leq C_a e^{-\frac{1}{2}|\zeta|^2}. \quad (48)$$

Note that this characterizes the case when $a > \frac{1}{2}$, not the case when $0 \leq a < \frac{1}{2}$. We cannot obtain Lemma 2 in the present paper directly from Lemma 5 in [2].

4 Construction of the Solution

The goal of this section is to give a more precise description of the dependence of the solution on the microscopic velocity and the spatial position between the two plates than that in [1].

Theorem 3 (Construction of the solution) For any $0 \leq a < \frac{1}{2}$, there exist $D_a > 0$ and $\kappa_a \geq e$ such that, for every $\kappa \geq \kappa_a$, there exists a solution ϕ for (12) with the following property

$$|\phi(x_1, \zeta) - \phi_0(x_1, \zeta)| \leq D_a \frac{\ln \kappa}{\kappa} g(x_1, \zeta) e^{-a|\zeta|^2}, \quad (49)$$

where $g \geq 0$ is defined in (22) and satisfies (23), and ϕ_0 is given in (14). The constant κ_a is given by the defining property (53).

Proof We construct a sequence through the following iteration:

$$\begin{aligned} \phi_0(x_1, \zeta) &= H(h)(x_1, \zeta), \\ \phi_n(x_1, \zeta) &= H\left(\frac{1}{\kappa} K(\phi_{n-1})\right)(x_1, \zeta), \quad n = 1, 2, \dots \end{aligned} \quad (50)$$

Note that, with (19) and (50), the leading term ϕ_0 is the same as that given in (14). Formally, $\sum_0^\infty \phi_n$ is the solution of (18).

For every fixed $0 \leq a < \frac{1}{2}$, we have $|h| \leq C'_a e^{-a|\zeta|^2}$. With this, we can apply Corollary 1 and Lemma 2 repeatedly and then use (21) to obtain

$$|\phi_n(\zeta)| < C'_a \left(C_a \frac{\ln \kappa}{\kappa}\right)^n g(x_1, \zeta) e^{-a|\zeta|^2}. \quad (51)$$

We let $\kappa = \kappa_a$ be the bigger solution for

$$C_a \frac{\ln \kappa}{\kappa} = \frac{1}{2}, \quad (52)$$

if it exists. If not, we let $\kappa_a = e$. Then we have

$$C_a \frac{\ln \kappa}{\kappa} \leq \frac{1}{2}, \text{ for } \kappa \geq \kappa_a. \quad (53)$$

Therefore $\sum_0^\infty \phi_n$ converges pointwisely, and we see from (51) that the limiting function ϕ satisfies

$$|\phi(x_1, \zeta)| \leq 2C'_a g(x_1, \zeta) e^{-a|\zeta|^2}. \quad (54)$$

Estimate (51) also gives the estimate of the remainder term of the iteration for $\kappa \geq \kappa_a$:

$$|\phi(\zeta) - \sum_{n=0}^N \phi_n(\zeta)| \leq 2C'_a \left(C_a \frac{\ln \kappa}{\kappa} \right)^{N+1} g(x_1, \zeta) e^{-a|\zeta|^2}. \quad (55)$$

Estimate (49) in the theorem follows from (55) by taking $N = 0$ and $D_a = 2C'_a C_a$. $\square$

5 Asymptotic Formula of the Flow Velocity

In the present section, we will derive the asymptotic formula (15) for the fluid velocity q_2 along the plate and prove Theorem 1.

We first show that the main part of q_2 that is explicitly written in (15) comes from the leading term ϕ_0 in the series expansion of ϕ . From the series expansion (50) of Section 4, we have

$$q_2(x_1) = \sum_{n=0}^\infty q_2^n(x_1) = \sum_{n=0}^\infty \varepsilon \int \zeta_2 E^{\frac{1}{2}} \phi_n(x_1, \zeta) d\zeta. \quad (56)$$

With the formula (14) for ϕ_0 , we can explicitly write down the leading term:

$$\frac{1}{\varepsilon} q_2^0(x_1) = \pi^{-\frac{3}{2}} \int_{\zeta_1 > 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\zeta_2^2 (\frac{5}{2} - |\zeta|^2)}{\zeta_1} e^{-|\zeta|^2 - \frac{\nu}{\kappa \zeta_1} |x_1 - s|} ds d\zeta. \quad (57)$$

