

A Version of Evans-Perkins Type Stochastic Representation Formula for Historical Superprocesses*

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1 Introduction

The purpose of this article is to introduce a version of Evans-Perkins type stochastic representation formula for a generalized $\{\gamma, a, b, g\}$ -historical superprocess (see the definition in §2). Here by Evans-Perkins type formula we mean an explicit stochastic integral representation for historical functional of a certain class, which is similar to and is a historical process counterpart of Itô-Clark formula (e.g. [U95, p.42]) in elementary stochastic calculus. The key idea of demonstration of the Itô-Clark type formula for historical superprocess is to derive a variant of stochastic integration by parts with respect to the historical process in the Perkins sense [P92].

The review of the Evans-Perkins theory [EP95] is a good point to start. There are two reasons why their integration by parts formula is so important. For one thing, it can provides with a new formula of transformations of stochastic integrals closely connected with the so-called historical processes. In addition, a generalization of formula itself is of independent interest, and it is very useful as a theoretical tool of stochastic calculus in the theory of measure-valued processes. For another, it has an extremely remarkable meaning on an applicational basis. By making use of the formula S.N. Evans and E.A. Perkins (1995) have succeeded in deriving a kind of Itô-Wiener chaos expansion for functionals of superprocesses [EP95].

S.N. Evans and E.A. Perkins have showed that any L^2 functional of superprocess may be represented as a constant C_0 plus a stochastic integral with respect to the associated orthogonal martingale measure M (e.g. [EP94]). Recently they have obtained the explicit representations involving multiple stochastic integrals for a quite general functional

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of the so-called Dawson-Watanabe superprocesses. Actually, the results are obtained in the setting of the historical process associated with the superprocess [EP95].

2 Notation and Preliminaries

Let $C = C^d = C([0, \infty), \mathbf{R}^d)$ denote the space of $\mathbf{R}^d$ -valued continuous paths on $\mathbf{R}_+ = [0, \infty)$ with the compact-open topology. $\mathcal{C} = \mathcal{B}(C)$ is its Borel σ -field and

$$\mathcal{C}_t = \mathcal{B}_t(C) = \sigma(y(s), s \leq t)$$

denotes its canonical filtration. For $y, w \in C^d$ and $s \geq 0$, we define the stopped path by $y^s(t) = y(t \wedge s)$ and let

$$y/s/w = \begin{cases} y(t), & \text{for } t < s, \\ w(t-s), & \text{for } t \geq s. \end{cases} \quad (1)$$

$M_F(C)$ is the space of finite measures on C with the topology of weak convergence and we define

$$M_F(C)^t := \{m \in M_F(C); y = y^t, m - a.s. y\}, \quad t \geq 0.$$

If P_x denotes Wiener measure on $(C, \mathcal{B}(C))$ starting at x , $\tau \geq 0$, and $m \in M_F(C)^\tau$, define $P_{\tau, m} \in M_F(C)$ by

$$P_{\tau, m}(A) := \int_C P_{y(\tau)}(\{w; y/\tau/w \in A\}) dm(y).$$

Let

$$\Omega_H[\tau, \infty) := \{H \in C([\tau, \infty), M_F(C)); H_t \in M_F(C)^t, \forall t \geq \tau\},$$

and put $\Omega_H := \Omega_H[0, \infty)$. We write $\mathcal{H}$ for the totality of Borel sets of Ω_H . We use the notation $H_t(\omega) = \omega(t)$ for $\omega \in \Omega_H$ as for the canonical realization of historical process.

Fix $0 \leq t_1 < \dots < t_n$ and $\psi \in C_b^2(\mathbf{R}^{nd})$. For $y \in C$ we set

$$\begin{aligned} \bar{y}(t) &= (y(t \wedge t_1), \dots, y(t \wedge t_n)), \\ \bar{\psi}(y) &\equiv \bar{\psi}(t_1, \dots, t_n)(y) = \psi(y(t_1), \dots, y(t_n)), \end{aligned}$$

and $\tilde{\psi}(t, y) = \bar{\psi}(y^t)$. ψ_i (resp. ψ_{ij}) stands for the first (resp. second) order partials $\partial_i \psi$ (resp. $\partial_{ij}^2 \psi$) of ψ . $\nabla \bar{\psi} : [0, \infty) \times C \rightarrow \mathbf{R}^d$ is the $(\mathcal{C}_t)$ -predictable process whose j -th component at (t, y) is given by

$$\sum_{i=0}^{n-1} \mathbf{I}(t < t_{i+1}) \psi_{id+j}(\bar{y}(t)).$$

While, for $1 \leq i, j \leq d$, $\bar{\psi}_{ij} : [0, \infty) \times C \rightarrow \mathbf{R}$ is the $(\mathcal{C}_t)$ -predictable process defined by

$$\bar{\psi}_{ij}(t, y) := \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \mathbf{I}(t < t_{k+1} \wedge t_{l+1}) \partial_{kd+i} \partial_{ld+j}(\bar{y}(t)).$$

Let us define the domains

$$D_0 := \bigcup_{n=1}^{\infty} \left\{ \bar{\psi}(t_1, \dots, t_n); 0 \leq t_1 < \dots < t_n, \bar{\psi} \in C_0^\infty(\mathbf{R}^{nd}) \right\} \cup \{1\},$$

$$\tilde{D}_0 := \left\{ \tilde{\psi}; \tilde{\psi}(t, y) = \bar{\psi}(y^t) \text{ for some } \bar{\psi} \in D_0 \right\}.$$

Let $\bar{\Omega} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq \tau}, \mathbf{P})$ be a filtered probability space and let $(\omega, y) = (\omega, y_1, \dots, y_d)$ denote sample points in $\bar{\Omega} = \Omega \times C^d$. Here $\tau \geq 0$ is fixed. When f is a function on $[\tau, \infty) \times \bar{\Omega}$ taking values in a normed linear space $(E, \|\cdot\|)$, then a bounded $(\mathcal{F}_t)$ -stopping time T is a reducing time for f if and only if

$$\mathbf{I}(\tau < t \leq T) \|f(t, \omega, y)\|$$

is uniformly bounded. In addition we say that a sequence $\{T_n\}$ reduces f if and only if each T_n reduces f and $T_n \nearrow \infty$ holds $\mathbf{P}$ -a.s. We say that f is locally bounded if such a sequence $\{T_n\}$ exists. We assume that

(LB) $\gamma \in [0, \infty), a \in S^d, b \in \mathbf{R}^d$ and $g \in \mathbf{R}$ are $(\hat{\mathcal{F}}_t^*)$ -predictable processes on $[\tau, \infty) \times \bar{\Omega}$ such that $\Lambda = (\gamma, a, b, g\gamma^{-1} \mathbf{I}(\hat{g} \neq))$ is locally bounded.

Notice that the above assumption implies that g is locally bounded.

Now we introduce the martingale problem formulation of historical processes in stochastic calculus on historical trees (cf. [P92], [P95]). For $\tau \geq 0$ and $m \in M_F(C)^\tau$, we define

$$A_{\tau, m} \tilde{\psi}(t, y) \equiv A(\bar{\psi})(t, y) := \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij}(t, y) \bar{\psi}_{ij}(t, y) + b(t, \omega, y) \cdot \nabla \bar{\psi}(t, y) + g(t, \omega, y) \bar{\psi}(y^t)$$

for $\bar{\psi} \in D_0$. We write $\langle \mu, f \rangle$ or sometimes $\mu(f)$ for the integral $\int f d\mu$ when μ is a measure and f is a suitable μ -integrable function. Suggested by [DkTn98], we may define

Definition 1 (cf. [P95], §2) *A predictable process $K = \{K_t, t \geq \tau\}$ on $\bar{\Omega}$ with sample paths a.s. in $\Omega_H[\tau, \infty)$ is a generalized $\{\gamma, a, b, g\}$ -historical process (GHP) (or $(A, -\gamma\lambda^2/2)$ -historical process) if and only if $K_t \in M_F(C)^t$ for all $t \geq \tau$, a.s. and $P[K_\tau(1)] < \infty$, and if there exists a probability measure $\mathcal{P}$ on $\Omega_H[\tau, \infty)$ such that it satisfies the martingale problem (MP) with initial data $\{\tau, m\}$ and $\{\gamma, a, b, g\}$: for $\forall \bar{\psi} \in D_0$,*

$$Z_t(\bar{\psi}) = \langle K_t, \bar{\psi} \rangle - \langle m, \bar{\psi} \rangle - \int_\tau^t \langle K_s, A(\bar{\psi})(s) \rangle ds, \quad t \geq \tau, \quad (2)$$

is a continuous $(\mathcal{F}_t)$ -local martingale satisfying $Z_\tau(\bar{\psi}) = 0$ and

$$\langle Z(\bar{\psi}) \rangle_t = \int_\tau^t \int \gamma(s, \omega, y) \bar{\psi}(y)^2 K_s(dy) ds, \quad \forall t \geq \tau, \quad \text{a.s.}$$

Remark. The existence and uniqueness of the law of K is essentially due to [F88] (cf. [DIP89]).

Set $T_s = [s, \infty)$, and in particular $T_0 = [\tau, \infty)$. Define $C(M_F(C)) := C(T_0; M_F(C))$, and we write $C(t) = (\tau, t] \times C$ for the integral domain. When $\mathcal{F}$ is the σ -field or the usual filtration, then $f \in \mathcal{F}$ indicates that the function f is $\mathcal{F}$ -measurable and $\mathcal{P}(\mathcal{F})$ is the totality of $(\mathcal{F})$ -predictable functions, and $b\mathcal{P}(\mathcal{F})$ denotes the whole space of functions that are all bounded elements of $\mathcal{P}(\mathcal{F})$. We use the symbol $U(M_F(C))$ for an admissible subset of the space $C(C(M_F(C)); \mathbf{R})$; more precisely $U(M_F(C))$ is the totality of real valued continuous functions F on $C(M_F(C))$ such that for some compactly supported finite measure $L(dt)$ on T_0 , the estimate

$$|\Delta F(h, g)| \leq \int_{T_0} g(t, C) L(dt)$$

holds for all $h, g \in C(M_F(C))$, where we define $\Delta F(x, y) := F(x + y) - F(x)$.

3 Predictable Representation Property

Let $\{T_N\}$ be a reducing sequence. Take a sequence $\{\bar{\psi}_n\}$, $\bar{\psi}_n \in D_0$ such that $\bar{\psi}_n$ converges bounded pointwise (*bp* for short) to ψ , namely,

$$\bar{\psi}_n \rightarrow \psi, \quad bp \quad (n \rightarrow \infty).$$

An application of dominated convergence theroem together with the local boundedness of γ implies that

$$\langle Z(\bar{\psi}_n - \bar{\psi}_m) \rangle_t \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

for $\forall t \geq \tau$, a.s. Therefore we obtain

Proposition 1 *There is an a.s. continuous adapted process $\{Z_t(\psi); t \geq \tau\}$ such that*

$$\sup_{\tau \leq t \leq N} |Z_t(\bar{\psi}_n) - Z_t(\psi)| \rightarrow 0$$

holds in probability (w.r.t. P) as $n \rightarrow \infty$ for $\forall N > \tau$.

To proceed our discussion, we need the following lemmas.

Lemma 1 (cf. Corollary 2.2, p.11, [P95]) *Let T be a reducing time for (γ, g) . Then we have*

$$(a) \quad 0 < P[K_T(1)] \leq P[\sup_{\tau \leq t \leq T} |K_t(1)| + \langle Z(1) \rangle_T] < \infty.$$

(b) *If $P[K_\tau(1)^p] < \infty$ for $p \in N$, then*

$$P \left\{ \left(\sup_{\tau \leq t \leq T} |K_t(1)| \right)^p + \langle Z(1) \rangle_T^p \right\} < \infty.$$

Lemma 2 (cf. [EP94, p.123]) *D_0 is dense in $b\mathcal{B}(C)$ relative to the bounded pointwise convergence topology.*

We may use Lemma 1 to obtain

$$\sup_{\tau \leq t \leq T_N} |Z_t(\bar{\psi}_n) - Z_t(\psi)| \rightarrow 0 \quad \text{in } L^2$$

as $n \rightarrow \infty$, for $\forall N \in \mathbb{N}$. Clearly $Z_t(\psi)$ is a continuous $(\mathcal{F}_t)$ -local martingale whose quadratic variation process is given by

$$\langle Z(\psi) \rangle_t = \int_{\tau}^t \int_C \gamma(s, \omega, y) \psi(y)^2 K_s(dy) ds. \quad (3)$$

By virtue of Lemma 2, it is a routine work to show that this Z_t extends to an orthogonal martingale measure

$$\{Z_t(\psi); t \geq \tau, \psi \in b\mathcal{B}(C)\}.$$

Consequently, the mapping $t \mapsto Z_t(\psi)$ is a continuous local martingale satisfying Eq.(3) for each $\psi \in b\mathcal{B}(C)$, and $\psi \mapsto Z_{t \wedge T_N}(\psi)$ is an L^2 -valued measure on $\mathcal{B}(C)$ for each $t \geq \tau$, $N \in \mathbb{N}$. By a trivial localization argument, we may define the stochastic integral

$$Z_t(\psi) = \int_{\tau}^t \int \psi(s, \omega, y) dM(s, y) \quad (4)$$

($\exists$ an orthogonal martingale measure $M = M^K$ in the sense of Walsh [W86, Chapter 2]) such that

$$\langle Z(\psi) \rangle_t = \int_{\tau}^t \langle K_s, \gamma(s, \omega) \psi(s, \omega)^2 \rangle ds, \quad (5)$$

$\forall t \geq \tau$, a.s., as long as ψ belongs to $L_{loc}^2(K, \mathbf{P})$. Here $L_{loc}^2(K, \mathbf{P})$ denotes the L^2 space of $(\mathcal{F}_t \times C)_{t \geq \tau}$ -predictable functions f and

$$\int_{\tau}^t \int \gamma(s, y) f(s, y)^2 K_s(dy) ds < \infty$$

for $\forall t \geq \tau$, $\mathbf{P}$ -a.s.

We write $f \in L^2(K, \mathbf{P})$ (resp. $L_{\infty}^2(K, \mathbf{P})$) if, in addition,

$$\mathbf{P} \left\{ \int_{\tau}^t \int \gamma(s, \omega, y) f(s, \omega, y)^2 K_s(dy) ds \right\} < \infty, \quad \forall t > 0,$$

respectively,

$$\mathbf{P} \left\{ \int_{\tau}^{\infty} \int \gamma(s, \omega, y) f(s, \omega, y)^2 K_s(dy) ds \right\} < \infty.$$

Theorem 1 (Predictable Representation Property) *If $V \in L^2(\Omega, \mathcal{F}, P)$, then there is an f in $L_{\infty}^2(K, P)$ such that*

$$V = P[V] + \int_{\tau}^{\infty} \int f(s, \omega, y) dM^K(s, y), \quad P - a.s. \quad (6)$$

The proof of Theorem 1 will be given in the succeeding section.