In (57) above, $\nu = \nu(|\zeta|)$. We will use this abbreviation thereafter. By direct calculation,

$$\begin{aligned} \frac{1}{\varepsilon} q_2^0(x_1) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \rho^4 \left(\frac{5}{2} - \rho^2 \right) e^{-\rho^2} \frac{\kappa}{\nu(\rho)} \left[\frac{4}{3} - \frac{2}{3} e^{-\frac{a_+}{\kappa}} - \frac{2}{3} e^{-\frac{a_-}{\kappa}} \right. \\ &\quad \left. - \frac{1}{6} \frac{a_+}{\kappa} \left(1 - \frac{a_+}{\kappa} \right) e^{-\frac{a_+}{\kappa}} - \frac{1}{6} \frac{a_-}{\kappa} \left(1 - \frac{a_-}{\kappa} \right) e^{-\frac{a_-}{\kappa}} \right. \\ &\quad \left. + \left(\frac{a_+}{\kappa} - \frac{1}{6} \left(\frac{a_+}{\kappa} \right)^3 \right) E_1 \left(\frac{a_+}{\kappa} \right) + \left(\frac{a_-}{\kappa} - \frac{1}{6} \left(\frac{a_-}{\kappa} \right)^3 \right) E_1 \left(\frac{a_-}{\kappa} \right) \right] d\rho, \end{aligned} \quad (58)$$

where

$$a_+(\rho) := \frac{\nu(\rho) |x_1 + \frac{1}{2}|}{\rho}, \quad a_-(\rho) := \frac{\nu(\rho) |x_1 - \frac{1}{2}|}{\rho}, \quad (59)$$

and E_1 is the exponential integral defined as follows

$$E_1(y) = \int_y^\infty \frac{1}{t} e^{-t} dt. \quad (60)$$

The series expansion for E_1 is

$$E_1(y) = -\gamma - \ln(y) - \sum_{k=1}^\infty \frac{(-1)^k y^k}{k k!}, \quad (61)$$

where γ is the Euler's constant. We are interested in the asymptotic formula up to an error of $O(\frac{1}{\kappa})$ for large κ .

Because of (4),

$$\frac{a_{\pm}(\rho)}{\kappa} \leq \begin{cases} 2\nu_1\kappa^{-\frac{1}{2}} & \text{when } \kappa^{-\frac{1}{2}} < \rho \leq 1, \\ 2\nu_1\kappa^{-1} & \text{when } 1 < \rho. \end{cases}$$

For $\rho > \kappa^{-\frac{1}{2}}$, we use the series expansion for $e^{-\frac{a_{\pm}}{\kappa}}$ and $E_1(\frac{a_{\pm}}{\kappa})$ and then integrating term by term, we can evaluate the integral. To obtain the error of $O(\kappa^{-1})$, we need the first two terms in the expansions, i.e. $e^{-y} \sim 1 - y$ and $E_1(-y) \sim -\gamma - \ln y$. On the other hand, for $\rho < \kappa^{-\frac{1}{2}}$, the integrand in (58) is of the order $O(\rho^4\kappa)$. If the above truncated expansion for $e^{-\frac{a_{\pm}}{\kappa}}$ and $E_1(\frac{a_{\pm}}{\kappa})$ is used, the integrand is of the order $O(\rho^3 \ln \rho, \rho^3 \ln \kappa)$. Therefore, the integral for $0 < \rho < \kappa^{-\frac{1}{2}}$ is controlled by C/κ , irrespective of whether we integrate the integrand with or without using the truncated expansion. Hence, by using the truncated expansion for the entire domain, we obtain the asymptotic formula

$$\begin{aligned} \frac{1}{\varepsilon} q_2^0(x_1) &= \frac{1}{4\sqrt{\pi}} \left(\ln \kappa + \frac{1}{2} - \gamma - C_T - |x_1 + \frac{1}{2}| \ln |x_1 + \frac{1}{2}| - |x_1 - \frac{1}{2}| \ln |x_1 - \frac{1}{2}| \right) \\ &\quad + O\left(\frac{1}{\kappa}\right), \end{aligned} \quad (62)$$

where

$$C_T = 4 \int_0^\infty \rho^3 \left(\frac{5}{2} - \rho^2 \right) e^{-\rho^2} \ln \frac{\nu(\rho)}{\rho} d\rho. \quad (63)$$