Remark. The predictable representation property was proved by Evans-Perkins (1994) [EP94, Theorem 1.1] for the $(Y, -\lambda^2/2)$ -superprocess with a Hunt process Y as its underlying process. In [EP95] a variant of the stochastic integral representation formula of the above type was proved for the $(Y, -\lambda^2/2)$ -historical process with a Markov process Y .

4 Proof of Theorem 1

If $f \in b\mathcal{B}(C)$, then the moment $\mathbf{P}[K_t(f)]$ is uniformly bounded as t ranges over a compact subset of $[\tau, \infty)$. We have the following explicit formula for the moment, namely,

Lemma 3 $P[K_t(f)] = P_{\tau, \nu}[f(Y^t)]$ holds for every f in $b\mathcal{B}(C)$ under $\nu \in M_F(C)^\tau$, where Y^t is the corresponding stopped path-valued process.

We set $\hat{E} := \{(s, y) \in [\tau, \infty) \times C; y^s = y\}$ and define a measure $Q_{s, y}$ on $(C, \mathcal{C})$ by

$$Q_{s, y}(A) := P_{y(s)}\{w \in C; (y/s/w) \in A\}, \quad A \in \mathcal{C}, \quad (s, y) \in \hat{E}.$$

Then a similar argument as in [F88] (cf. Theorem 2.1.3, [DP91]) allows us to show

Proposition 2 Assume that $T_{s, t}f(y) := P_{y(s)}[f(y/s/Y^{t-s})]$ satisfies the semigroup property for $(s, y) \in \hat{E}$, $t \geq s$, and $f \in b\mathcal{B}(C)$. Then we have

$$P[\exp\{-\langle K_t, f \rangle\}] = \exp\{-\langle m, V_{\tau, t}f \rangle\},$$

for all $f \in b\mathcal{B}(C)$ and $m \in M_F(C)$. Moreover, $\{V_{\tau, t}\}$ forms a semigroup on $b\mathcal{B}(C)$, and $V_{s, t}f(y) \equiv v_{s, t}(y)$ is Borel measurable as a function of (s, y, t) in $\hat{E} \times [\tau, \infty)$ with $t \geq s$, and is the unique solution of

$$v_{s, t}(y) = P_{y(s)}[f(y/s/Y^{t-s})] - \frac{1}{2} \int_{\tau}^{t-s} P_{y(s)}[\gamma(u, \omega, y)v_{u+s, t}(y/s/Y^u)] du.$$

Proof of Lemma 3. According to the same discussion as in Theorem 2.1.5 [DP91, p.19], we can deduce from Proposition 2 that under $\nu \in M_F(C)^\tau$

$$\mathbf{P}[\langle K_t, f \rangle] = \langle \nu, G_{\tau, t}f \rangle, \quad \text{--- --- ---} (*)$$

where $G_{s, t}f(y) = Q_{s, y}[f(Y_t)]$. A simple computation reads

$$\begin{aligned} \langle \nu, G_{\tau, t}f \rangle &= \int_C Q_{\tau, y}[f(Y_t)] \nu(dy) \\ &= \int_C \left\{ \int_{C^t} f(Y_t) P_{y(\tau)}\{w \in C; (y/\tau/w) \in d\zeta\} \right\} \nu(dy) \\ &= \int_{C^t} f(Y^t) \int_C P_{y(\tau)}\{(y/\tau/Y) \in d\zeta\} \nu(dy) \\ &= \int_{C^t} f(Y^t) P_{\tau, \nu}(dy) = P_{\tau, \nu}[f(Y^t)], \end{aligned}$$

because we made use of the Fubini theorem in the second line. By (*), this concludes the proof. Q.E.D.

Suggested by the argument [MP92, pp.331-332] (also see [EP95, pp.1779-1780]), we define

$$F^{\tau, \nu} := \{\varphi \in b(\mathcal{B}([\tau, \infty)) \times C); \varphi(t, y) = \varphi(t, y^t) \text{ for all } t \geq \tau, \\ \text{the map } t \mapsto \varphi(t, Y) \text{ is } P_{\tau, \nu} - \text{a.s. right continuous, } \forall t \geq \tau\}$$

under $\nu \in M_F(C)^t$, and $\tilde{F}^{\tau,\nu}$ is the set of bounded functions ψ in $\mathcal{B}([\tau, \infty)) \times \mathcal{F} \times \mathcal{C}$ such that

$$\psi(\cdot, \omega, \cdot) \in F^{\tau,\nu}, \quad P - \text{a.s.},$$

and the condition (C) is compatible with the definition of K in §2.

(C) For $H_t \in M_F(C)^t$, $\mathbf{P}$ -a.s. for all $t \geq \tau$ with Y as its corresponding path-valued process, and for all $\varphi \in F^{\tau,\nu}$,

$$M_t(\varphi) := \langle H_t, \varphi(t, \cdot) \rangle - \langle \nu, \varphi(\tau, \cdot) \rangle - \int_{(\tau, t]} \langle H_s, \psi(s, \omega, \cdot) \rangle ds, \quad t \geq \tau, \quad \text{under } \nu \in M_F(C)^\tau,$$

is a continuous $(\mathcal{F}_t)_{t \geq \tau}$ martingale for which $M_\tau(\varphi) = 0$ and

$$\langle M(\varphi) \rangle_t = \int_{(\tau, t]} \int_C \gamma(s, \omega, y) \varphi(s, y)^2 H_s(dy) ds.$$

Let $A^{\tau,\nu}$ denote the set of pairs (φ, ψ) in $F^{\tau,\nu} \times \tilde{F}^{\tau,\nu}$ such that

$$Z_t := \varphi(t, Y) - \varphi(\tau, Y) - \int_{(\tau, t]} \psi(s, Y) ds, \quad t \geq \tau,$$

is a $(\bar{\mathcal{C}}_t^\nu)_{t \geq \tau}$ -martingale under $P_{\tau,\nu}$, where $\bar{\mathcal{C}}_t^\nu$ is the σ -field generated by $\mathcal{C}_{t+}$ and the $P_{\tau,\nu}$ -null sets in C .

Proposition 3 *There exists for each $n \in N$ a function $g_n = g_n(t, \omega, y)$ in $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$ such that*

$$V = P[V] + \lim_{n \rightarrow \infty} \int_{(\tau, \infty)} \int_C g_n(s, \omega, y) dM^K(s, y),$$

with $L^2(P)$ -convergence.

Proof. Recall the condition (C). By virtue of Theorem 2 and Proposition 2 of Jacod (1977) [J77] (e.g. [EP94, p.124] or [EP95, p.1796]), we can deduce that for each $n \in N$ there exist suitable pairs

$$(\varphi_n^1, \psi_n^1), \dots, (\varphi_n^{N(n)}, \psi_n^{N(n)}) \in A^{\tau, m},$$

(relative to K_t), $\xi_n^1, \dots, \xi_n^{N(n)} \in b\mathcal{P}(\mathcal{F}_t)$, and $\{t_n\}_n \subset (\tau, \infty)$ such that $t_n \nearrow \infty$ (as $n \rightarrow \infty$) and

$$V = \mathbf{P}[V] + \lim_{n \rightarrow \infty} \int_{(\tau, t_n]} \int_C \sum_k \xi_n^k(s, \omega) \varphi_n^k(s, y) dM^K(s, y),$$

where the convergence is in $L^2(\mathbf{P})$. Moreover, we can choose a bounded $(\mathcal{C}_t)_{t \geq \tau}$ -predictable function η such that

$$\int \int_{C(t)} \xi(s, \omega) \varphi(s, y) dM^K(s, y) = \int \int_{C(t)} \xi(s, \omega) \eta(s, y) dM^K(s, y), \quad P - \text{a.s.}, \quad \forall t \geq \tau,$$

for each $(\varphi, \psi) \in A^{\tau, m}$ and each ξ in $b\mathcal{P}(\mathcal{F}_t)$, and also that the y -section

$$\{(s, y) \in [\tau, \infty) \times C; \varphi(s, y) \neq \eta(s, y)\}$$

is a countable set. By the property of stochastic integral and the Fubini type theorem, we readily obtain

$$\begin{aligned} P \left| \int \int_{C(t)} \xi \varphi dM^K - \int \int_{C(t)} \xi \eta dM^K \right|^2 &= P \left[\int \int_{C(t)} \gamma \xi(s)^2 \{\varphi(s, y) - \eta(s, y)\}^2 dK_s ds \right] \\ &\leq C_0 \cdot P \left[\int_{\tau}^t \int_C \{\varphi(s, y) - \eta(s, y)\}^2 K_s(dy) ds \right] \\ &= C_0 \int_{\tau}^t P [K_s(|\varphi_s - \eta_s|^2)] ds. \end{aligned}$$

for some constant C_0 . By Lemma 3, the last term in the above can be replaced by

$$\int_{\tau}^t P_{\tau, m} [|\varphi(s, Y^s) - \eta(s, Y^s)|^2] ds,$$

which, indeed, becomes null if we apply the Fubini theorem again because we employed the condition

$$\int_{(\tau, t]} \{\varphi(s, Y) - \eta(s, Y)\}^2 ds = 0, \quad \forall t > \tau, \quad P_{\tau, m} - a.s.$$

So that, by making use of the above-mentioned η , we have only to set

$$g_n(s, \omega, y) = \sum_k \xi_n^i(s, \omega) \eta_n^i(s, y)$$

for each n . This completes the proof. Q.E.D.

By virtue of the arguments in the proof of Proposition 3, we have that

$$\begin{aligned} 0 &= \lim_{n, k \rightarrow \infty} \left\| \int \int_{C(\infty)} g_n(s, \cdot, y) dM^K(s, y) - \int \int_{C(\infty)} g_k(s, \cdot, y) dM^K(s, y) \right\|_{L^2(P)}^2 \\ &= \lim_{n, k \rightarrow \infty} \|g_n(\cdot) - g_k(\cdot)\|_{L_{\infty}^2(K, P)}^2. \end{aligned}$$

Hence there exists a limit function f in $L_{\infty}^2(K, P)$ such that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} P \left[\int_{\tau}^{\infty} \int_C \gamma(s, \omega, y) \{g_n(s, \omega, y) - f(s, \omega, y)\}^2 K_s(dy) ds \right] \\ &= \lim_{n \rightarrow \infty} P \left| \int \int_{C(\infty)} g_n(s, \omega, y) dM^K(s, y) - \int \int_{C(\infty)} f(s, \omega, y) dM^K(s, y) \right|^2. \end{aligned}$$

Immediately this implies from Proposition 3 that

$$\begin{aligned} V &= P[V] + \lim_{n \rightarrow \infty} \int \int_{C(\infty)} g_n(s, \omega, y) dM^K(s, y) \\ &= \int \int_{C(\infty)} f(s, \omega, y) dM^K(s, y). \end{aligned}$$

which completes the proof of Theorem 1.

5 Canonical Measure and Campbell Measure

For $y \in D = D(\mathbf{R}_+; \mathbf{R}^d)$, we define $y^{t-}(s)$ as $y(s)$ itself if $s < t$ and as $y(t-)$ if $s \geq t$. $Q(s, y)$ is a σ -finite measure on $C(M_F(D))$ such that

$$Q\left(s, y^{s-}; \{h \in C(M_F(D)); \tau \leq \exists t \leq s, h(t) \neq 0\}\right) = 0,$$

which can be defined by the canonical measure $R(\tau, t, y; d\zeta)$ [D93] associated with the law of $K_t = K(t)$ and the path restriction mapping π (cf. §2, pp.1781-1782 in [EP95]) together with a discussion involved with the Dawson-Perkins theory(1991) (e.g. Theorem 2.2.3(pp.27-28) and Proposition 3.3(pp.38-39) in [DP91]). Here R is characterized by

$$\log \mathcal{P}_{s, \delta_y}[\exp\langle K_t, -\varphi \rangle] = \int_{M_F(M_F(C))} (e^{-\langle \zeta, \varphi \rangle} - 1) R(s, t, y; d\zeta)$$

(cf. Lemma 1 in [Dk99c]; see also [DP91, Proposition 3.3, pp.38-39]). Let F be a real valued Borel function on $C(M_F(C))$. Assume that

$$I_{s,y}^Q[\Delta F](h) := \int_{C(M_F(C))} \Delta F(h, g) Q(s, y^{s-}; dg) \quad (7)$$

is well-defined and bounded below for all $s > \tau$, $y \in C$, and $h \in C(M_F(C))$. For a bounded $(\mathcal{F}_t)$ -stopping time T , we define the Campbell measure P_T associated with $K(t)$ by

$$P_T(A \times B) := \mathbf{P}(K(T, A) \cdot \mathbf{I}_B\{K(T)\})/m(C) \quad (8)$$

for any $A \times B \in (C \times \Omega, \mathcal{C} \times \mathcal{F})$ (cf. [P95], p.21; or [DP91], p.62). Notice that $K_\tau = m$. Since the mapping $(s, y, \omega) \mapsto I_{s,y}^Q[\Delta F](K(\omega))$ is bounded below and measurable with respect to the product of the predictable σ -field associated with the filtration $(\mathcal{C}_t)$ and the σ -field $\mathcal{F}$, we can apply Lemma 2.2(p.1783) [EP95] together with the projection operation argument and the predictable section theorem (e.g. Theorem 2.14(p.19) or Theorem 2.28(p.23), [JS87]; see also [E82], pp.50-52), to deduce that there exists a $(\mathcal{C}_t \times \mathcal{F}_t)_{t \geq \tau}$ -predictable function $Pr[F](s, y, \omega) : (\tau, \infty) \times C \times \Omega \rightarrow \mathbf{R}$ such that

$$P_T\{I^Q[\Delta F](T)/(\mathcal{C} \times \mathcal{F})_T\} = Pr[F](T, \omega, y) \quad (9)$$

holds P_T -a.s. for all bounded $(\mathcal{F}_t)$ -predictable stopping times $T > s$. It is quite interesting to note that in particular

$$\mathbf{P} \int_C I^Q[\Delta F](T, y) K(T, dy) = \mathbf{P} \int_C Pr[F](T, y) K(T, dy).$$

We shall introduce an approximation map. For each $l \in \mathbf{N}$, let us choose a partition $\Delta(l) = \{t^{(l)}(j); 1 \leq j \leq k[l]\}$ such that $\tau = t^{(l)}(0) < t^{(l)}(1) < \dots < t^{(l)}(k[l]) < \infty$,

$$\lim_{l \rightarrow \infty} \sup_k \Delta t[l; k] = 0 \quad \text{and} \quad \lim_{l \rightarrow \infty} t^{(l)}(k[l]) = +\infty.$$

The approximation map $W[l]$ from $C(M_F(C))$ into $C(M_F(C))$ is defined by

$$W[l](g)(t) := \{Sb(t^{(l)}(i+1)) \cdot g(t^{(l)}(i)) - Sb(t^{(l)}(i)) \cdot g(t^{(l)}(i+1))\} \Delta t[l; i]^{-1}$$

if $t \in [t^{(l)}(i), t^{(l)}(i+1))$, and $:= g(t^{(l)}(k[l]))$ if $t \geq t^{(l)}(k[l])$, for any element g of $C(M_F(C))$ with $Sb(k) = k - t$. Immediately we get