To prove Theorem 1, it remains to show that the remainder term $\sum_{n=1}^\infty q_2^n$ is of the order $(\ln \kappa)^2/\kappa$. To this end, we write q_2^n , $n \geq 1$ by the use of (50) as

$$\begin{aligned} \frac{1}{\varepsilon} q_2^n(x_1) &= \frac{1}{\kappa} \int \zeta_2 E^{\frac{1}{2}} H(K(\phi_{n-1})) d\zeta \\ &= \frac{1}{\kappa} \int_{-\frac{1}{2}}^{x_1} \left(\int_{\zeta_1 > 0} E^{\frac{1}{2}} \frac{\zeta_2}{|\zeta_1|} e^{-\frac{\nu}{k|\zeta_1|}(x_1-s)} K(\phi_{n-1}) d\zeta \right) ds \\ &\quad + \frac{1}{\kappa} \int_{x_1}^{\frac{1}{2}} \left(\int_{\zeta_1 < 0} E^{\frac{1}{2}} \frac{\zeta_2}{|\zeta_1|} e^{-\frac{\nu}{k|\zeta_1|}(s-x_1)} K(\phi_{n-1}) d\zeta \right) ds \\ &= \frac{1}{\kappa} \int_{-\frac{1}{2}}^{x_1} \left(\int_{\zeta_1 > 0} E^{\frac{1}{2}} \frac{\zeta_2}{\zeta_1} e^{-\frac{\nu}{k\zeta_1}(x_1-s)} K(\phi_{n-1})(\zeta_1, \zeta_2, \zeta_3) d\zeta \right) ds \\ &\quad + \frac{1}{\kappa} \int_{x_1}^{\frac{1}{2}} \left(\int_{\zeta_1 > 0} E^{\frac{1}{2}} \frac{\zeta_2}{\zeta_1} e^{-\frac{\nu}{k\zeta_1}(s-x_1)} K(\phi_{n-1})(-\zeta_1, \zeta_2, \zeta_3) d\zeta \right) ds. \end{aligned} \quad (64)$$

From Lemma 2 and (51), with some $0 \leq a < \frac{1}{2}$ fixed, we have the estimate

$$\left| \frac{1}{\kappa} K(\phi_{n-1}) \right| < C'_a \left(C_a \frac{\ln \kappa}{\kappa} \right)^n e^{-a|\zeta|^2}, \quad (65)$$

hence

$$\begin{aligned}
\frac{1}{\varepsilon} |q_2^n(x_1)| &\leq C_a'' \left(C_a \frac{\ln \kappa}{\kappa} \right)^n \int_{\zeta_1 > 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\zeta_1} e^{-\frac{v_0}{\kappa \zeta_1} |x_1 - s| - \frac{|\zeta|^2}{4}} ds d\zeta \\
&\leq C_a'' \left(C_a \frac{\ln \kappa}{\kappa} \right)^n \int_{\zeta_1 > 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\zeta_1} e^{-\frac{v_0}{\kappa \zeta_1} |x_1 - s| - \frac{|\zeta|^2}{4}} ds d\zeta \\
&= 4\pi C_a'' \left(C_a \frac{\ln \kappa}{\kappa} \right)^n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^\infty \frac{1}{\zeta_1} e^{-\frac{v_0}{\kappa \zeta_1} |x_1 - s| - \frac{\zeta^2}{4}} d\zeta_1 ds \\
&=: 4\pi C_a'' \left(C_a \frac{\ln \kappa}{\kappa} \right)^n A.
\end{aligned}$$

Here, we have used $v(|\zeta|) > v_0$, (4), from the first to the second line, and have carried out the integrations with respect to ζ_2 and ζ_3 from the second to the third line. Finally, in order to estimate A , we divide the domain of its inner integration into three parts $(0, \frac{v_0|x_1-s|}{\kappa})$, $(\frac{v_0|x_1-s|}{\kappa}, 2)$, and $(2, \infty)$, and call the corresponding inner integrals A_1 , A_2 , and A_3 , respectively. Each of them is estimated as follows:

$$\begin{aligned}
A_1 &\leq \frac{\kappa}{v_0|x_1-s|} \int_0^{\frac{v_0|x_1-s|}{\kappa}} \frac{v_0|x_1-s|}{\kappa \zeta_1} e^{-\frac{v_0}{\kappa \zeta_1} |x_1-s|} d\zeta_1 \leq C, \\
A_2 &\leq \int_{\frac{v_0|x_1-s|}{\kappa}}^2 \frac{1}{\zeta_1} d\zeta_1 \leq C(|\ln|x_1-s|| + \ln \kappa), \\
A_3 &\leq \int_2^\infty \frac{1}{2} e^{-\frac{\zeta^2}{4}} d\zeta_1 \leq C.
\end{aligned}$$

Therefore, after the integration with respect to s , we obtain

$$A \leq C \ln \kappa, \quad (66)$$

and thus the remainder, $\frac{1}{\varepsilon}(q_2 - q_2^0)$, is of order $O(\frac{(\ln \kappa)^2}{\kappa})$. This establishes the asymptotic formula (15) and completes the proof of Theorem 1.