Lemma 4 (cf. Lemma 4, [DK98a]) *Let F be an element of $C(C(M_F(C)); R)$. Then for all $g \in C(M_F(C))$*

$$\lim_{l \rightarrow \infty} (F \circ W[l])(g) = F(g).$$

6 Random Measures and Assumptions

We shall introduce the assumptions for our main results (Theorem 2, Theorem 3 and Theorem 4) which are stated in the succeeding section. C^t denotes the image of C under the map: $y \mapsto y^t$. We define a measure $K^*[s, t]$ on C^s by $K^*[s, t](F) := K_t(\{y : y^s \in F\})$. Then the measure $K^*[s, t]$ is atomic with a finite set of atoms, and we write $L[s, t](\subset C^s)$ for the locations of these atoms. For $s \in (a, b]$, let $\lambda_s[\varphi]$ be the random measure on C that places mass $\varphi(s, y)$ at each point y in $(L[b, c])^s = L[s, c]$. With some localization arguments in stochastic calculus, the Perkins-Girsanov theorem of Dawson type [P95] guarantees the existence of a probability measure $\mathbf{Q}_N$ on $(\Omega, \mathcal{F})$ such that

$$\begin{aligned} \left. \frac{d\mathbf{Q}_N}{d\mathbf{P}} \right|_{\mathcal{F}_t} = \exp \left\{ \int_{\tau}^{t \wedge T_N} \int g \gamma^{-1}(s) \mathbf{I}(g(s) \neq 0) dM^K(s, y) \right. \\ \left. - \frac{1}{2} \int_{\tau}^{t \wedge T_N} \int g^2 \gamma^{-1}(s) \mathbf{I}(g(s) \neq 0) K_s(dy) ds \right\}. \end{aligned}$$

For brevity's sake we rather write $\mathcal{E}(t \wedge T_N)$ than the above. On this account, $K_{\cdot \wedge T_N}$ satisfies the martingale problem (MP) $[\gamma_N, a_N, b_N, 0]$ instead of (MP) $[\gamma, a, b, g]$, where we set $f_N := f \cdot \mathbf{I}(\tau < t \leq T_N)$. Moreover, for $s \in (a, b]$, $y \in C^s$, the symbol $\mathcal{M}[s, y]$ denotes the mapping of the set of functions $\{m : (\tau, \infty) \rightarrow M_F(C)\}$ into itself and is defined as follows: i.e., $\{\mathcal{M}[s, y]m\}_t(F)$ is equal to $m_t(F)$ if $t < s$, or is equal to $m_t(\{y' \in F : (y')^s \neq y\})$ if $t \geq s$.

Let us now introduce assumptions for our principal results.

(A.1) $g : [\tau, \infty) \times \Omega \times C \rightarrow \mathbf{R}$ is a $(\mathcal{F}_t \times \mathcal{C}_t)^*$ -predictable process such that $g\gamma^{-1} \cdot \mathbf{I}(g \neq 0)$ is locally bounded.

(A.2) For any predictable function f on $[\tau, \infty) \times I \times C^* \times \Omega$, the counting measure n^* satisfies

$$\mathbf{P} \int_{C^*} n^*((s, t] \times I) G_t(dx) = m(C^*)(t - s)$$

where G_t is a marked historical process corresponding to K and N_t is the martingale measure associated with G_t (cf. §7 for details).

(A.3) There exists a random measure Λ_φ on $(\tau, \infty) \times C$ such that

$$\int \int_{C(\infty)} f(s, y) \Lambda_\varphi(ds \otimes dy) = \int_{a+}^b \int_C f(s, y) \lambda_s[\varphi](dy) ds$$

holds for any suitable predictable function f .

(A.4) $\Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1}$ is uniformly bounded in s , K_s -a.e. y , $\mathbf{Q}_N$ -a.s.

(A.5) There exists some constant $C_0 (> 0)$ such that

$$\int \int_{C(t)} \Psi(s, y)^2 \mathcal{E}(t \wedge T_N)^{-2} \gamma(s, y) K_s(dy) ds \leq C_0$$

holds $\mathbf{Q}_N$ -a.s., for all $t \geq \tau$.

Note that we shall assume (A.1)-(A.5) hereafter all through the whole paper.

7 Stochastic Integration Formulae : Main Results

The followings are our main results in this paper. The first one is a finite dimensional version of Evans-Perkins type stochastic integration by parts formula. Let K be a predictable measure-valued process whose law is specified by a general martingale problem (MP)[$\tau, K_\tau, \gamma, a, b, g$].

Theorem 2 (cf. [Dk98b]) *Assume that $\Phi : C(M_F(C)) \rightarrow R$ is a cylinder function with bounded representing function $\varphi : [M(C)]^k \rightarrow R$ and base $\tau < t(1) < \dots < t(k)$, such that*

$$|\Delta\varphi(\alpha, \beta)| \leq c_0 \sum_j \beta_j(C)$$

for some positive constant c_0 , for all $\alpha, \beta = (\beta_j) \in [M(C)]^k$. Then for $t > \tau$

$$P \left\{ \Phi(K) \int \int_{C(t)} \Psi(s, y) dM^K(s, y) \right\} = P \int \int_{C(t)} Pr[\Phi](s, y) \Psi(s, y) \gamma(s, y) K_s(dy) ds$$

holds where Ψ is a bounded $(\mathcal{C}_t \times \mathcal{F}_t)_{t \geq \tau}$ -predictable function, K_t is a GHP, and $Pr[\Phi]$ is a predictable function determined by (9) in accordance with the given Φ .

Remark 1. The assertion of the above theorem is quite similar to Theorem 2.4(p.1785, §2, [EP95]).

Theorem 3 (Stochastic Integration By Parts) *Let $F \in U(M_F(C))$. If Ψ is an element of $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$, then for all $t > s$,*

$$\begin{aligned} P \left\{ F(K) \int \int_{C(t)} \Psi(s, y) dM^K(s, y) \right\} \\ = P \int \int_{C(t)} Pr[F](s, y) \gamma(s, y) \Psi(s, y) K_s(dy) ds. \end{aligned} \quad (10)$$

Remark 2. Note that it is not hard to extend the assertion in Theorem 2 to the case of a more general functional $F(K)$. As a matter of fact, once the integral formula as given in Theorem 2 is established, it is a kind of routine work to generalize it (cf. §3, [Dk98a]). We shall refer to this generalization in §9.