6 Singularity of the Flow Velocity

In the present section, we prove Theorem 2 using Lemma 4 and Lemma 5. We consider term by term in the series (56):

$$\frac{1}{\varepsilon} \partial_{x_1} q_2^n(x_1) = \partial_{x_1} \int \zeta_2 E^{\frac{1}{2}} \phi_n(x_1, \zeta) d\zeta, \quad n = 0, 1, \dots \quad (67)$$

Lemma 4 *The leading term in (67) has the following singularity:*

$$\frac{1}{\varepsilon} \partial_{x_1} q_2^0 = \frac{1}{4\sqrt{\pi}} \left(\ln \left| \frac{1}{2} - x_1 \right| - \ln \left| \frac{1}{2} + x_1 \right| \right) + O\left(\frac{1}{\kappa}\right). \quad (68)$$

Proof Direct calculation yields

$$\begin{aligned} \frac{1}{\varepsilon} \partial_{x_1} q_2^0 &= \pi^{-\frac{3}{2}} \int_{\zeta_1 > 0} \frac{\zeta_2^2 (\frac{5}{2} - |\zeta|^2)}{\zeta_1} e^{-|\zeta|^2 - \frac{\nu}{\kappa \zeta_1} |\frac{1}{2} + x_1|} d\zeta \\ &\quad - \pi^{-\frac{3}{2}} \int_{\zeta_1 < 0} \frac{\zeta_2^2 (\frac{5}{2} - |\zeta|^2)}{\zeta_1} e^{-|\zeta|^2 - \frac{\nu}{\kappa \zeta_1} |\frac{1}{2} - x_1|} d\zeta \\ &=: I_+ - I_-. \end{aligned}$$

The integrals $I_{\pm}$ can be reduced to

$$\begin{aligned} I_{\pm} &= \pi^{-\frac{1}{2}} \int_0^{\infty} \rho^3 \left(\frac{5}{2} - \rho^2 \right) e^{-\rho^2} \left(E_1 \left(\frac{a_{\pm}}{\kappa} \right) - \frac{1}{2} e^{-\frac{a_{\pm}}{\kappa}} \right. \\ &\quad \left. + \frac{1}{2} \frac{a_{\pm}}{\kappa} e^{-\frac{a_{\pm}}{\kappa}} - \frac{1}{2} \frac{a_{\pm}^2}{\kappa^2} E_1 \left(\frac{a_{\pm}}{\kappa} \right) \right) d\rho, \end{aligned}$$

where $a_{\pm}$ are the ones defined in (59). Thus, by a similar argument to that in the first half of Section 5, we may use the series expansions for E_1 , (61), and the exponential function both in I_+ and I_- to complete the proof of the lemma. $\square$

In view of Lemma 4, the remaining task for the proof of Theorem 2 is to show that

$$\frac{1}{\varepsilon} (\partial_{x_1} q_2 - \partial_{x_1} q_2^0) = O \left(\frac{\ln \kappa}{\kappa} \right) \left(\ln \left| \frac{1}{2} - x_1 \right| - \ln \left| \frac{1}{2} + x_1 \right| + O(\ln \kappa) \right),$$

which is obtained straightforwardly from the following lemma:

Lemma 5 *For $n \geq 1$, there exist positive constants C_0 and C_1 independent of n such that*

$$\frac{1}{\varepsilon} |\partial_{x_1} q_2^n(x_1)| \leq C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^n \left(\ln \kappa + \left| \ln \left| \frac{1}{2} + x_1 \right| \right| + \left| \ln \left| \frac{1}{2} - x_1 \right| \right| \right). \quad (69)$$