Theorem 4 (A Variant of Evans-Perkins Type Formula) *Let $F \in U(M_F(C))$.*

$$F(K) = P[F(K)] + \int_{\tau+}^{\infty} \int Pr[F](s, y) dM^K(s, y) \quad (11)$$

where $Pr[F](s, y)$ is a $\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$ -measurable version (relative to P_T) of

$$P_T \left[\int_{C(M_F(C))} \Delta F(K, h) Q(s, y^{s-}; dh) / (\mathcal{D} \times \mathcal{F})_T \right].$$

8 Marked Historical Processes and Girsanov-Dawson-Perkins Theorem

Set $I = [0, 1]$, $E^* = C \times I$ and $C^* = C(\mathbf{R}_+, E^*)$, and let $\mathcal{C}^*$ (resp. $\mathcal{C}_t^*$) be the Borel σ -field (resp. the canonical filtration) of C^* . Put $x = (y, n) \in E^*$. Let G be the corresponding counterpart historical process of K starting at (τ, μ) , defined on the stochastic basis $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbf{P}^*)$. Suppose that $\varphi : (\tau, \infty) \times C \times \Omega \rightarrow I$ be an element of $\mathcal{P}(\mathcal{C}_t \times \mathcal{H}_t)$. Given any cadlag function $n : \mathbf{R}_+ \rightarrow I$, we can construct a σ -finite counting measure n^* on $\mathbf{R}_+ \times I$ by assigning an atom of mass one to each point (s, z) such that $n(s) - n(s-) = z \neq 0$. Put

$$A(t, x, \omega) := n^*(\{(s, z) \in [\tau, t] \times I; \varphi(s, y, \omega) > z\}) \quad (12)$$

and $B(t, x, \omega) = \mathbf{I}\{A(t, x, \omega) = 0\}$. Then we can define an $M_F(C)$ -valued process $K[\varphi](t)$ by

$$K[\varphi; J](t) := \int_{C^*} \mathbf{I}\{J\}(y) B(t, x) G_t(dx). \quad (13)$$

Put

$$I_1(\varphi, N) = \int \int_{C^*(t)} \varphi(s, y) dN(s, x), \quad \text{and} \quad I_2(\varphi, G) = \int \int_{C^*(t)} \gamma(s, y) \varphi(s, y)^2 G_s(dx) ds$$

with $C^*(t) = (\tau, t] \times C^*$. Then we define

$$\Lambda[\varphi](t) := \exp \left\{ I_1(\varphi, N) - \frac{1}{2} I_2(\varphi, G) \right\}. \quad (14)$$

Note that $\Lambda[\varphi](t)$ is a $\mathcal{H}_t$ -martingale. The new probability space $(\Omega, \mathcal{H}, \mathbf{P}^*[\varphi])$ is defined by $\mathbf{P}^*[\varphi]\{F\} := \mathbf{P}^*\{F \cdot \Lambda[\varphi](t)\}$ (cf. [Dk98a]) for any $F \in b\mathcal{H}_t$ with

$$\mathcal{H} := \bigvee_{t \geq \tau} \mathcal{H}_t \quad (15)$$

(see Theorem 2.1(pp.125-126) and Theorem 2.3b(p.127), [EP94]). It is easy to show the following proposition if we apply Dawson's Girsanov theorem [D93] (see also [P95]).

Proposition 4 (cf. Theorem 5.1, p.1798, [EP95]) *The law of $K[\varphi]$ under $P[\varphi]$ is equivalent to the law of K under P .*

9 Sketch of Proofs of Main Theorems

§9.1 Generalization of the Cylinder Function Case: Proof of Theorem 3

As mentioned in Remark 2 of §7, the essential part of an extension of the Evans-Perkins type integration formula is compressed into the study on its finite dimensional case, namely, Theorem 2. The general case easily follows from a kind of routine work [Dk98a]. We define a real valued function L^* on $C(M_F(C))$ by

$$L^*[g] := \int_{T_0} g(t, C) L(dt) = \langle L, g(\cdot, C) \rangle. \quad (16)$$

In connection with the measure L (see §2), we introduce the finite measure $L(l) \equiv L(l, dt)$ which concentrates its mass on $\{t^{(l)}(j); 0 \leq j \leq k[l]\}$ (cf. [Dk98a, p.5]). We have $(L^* \circ W[l])[g] = \langle L(l), g(\cdot, C) \rangle$ for $g \in C(M_F(C))$. Recall that

$$\int g(t, C) Q(s, y; dg) = \int \xi(C) R(s, t, y; d\xi) = 1$$

holds (cf. Lemma 3, [Dk99a]) with ease for $s < t$ from Lemma 3.4(pp.41-43), [DP91]. Then it is easy to verify the followings:

$$\mathbf{P} \int \int_{C(t)} \{Q(s, y^{s-}) L^*[g]\} K_s(dy) ds = \lim_{l \rightarrow \infty} \mathbf{P} \int \int_{C(t)} \{Q(s, y^{s-}) (L^* \circ W[l])[g]\} K_s(dy) ds$$

holds with $g \in C(M_F(C))$ for all $t > \tau$, and

$$\begin{aligned} & \mathbf{P} \int \int_{C(t)} Pr[F](s, y) Z(s, y) K_s(dy) ds \\ &= \lim_{l \rightarrow \infty} \mathbf{P} \int \int_{C(t)} Pr[F \circ W[l]](s, y) Z(s, y) K_s(dy) ds. \end{aligned} \quad (17)$$

holds for all $t > \tau$ if $Z \in \mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$. Since, for each $n \geq 1$, $\mathbf{P}\{K_t(C)^n\}$ is uniformly bounded on compact intervals, we can readily deduce that $\mathbf{P}\{(L^* \circ W[l])[K]^n\}$ is bounded in l for each $n \geq 1$. Moreover,

$$\mathbf{P} \left\{ F(K) \int \int_{C(t)} \Psi(s, y) dM(s, y) \right\} = \lim_{l \rightarrow \infty} \mathbf{P} \left\{ (F \circ W[l])(K) \int \int_{C(t)} \Psi(s, y) dM(s, y) \right\}.$$

To complete the extension discussion in this section we have only to observe that $F \circ W[l]$ satisfies all the conditions of Theorem 2 (cf. Lemma 22, pp.9-10, [Dk98a]). Thus we have a finite dimensional special case of stochastic integration by parts formula related to historical processes as far as Proposition 4 in §8 is valid. Hence, combining the above results, we obtain

$$\begin{aligned} \mathbf{P} \left\{ F(K) \int \int_{C(t)} \Psi(s, y) dM \right\} &= \lim_{l \rightarrow \infty} \mathbf{P} \left\{ (F \circ W[l])(K) \int \int_{C(t)} \Psi(s, y) dM \right\} \\ &= \lim_{l \rightarrow \infty} \mathbf{P} \int \int_{C(t)} Pr[F \circ W[l]] \gamma(s, y) \Psi(s, y) K_s(dy) ds \\ &= \mathbf{P} \int \int_{C(t)} Pr[F](s, y) \gamma(s, y) \Psi(s, y) K_s(dy) ds, \end{aligned}$$

which concludes Theorem 3.

§9.2 Stochastic Integration by Parts: Proof of Theorem 2

Since the complete proof is longsome and tiresome, computation in details will be sacrificed for the sake of simplicity and clearness. The basic idea is due to §7 in [Dk99a].

Thanks to (A.1), it suffices to verify the integral formula for a special $\{\gamma_N, a_N, b_n, 0\}$ -historical process $K_{\cdot \wedge T_N}$ under $\mathbf{Q}_N$ instead of the generalized K (GHP) with $\mathbf{P}$. Indeed, since $d\mathbf{P} = \mathcal{E}(t \wedge T_N)^{-1} d\mathbf{Q}_N$, what we have to show is as follows:

(The Modified Stochastic Integration By Parts Formula)

$$\begin{aligned} & \mathbf{Q}_N \left\{ \mathcal{E}(t \wedge T_N)^{-1} \cdot \Phi(K_{\cdot \wedge T_N}) \int \int_{C(t)} \Psi(s, y) dM(s, y) \right\} \\ &= \mathbf{Q}_N \left\{ \mathcal{E}(t \wedge T_N)^{-1} \int \int_{C(t)} Pr[\Phi](s, y) \gamma(s, y) \Psi(s, y) K_{s \wedge T_N}(dy) ds \right\}. \end{aligned}$$

Note that both sides above are well-defined by virtue of (A.4). Notice that Eq.(12)-(14) remains valid even for $\varphi = \Psi \cdot \mathcal{E}^{-1}$. Hence, by the arguments on exponential martingale formalism for the historical process, $\Lambda[\Psi \cdot \mathcal{E}^{-1}](t)$ is a $\mathcal{H}_t$ -martingale and the measure $\mathbf{Q}_N[\Psi \cdot \mathcal{E}^{-1}]$ is given by $\mathbf{Q}_N[\{\cdot\} \Lambda[\Psi \cdot \mathcal{E}^{-1}]]$. Then it follows from Dawson's Girsanov theorem (Proposition 3 in §8) that, for any positive ε ,

$$\mathbf{Q}_N\{\Phi(K_{\cdot \wedge T_N})\} = \mathbf{Q}_N[\varepsilon \Psi \mathcal{E}^{-1}]\{\Phi(K_{\cdot \wedge T_N}[\varepsilon \Psi \mathcal{E}^{-1}])\}.$$

Immediately,

$$\begin{aligned} & \mathbf{Q}_N \left\{ \Phi(K_{\cdot \wedge T_N}) \cdot (\Lambda[\varepsilon \Psi \mathcal{E}^{-1}](t) - 1) \right\} \\ &+ \mathbf{Q}_N \left\{ \left(\Phi(K_{\cdot \wedge T_N}[\varepsilon \Psi \mathcal{E}^{-1}]) - \Phi(K_{\cdot \wedge T_N}) \right) \cdot (\Lambda[\varepsilon \Psi \mathcal{E}^{-1}](t) - 1) \right\} \\ &= \mathbf{Q}_N \left\{ \Phi(K_{\cdot \wedge T_N}) - \Phi(K_{\cdot \wedge T_N}[\varepsilon \Psi \mathcal{E}^{-1}]) \right\}. \end{aligned}$$

For simplicity we denote by I_1 (resp. I_2) the first (resp. second) term at the left hand side of the above equality, and put

$I_3 =$ the right hand side with the minus sign.

Then we find that the convergence

$$\varepsilon^{-1} \cdot (\Lambda[\varepsilon \Psi \mathcal{E}^{-1}](t) - 1) \rightarrow \int \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y), \quad \mathbf{Q}_N - a.s. \quad (\varepsilon \rightarrow 0)$$

is true (cf. Lemma 8, [Dk99a]). Hence we readily obtain

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} I_1 = \mathbf{Q}_N \left\{ \Phi(K_{\cdot \wedge T_N}) \cdot \int \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y) \right\}.$$

Paying attention to the fact that

$$\lim_{\varepsilon \downarrow 0} K^*[\varepsilon \Psi \mathcal{E}^{-1}; C](t) = 0, \quad \mathbf{Q}_N - a.s.,$$

we can show that $\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} I_2 = 0$, as well.

It remains to treat the third term I_3 . In order to discuss the convergence of I_3 divided by ε , we need the following:

Key Lemma (cf. Lemma 12, [Dk99a])

$$\begin{aligned} \mathbf{Q}_N \int \int \left\{ \Phi(\mathcal{M}[s, y] K_{\cdot \wedge T_N}) - \Phi(K_{\cdot \wedge T_N}) \right\} \Lambda_{\Psi, \mathcal{E}^{-1}}(ds \otimes dy) \\ = - \mathbf{Q}_N \int \int Pr[\Phi] \gamma(s, y) \Psi(s, y) \mathcal{E}^{-1}(t \wedge T_N) dK_{s \wedge T_N}(y) ds. \end{aligned}$$

On the other hand, for $\varepsilon > 0$ we have

$$\begin{aligned} \mathbf{Q}_N [\Phi(K[\varepsilon \varphi]) - \Phi(K)] / \mathcal{F} \\ = \varepsilon \cdot e^{-\varepsilon \Lambda_\varphi((\tau, \infty) \times C)} \int \int_{C(\infty)} \{ \Phi(\mathcal{M}[s, y] K) - \Phi(K) \} \Lambda_\varphi(ds \otimes dy) + R(\varepsilon, \Phi, \varphi) \end{aligned} \quad (18)$$

where the residue function R satisfies $|R(\varepsilon, \Phi, \varphi)| \leq o(\varepsilon)$. From (18) we get the convergence

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} I_3 = -\mathbf{Q}_N \int \int_{C(t)} Pr[\Phi] \gamma(s, y) \cdot \Psi \mathcal{E}^{-1} dK_{s \wedge T_N} ds. \quad (19)$$

In fact, a simple application of the above-mentioned Key Lemma yields the required result. To complete the proof, we have only to combine the above results.

§9.3 Cluster Representation Argument: Proof of Key Lemma

For the proof of Key Lemma, although it is very technical, we are based on the cluster representation argument [D93] (see also [DP91]). For the details, we refer to the arguments stated in §8 in [Dk99a]. The following lemmas are merely essential parts of the discussion.

For any $y \in C^s$, $R(s, t, y)$ denotes the canonical measure (cf §5) in the theory of cluster random measures (e.g. [D93], [DP91]). Actually, R is a σ -finite measure such that

$$R(s, t, y; M_F(C)) = r_{s,t}.$$

Here the crucial point is that the total mass $r_{s,t}$ does not depend on y . So $r_{s,t}^{-1} dR(s, t, y)$ becomes a probability measure. It is interesting to note that K_t is a sum of independent nonzero clusters with laws $r_{s,t}^{-1} R(s, t, y; dh)$, conditional on $L[s, t]$ (see §6). Furthermore, conditional on $\mathcal{F}_s$, $L[s, t]$ can be regarded as a Poisson point process with intensity $r_{s,t} \gamma(s) K_s$. This is one of the most important points for the computation in terms of clusters growing from the points of $L[s, t_{l+1}]$ in what follows. We define a measure S by the following equation: for $\forall g \in b\mathcal{B}([M_F(C)]^{k-l} \rightarrow \mathbf{R})$,

$$\begin{aligned} \int g(\eta_{l+1}, \dots, \eta_k) S_{s,y}(d\eta_{l+1} \otimes \dots \otimes d\eta_k) \\ = \int g(h(t_{l+1}), \dots, h(t_k)) \cdot \mathbf{I}\{h(t_{l+1}) \neq 0\} Q(s, y; dh) \end{aligned}$$

where $Q(s, y; dh)$ is a σ -finite measure on $C(M_F(C))$ (cf. Eq.(7) in §5). $S_{s,y}^*$ is the normalization of $S_{s,y}$, given by $dS_{s,y}^* := r_{s,t_{l+1}}^{-1} dS_{s,y}$. Moreover, we define

$$\begin{aligned} \Xi(s; E) &:= \int \int \cdots (k-l) \cdots \int \varphi(K(t_1), \dots, K(t_l), \sum_{i=1}^m \eta_{l+1}^i, \dots, \sum_{i=1}^m \eta_k^i) \\ &\quad \times \bigotimes_{i=1}^m S_{s,y}^*(d\eta_{l+1}^i \otimes \cdots \otimes d\eta_k^i), \end{aligned}$$

where $E = \{y_1, \dots, y_m\} (\neq \emptyset)$.