Proof We start with the middle expression of (64):

$$\begin{aligned} \frac{1}{\varepsilon} q_2^n(x_1) &= \frac{1}{\kappa} \int_{\zeta_1 > 0} \int_{-\frac{1}{2}}^{x_1} \frac{\zeta_2}{|\zeta_1|} e^{-\frac{\nu}{\kappa |\zeta_1|} (x_1 - s)} E^{\frac{1}{2}} K(\phi_{n-1}) ds d\zeta \\ &\quad + \frac{1}{\kappa} \int_{\zeta_1 < 0} \int_{x_1}^{\frac{1}{2}} \frac{\zeta_2}{|\zeta_1|} e^{-\frac{\nu}{\kappa |\zeta_1|} |x_1 - s|} E^{\frac{1}{2}} K(\phi_{n-1}) ds d\zeta \\ &=: \frac{1}{\varepsilon} (q_{2+}^n + q_{2-}^n). \end{aligned}$$

We will deal with q_{2+}^n only, because the other term can be treated in the same way.

We first perform the integration by parts with respect to s to rewrite q_{2+}^n as

$$\begin{aligned} \frac{1}{\varepsilon} q_{2+}^n(x_1) &= \int_{\zeta_1 > 0} \frac{\zeta_2}{\nu} E^{\frac{1}{2}} \left(K(\phi_{n-1})(x_1) - e^{-\frac{\nu}{\kappa |\zeta_1|} (x_1 + \frac{1}{2})} K(\phi_{n-1}) \left(-\frac{1}{2} \right) \right) d\zeta \\ &\quad - \int_{\zeta_1 > 0} \int_{-\frac{1}{2}}^{x_1} \frac{\zeta_2}{\nu} e^{-\frac{\nu}{\kappa |\zeta_1|} (x_1 - s)} E^{\frac{1}{2}} \partial_s K(\phi_{n-1}) ds d\zeta, \end{aligned}$$

and then differentiate this to obtain

$$\begin{aligned} \frac{1}{\varepsilon} \partial_{x_1} q_{2+}^n(x_1) &= \int_{\zeta_1 > 0} \frac{\zeta_2}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1 + \frac{1}{2})} E^{\frac{1}{2}} \frac{1}{\kappa} K(\phi_{n-1}) \left(-\frac{1}{2}\right) d\zeta \\ &\quad + \int_{\zeta_1 > 0} \int_{-\frac{1}{2}}^{x_1} \zeta_2 \frac{1}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1 - s)} E^{\frac{1}{2}} \frac{1}{\kappa} \partial_s K(\phi_{n-1}) ds d\zeta \\ &=: I + II. \end{aligned} \quad (70)$$

For the former part I , with some $0 \leq a < \frac{1}{2}$ fixed, we use inequality (65) to obtain

$$|I| < C_a'' \left(C_a \frac{\ln \kappa}{\kappa} \right)^n \int_{\zeta_1 > 0} \frac{1}{\zeta_1} e^{-\frac{\nu}{\kappa\zeta_1}(x_1 + \frac{1}{2}) - \frac{|\zeta|^2}{4}} d\zeta. \quad (71)$$

Thus, by just following the analysis in the second half of Section 5, we have the estimate for I :

$$|I| < C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^n \left(\left| \ln |x_1 + \frac{1}{2}| \right| + \ln \kappa \right). \quad (72)$$

For the latter part II of (70), we resort to the following inequality:

$$|\partial_s K(\phi_{n-1})| \leq C \left(C_1 \frac{\ln \kappa}{\kappa} \right)^{n-1} \left(\ln \kappa + \left| \ln |s + \frac{1}{2}| \right| + \left| \ln |s - \frac{1}{2}| \right| \right),$$

which will be proved in Lemma 6, the last lemma in this paper. With this inequality,

$$\begin{aligned} |II| &< \frac{C'}{\kappa} \left(C_1 \frac{\ln \kappa}{\kappa} \right)^{n-1} \int_{-\frac{1}{2}}^{x_1} \int_{\zeta_1 > 0} \frac{1}{\zeta_1} e^{-\frac{\nu}{\kappa\zeta_1}(x_1 - s) - \frac{1}{4}|\zeta|^2} \\ &\quad \cdot \left(\ln \kappa + \left| \ln |s + \frac{1}{2}| \right| + \left| \ln |s - \frac{1}{2}| \right| \right) d\zeta ds \\ &< \frac{C_0}{\kappa} \left(C_1 \frac{\ln \kappa}{\kappa} \right)^{n-1} \int_{-\frac{1}{2}}^{x_1} \left(\left| \ln |x_1 - s| \right| + \ln \kappa \right) \\ &\quad \cdot \left(\ln \kappa + \left| \ln |s + \frac{1}{2}| \right| + \left| \ln |s - \frac{1}{2}| \right| \right) ds \\ &< \frac{C_1}{\kappa} \left(C_1 \frac{\ln \kappa}{\kappa} \right)^{n-1} (\ln \kappa)^2, \end{aligned}$$

where the second inequality follows from the analysis from (71) to (72). Therefore, we obtain

$$\left| \frac{1}{\varepsilon} \partial_{x_1} q_{2+}^n(x_1) \right| < C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^n \left(\left| \ln |x_1 + \frac{1}{2}| \right| + \ln \kappa \right).$$

The same analysis for q_{2-}^n yields

$$\left| \frac{1}{\varepsilon} \partial_{x_1} q_{2-}^n(x_1) \right| < C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^n \left(\left| \ln |x_1 - \frac{1}{2}| \right| + \ln \kappa \right).$$

Combining the last two inequalities, we complete the proof of the lemma. $\square$

Lemma 6 For $n \geq 0$, there exist positive constants C_0 and C_1 independent of n such that

$$|\partial_s K(\phi_n)| \leq C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^n \left(\ln \kappa + \left| \ln |s + \frac{1}{2}| \right| + \left| \ln |s - \frac{1}{2}| \right| \right). \quad (73)$$

Proof We will prove the lemma by mathematical induction.

Supposing that (73) holds for $0 \leq n \leq l-1$, we will prove it for $n = l$. To this end, we first use (50) in the definition of $K(\phi_l)$ to have

$$\begin{aligned} K(\phi_l)(s, \zeta) &= \int k(\zeta, \zeta_*) \phi_l(s, \zeta_*) d\zeta_* \\ &= \frac{1}{\kappa} \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) \int_{-\frac{1}{2}}^s \frac{1}{|\zeta_{*1}|} e^{-\frac{v_*}{\kappa|\zeta_{*1}|}(s-t)} K(\phi_{l-1})(t, \zeta_*) dt d\zeta_* \\ &\quad + \int_{\zeta_{*1} < 0} k(\zeta, \zeta_*) \int_s^{\frac{1}{2}} \frac{1}{|\zeta_{*1}|} e^{-\frac{v_*}{\kappa|\zeta_{*1}|}(t-s)} K(\phi_{l-1})(t, \zeta_*) dt d\zeta_* \\ &=: K(\phi_l)_+ + K(\phi_l)_-. \end{aligned} \quad (74)$$

In the above, $v_* = v(|\zeta_*|)$; this abbreviation is used hereafter. We will treat $K(\phi_l)_+$ only; $K(\phi_l)_-$ can be treated in the same way. Similar to the calculation for $\partial_{x_1} q_2^n$, we perform the integration by part with respect to t first and then differentiate the resulting integral by s to obtain

$$\begin{aligned} \partial_s K(\phi_l)_+ &= -\frac{1}{\kappa} \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa\zeta_{*1}}(s+\frac{1}{2})} K(\phi_{l-1})(-\frac{1}{2}, \zeta_*) d\zeta_* \\ &\quad + \frac{1}{\kappa} \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) \int_{-\frac{1}{2}}^s \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa\zeta_{*1}}(s-t)} \partial_t K(\phi_{l-1})(t, \zeta_*) dt d\zeta_* \\ &=: A + B. \end{aligned} \quad (75)$$

For the former term A , we first use the inequality (65) and then divide the domain of integration into two parts:

$$\begin{aligned} |A| &< C'_a \left(C_a \frac{\ln \kappa}{\kappa} \right)^l \int_{\zeta_{*1} > 0} |k(\zeta, \zeta_*)| \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa\zeta_{*1}}(s+\frac{1}{2})-a|\zeta_*|^2} d\zeta_* \\ &= C'_a \left(C_a \frac{\ln \kappa}{\kappa} \right)^l \int_{\zeta_{*1} > 0, |\zeta_* - \zeta| > 1} |k(\zeta, \zeta_*)| \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa\zeta_{*1}}(s+\frac{1}{2})-a|\zeta_*|^2} d\zeta_* \\ &\quad + C'_a \left(C_a \frac{\ln \kappa}{\kappa} \right)^l \int_{\zeta_{*1} > 0, |\zeta_* - \zeta| < 1} |k(\zeta, \zeta_*)| \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa\zeta_{*1}}(s+\frac{1}{2})-a|\zeta_*|^2} d\zeta_* \\ &=: C'_a \left(C_a \frac{\ln \kappa}{\kappa} \right)^l (A_1 + A_2). \end{aligned}$$