Take the mass φ as $(\Psi\mathcal{E}^{-1})(s, y)$ at each point y (cf. §6). For simplicity we set

$$\Delta[\Phi](\mathcal{M}; s, y, K) := \Phi(\mathcal{M}[s, y]K_{\cdot \wedge T_N}) - \Phi(K_{\cdot \wedge T_N}).$$

Recall the assumption (A.3). Immediately we can get

$$\begin{aligned} \mathbf{Q}_N &\int \int_{C(\infty)} \Delta[\Phi](\mathcal{M}; s, y, K) \Lambda_{\Psi\mathcal{E}^{-1}}(ds \otimes dy) \\ &= \mathbf{Q}_N \int_{a+}^b \int_C \Delta[\Phi](\mathcal{M}; s, y, K) \lambda_s[\Psi\mathcal{E}^{-1}](dy) ds \\ &= \int_{a+}^b ds \mathbf{Q}_N \left\{ \sum_{y \in L[s, u]} \Delta[\Phi](\mathcal{M}; s, y, K) \cdot (\Psi\mathcal{E}^{-1})(s, y) \right\}. \end{aligned}$$

In the following calculation, we may take much advantage of those concepts such as i) the Markov property of K_t ; ii) the infinite divisibility of the law of historical process; iii) the Poisson nature of the location $L[s, t_{l+1}]$. Hence we can proceed with the computation. In fact,

$$\begin{aligned} &\mathbf{Q}_N \left\{ \sum_{y \in L[s, u]} \Delta[\Phi](\mathcal{M}; s, y, K) \cdot (\Psi\mathcal{E}^{-1})(s, y) \right\} \\ &= \mathbf{Q}_N \left\{ \mathbf{P} \left[\sum_{y \in L[s, u]} \mathbf{P}\{\Delta[\Phi] \cdot \Psi\mathcal{E}^{-1} | \mathcal{F}_s \vee \sigma(L[s, u])\} \middle| \mathcal{F}_s \right] \right\} \\ &= \mathbf{Q}_N \left\{ \mathbf{P} \left[\sum_{y \in L[s, u]} \{\Xi(s; L[s, u] \setminus \{y\}) - \Xi(s; L[s, u])\} \cdot \Psi\mathcal{E}^{-1} \middle| \mathcal{F}_s \right] \right\} \quad (20) \end{aligned}$$

It is easy to see the following lemma.

Lemma 5 *The last expression of (20) is equivalent to*

$$\begin{aligned} &\mathbf{Q}_N \int_C (\Psi\mathcal{E}^{-1})(s, y) \cdot r_{s,t_{l+1}} \gamma(s, y) K_{s \wedge T_N}(dy) \left[\exp(-r_{s,t_{l+1}} K_s(C)) \cdot \right. \\ &\quad \times \sum_{m=0}^{\infty} \frac{1}{m!} \int \int \cdots (m) \cdots \int_{[C]^m} \{\Xi(s; \{y_1, \dots, y_m\}) - \Xi(s; \{y_1, \dots, y_m, y\})\} \cdot \\ &\quad \times (r_{s,t_{l+1}})^m K_s^{\otimes m}(dy_1, \dots, dy_m) \left. \right]. \end{aligned}$$

A simple computation implies that the integral expression in Lemma 5 is also equal to

$$\begin{aligned} \mathbf{Q}_N & \int_C (\Psi \mathcal{E}^{-1})(s, y) \gamma(s, y) K_{s \wedge T_N}(dy) \cdot \left[\int \int \cdots (k-l) \cdots \int_{[M_F(C)]^{k-l}} \right. \\ & \times \mathbf{P}\{\varphi(K(t_1), \dots, K(t_k)) - \varphi(K(t_1), K(t_l), K(t_{l+1}) + \eta_{l+1}, \dots, K(t_k) + \eta_k) | \mathcal{F}_s\} \\ & \times \left. r_{s, t_{l+1}} \cdot S_{s, y^{s-}}^* (d\eta_{l+1} \otimes \cdots \otimes d\eta_k) \right]. \end{aligned} \quad (21)$$

While, taking (7), (8) in §5, the Campbell measure theory, and predictable section argument into consideration, we readily obtain

Lemma 6 *The following equality holds for all s, y :*

$$\begin{aligned} Pr [\Phi](s, y) &= \int \int \cdots (k-l) \cdots \int r_{s, t_{l+1}} \cdot S_{s, y^{s-}}^* (d\eta_{l+1} \otimes \cdots \otimes d\eta_k) \cdot \\ & \times P\{\varphi(K(t_1), \dots, K(t_l), K(t_{l+1}) + \eta_{l+1}, \dots, K(t_k) + \eta_k) - \varphi(K(t_1), \dots, K(t_k)) | \mathcal{F}_s\}. \end{aligned}$$

Therefore, an application of the above assertion with Lemma 5 implies

$$\begin{aligned} & - \mathbf{Q}_N \int \int_{C(t)} Pr[\Phi](\gamma \cdot \Psi \mathcal{E}^{-1})(s, y) dK_{s \wedge T_N} ds \\ &= \int_{\tau+}^t ds \left\{ \mathbf{Q}_N \int_C (-Pr[\Phi]) \gamma \cdot \Psi \mathcal{E}^{-1} dK_{s \wedge T_N} ds \right\} = \int_{\tau+}^t Eq.(21) ds = \int_{\tau+}^t Eq.(20) ds \\ &= \mathbf{Q}_N \int \int_{C(t)} \Delta[\Phi](\mathcal{M}; s, y, K) \Lambda_{\Psi \mathcal{E}^{-1}}(ds \otimes dy), \end{aligned}$$

which completes the proof.

10 Evans-Perkins Type Formula: Proof of Theorem 4

Since $\mathbf{P}[K_t(C)^2]$ is uniformly bounded on compact intervals, our major premise guarantees the finiteness of the quantity $\mathbf{P}[F(K)^2]$. Therefore we can apply Theorem 1 (§3) for $F(K)$ to obtain that

$$F(K) = \mathbf{P}[F(K)] + \int_{\tau}^{\infty} \int_C f(s, y) dM^K(s, y), \mathbf{P} - a.s. \quad (22)$$

holds for some f in $L_{\infty}^2(K, \mathbf{P})$. While, it follows from the covariance formula in the theory of stochastic integration that

$$\begin{aligned} \mathbf{P} & \left[\left(\int \int_{C(\infty)} f(s, y) dM^K(s, y) \right) \left(\int \int_{C(t)} \Psi(s, y) dM^K(s, y) \right) \right] \\ &= \mathbf{P} \left[\int_{\tau}^t \int_C f(s, y) \Psi(s, y) \gamma(s, y) K_s(dy) ds \right] \end{aligned} \quad (23)$$

for all $t > \tau$ and Ψ in $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$. Rewriting the left hand side of Eq.(23) we get

$$\mathbf{P} \left[F(K) \int_{\tau}^t \int_C \Psi(s, y) dM^K(s, y) \right] \quad (24)$$

by employing the predictable representation property (22). Hence we may apply Theorem 3 (§7) to rewrite (24), because the stochastic integration by parts formula is valid for any bounded $(\mathcal{C}_t \times \mathcal{F}_t)$ -predictable functions. So that, from (23)

$$\mathbf{P} \int \int_{C(t)} f(s, y) \Psi(s, y) \gamma(s, y) dK_s ds = \mathbf{P} \int \int_{C(t)} Pr[F](s, y) \Psi(s, y) \gamma(s, y) dK_s ds.$$

On this account, the general theory of Hilbert spaces shows that

$$\mathbf{P} \int_{\tau}^t \int_C \{f(s, y) - Pr[F](s, y)\}^2 \gamma(s, y) K_s(dy) ds = 0.$$

Therefore the uniqueness argument allows us to conclude that $\int \int_{C(t)} f dM$ is equivalent to $\int \int_{C(t)} Pr[F] dM$, $\mathbf{P}$ -a.s. Note that $Pr[F](s, y)$ become null for K_s -a.s. y , for any $s > t$, by its construction, as long as we choose t largely enough for the support of m to be contained in $[\tau, t]$. Consequently, the above integral $\int \int Pr[F] dM$ can be replaced by $\int \int_{C(\infty)} Pr[F] dM$, which completes the proof. This goes quite similarly as in the proof of Theorem 2.5 in [EP95].

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