Obviously from the second expression for $k_1(\zeta, \zeta_*)$ in (3), $k(\zeta, \zeta_*)$ is bounded in the domain of integration for A_1 ; thus A_1 can be estimated by the same process as that from (71) to (72):

$$A_1 < C \int_{\zeta_{*1} > 0} \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa\zeta_{*1}}(s+\frac{1}{2})-a|\zeta_*|^2} d\zeta_* < C \left(\left| \ln |s + \frac{1}{2}| \right| + \ln \kappa \right).$$

For the other part A_2 , we change to the cylindrical coordinates centered at ζ with ζ_1 in the z direction:

$$\begin{aligned}
A_2 &= \int_{\zeta_{*1} > 0, |\zeta_* - \zeta| < 1} |k(\zeta, \zeta_*)| \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa \zeta_{*1}}(s + \frac{1}{2}) - a|\zeta_*|^2} d\zeta_* \\
&< C \int_{|\zeta_* - \zeta| < 1} \frac{1}{|\zeta_* - \zeta|} \frac{1}{|\zeta_{*1}|} e^{-\frac{v_0}{\kappa |\zeta_{*1}|}(s + \frac{1}{2})} d\zeta_* \\
&< C \int_{-1}^1 \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{\rho^2 + z^2}} \frac{1}{|z + \zeta_1|} e^{-\frac{v_0}{\kappa |z + \zeta_1|}(s + \frac{1}{2})} \rho d\rho d\theta dz \\
&= C' \int_{-1}^1 \frac{1}{|z + \zeta_1|} e^{-\frac{v_0}{\kappa |z + \zeta_1|}(s + \frac{1}{2})} dz < C' \left(\left| \ln |s + \frac{1}{2}| \right| + \ln \kappa \right).
\end{aligned}$$

Here again $v(|\zeta_*|) > v_0$, (4), has been used. Thus, we obtain the estimate for $|A|$:

$$|A| < C_a \left(C_1 \frac{\ln \kappa}{\kappa} \right)^l \left(\left| \ln |s + \frac{1}{2}| \right| + \ln \kappa \right). \quad (76)$$

For the latter term B in (75), we use the induction hypothesis to yield

$$\begin{aligned}
|B| &= \frac{1}{\kappa} C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^{l-1} \int_{\zeta_{*1} > 0} |k(\zeta, \zeta_*)| \int_{-\frac{1}{2}}^s \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa \zeta_{*1}}(s-t)} \\
&\quad \cdot \left(\ln \kappa + \left| \ln |t + \frac{1}{2}| \right| + \left| \ln |t - \frac{1}{2}| \right| \right) dt d\zeta_*.
\end{aligned}$$

By following the same argument for the evaluation of A , we can manage the integration with respect to ζ_* to obtain

$$\begin{aligned}
|B| &< \frac{1}{\kappa} C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^{l-1} \int_{-\frac{1}{2}}^s \left(\ln \kappa + \left| \ln |s - t| \right| \right) \\
&\quad \cdot \left(\ln \kappa + \left| \ln |t + \frac{1}{2}| \right| + \left| \ln |t - \frac{1}{2}| \right| \right) dt \\
&< C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^l \ln \kappa.
\end{aligned} \quad (77)$$

Thus, combining (76) and (77), we have the estimate

$$|\partial_s K(\phi_l)_+| < C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^l \left(\left| \ln |s + \frac{1}{2}| \right| + \ln \kappa \right). \quad (78)$$

The parallel argument for $\partial_s K(\phi_l)_-$ leads to

$$|\partial_s K(\phi_l)_-| < C_0 \left(C_1 \frac{\ln \kappa}{\kappa} \right)^l \left(\left| \ln |s - \frac{1}{2}| \right| + \ln \kappa \right). \quad (79)$$

Therefore, (73) holds for $n = l$.

Finally, we need to prove that (73) holds for $n = 0$. As above, we treat $\partial_s K(\phi_0)_+$ only. The expression of $\partial_s K(\phi_0)_+$ is equation (75) with $\frac{1}{\kappa} K(\phi_{l-1})$ being replaced with h , (19):

$$\partial_s K(\phi_0)_+ = - \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) \frac{1}{\zeta_{*1}} e^{-\frac{v_*}{\kappa \zeta_{*1}}(s + \frac{1}{2})} h(\zeta_*) d\zeta_*,$$

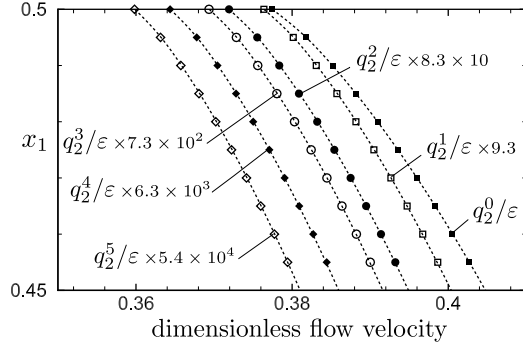


Fig. 1 Profiles of q_2^n/ε ($n = 0, 1, 2, 3, 4, 5$) near the boundary at $x_1 = 1/2$ in the case of $\kappa = 20$.

Table 1 Data of coefficients of the curve (80).

n	$C_0^{(n)}(\kappa = 20)$	$C_1^{(n)}(\kappa = 20)$	$C_0^{(n)}(\kappa = 100)$	$C_1^{(n)}(\kappa = 100)$
0	3.7747×10^{-1}	1.375×10^{-1}	5.9742×10^{-1}	1.405×10^{-1}
1	4.04715×10^{-2}	1.3×10^{-2}	1.97323×10^{-2}	4.285×10^{-3}
2	4.481×10^{-3}	1.39×10^{-3}	6.6565×10^{-4}	1.418×10^{-4}
3	5.06×10^{-4}	1.535×10^{-4}	2.26875×10^{-5}	4.74×10^{-6}
4	5.784×10^{-5}	1.725×10^{-5}	—	—
5	6.664×10^{-6}	1.975×10^{-6}	—	—

where it is used that h is independent of t . Because h also satisfies inequality (65) with $n = 0$, the estimate for $\partial_s K(\phi_0)_+$ can be obtained from that for A in (76) with $l = 0$. The estimate for $\partial_s K(\phi_0)_-$ is obtained in the same way, and we finish the proof of the lemma. $\square$

7 Numerical demonstration of the singularity

In [9], the equation (50) has been solved numerically to obtain the mass flow rate in the channel. Here we have recomputed it by the same numerical code with a grid system refined near the boundary, in order to make clear the profile of flow velocity there.

Figure 1 shows the profiles of q_2^n/ε ($n = 0, 1, 2, 3, 4, 5$) near the boundary at $x_1 = 1/2$ in the case of $\kappa = 20$. They are indicated by symbols. In the figure, the curves given by

$$C_0^{(n)}(\kappa) - C_1^{(n)}(\kappa) \left(\left| x_1 + \frac{1}{2} \right| \ln \left| x_1 + \frac{1}{2} \right| + \left| x_1 - \frac{1}{2} \right| \ln \left| x_1 - \frac{1}{2} \right| \right), \quad (80)$$

are also shown by dotted lines. The curve with coefficients $C_0^{(n)}(\kappa)$ and $C_1^{(n)}(\kappa)$, which are listed in Table 1, agrees with q_2^n/ε . This means that the structure of the singularity is kept at each stage of the Picard iteration, which is more than that stated by Theorem 1. Incidentally, for $\kappa = 20$, the partial sum $\sum_{n=0}^5 q_2^n$ of the series has already converged to q_2 actually.

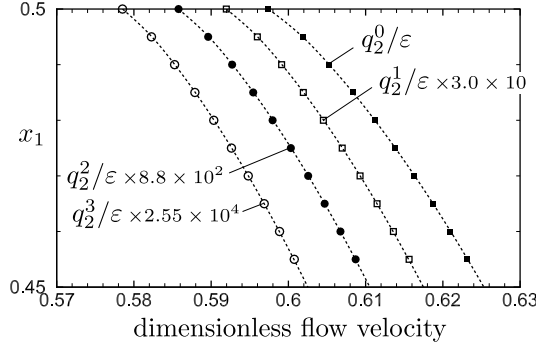


Fig. 2 Profiles of q_2^n/ε ($n = 0, 1, 2, 3$) near the boundary at $x_1 = 1/2$ in the case of $\kappa = 100$.

Figure 2 shows the numerical results corresponding to Fig. 1 in the case of $\kappa = 100$. Since the convergence is faster than the case of $\kappa = 20$, the iteration has been terminated at $n = 3$. Again, the profile of q_2^n near the boundary is quite well fitted by the curve (80). The coefficients in this case are also listed in Table 1.

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