

On ultraproducts of o-minimal structures

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概要

abstract We consider about ultraproducts of o-minimal structures. Such structures are definably complete locally o-minimal. We try to characterize them.

1. Introduction

At first we recall some definitions and fundamental facts.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is *o-minimal* if every definable subset of M^1 is a finite union of points and intervals.

M is *locally o-minimal* if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is *uniformly locally o-minimal* if for any formula $\varphi(x, \bar{y})$ over $\emptyset$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of points and intervals for any $\bar{b} \in M^n$ where $\varphi(M, \bar{b})$ is the realization set of $\varphi(x, \bar{b})$ in M .

M is *definably complete* if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Example 2. [1], [2]

$(\mathbb{R}, +, <, \mathbb{Z})$ where $\mathbb{Z}$ is the interpretation of a unary predicate, and $(\mathbb{R}, +, <, \sin)$ are definably complete locally o-minimal structures.

Fact 3. [1] *Definably complete local o-minimality is preserved under elementary equivalence.*

Thus we argue in a sufficiently large saturated model $\mathcal{M}$.

We recall the definition of ultraproduct. In this note, we consider ultraproducts of structures only and ultrafilters are always nonprincipal.

Definition 4. Let I be an infinite set and $\mathcal{U}$ be a nonprincipal ultrafilter over I .

And let M_i ($i \in I$) be structures of some fixed language L .

Consider the equivalence relation $\equiv_{\mathcal{U}}$ on the Cartesian product $C = \prod_{i \in I} M_i$ such that for $f, g \in C$, $f \equiv_{\mathcal{U}} g$ if and only if $\{i \in I : f(i) = g(i)\} \in \mathcal{U}$.

We define the *ultraproduct of M_i modulo $\mathcal{U}$* be the set of all equivalence classes of $\equiv_{\mathcal{U}}$, that is,

$$\prod_{i \in I} M_i / \mathcal{U} = \{f_{\mathcal{U}} : f \in \prod_{i \in I} M_i\}.$$

And we recall a fundamental theorem i.e. Łoś' theorem.

Theorem 5. *Let N be an ultraproduct $\prod_{i \in I} M_i / \mathcal{U}$ and let I be the index set. Then ;*

(i) *For any term $t(x_1, \dots, x_n)$ of a language L and elements $f_{\mathcal{U}}^1, \dots, f_{\mathcal{U}}^n \in N$, we have*

$$t_N[f_{\mathcal{U}}^1, \dots, f_{\mathcal{U}}^n] = \langle t_{M_i}[f^1(i), \dots, f^n(i)] : i \in I \rangle_{\mathcal{U}}.$$

(ii) *Given any formula $\varphi(x_1, \dots, x_n)$ of L and $f_{\mathcal{U}}^1, \dots, f_{\mathcal{U}}^n \in N$, we have*

$$N \models \varphi[f_{\mathcal{U}}^1, \dots, f_{\mathcal{U}}^n] \text{ if and only if } \{i \in I : M_i \models \varphi[f^1(i), \dots, f^n(i)]\} \in \mathcal{U}.$$

(iii) *For any sentence φ of L , $N \models \varphi$ if and only if $\{i \in I : M_i \models \varphi\} \in \mathcal{U}$.*

We try to characterize ultraproducts of locally o-minimal structures. At first we consider about ultraproducts of o-minimal structures.

We verify some elementary facts.

Fact 6. *There are ultraproducts of o-minimal structures which are not o-minimal.*

However ultraproducts of o-minimal structures are locally o-minimal, and definably complete, and infinite 1-types are complete by order formulas, that is,

Let M be a model and $\phi(x, \bar{m}) \in L(M)$ be a one-variable formula. Then there is $b \in M$ such that either for any $c > b$, $M \models \phi(c, \bar{m})$ or for any $c > b$, $M \models \neg\phi(c, \bar{m})$ (it is also true for the lower side).

Some people call these property DCTC, that is, definably complete and type complete (they contain local o-minimality in TC).

I show a poor example.

Example 7. *Let L be the language of ordered fields together with a unary predicate $P(x)$.*

Each $L(P)$ -structure $M_n = (\mathbb{R}, \{0, 1, \dots, n\})$ is o-minimal, but their ultraproducts $M_{\mathcal{U}} = (\mathbb{R}_{\mathcal{U}}, \mathbb{N}_{\mathcal{U}})$ is not o-minimal.

We recall some notations and facts.

Definition 8. *Let L be a language containing a binary predicate $<$ to be interpreted as a dense linear order.*

*Some people call ultraproducts of o-minimal structures *ultra-o-minimal* structure.*

*And they call an L -structure M *pseudo-o-minimal* if it is a model of $T^{\text{omin}}(L)$, that is, the collection of L -sentences that hold true in every o-minimal L -structure.*

A.Rennet showed that T^{omin} is strictly strong than DCTC in [4].

2. Structural complexity in ultraproducts of o-minimal structures

At first we recall characterization by H.Schoutens. Before that, we recall some definitions from [15].

Definition 9. Let M be an o-minimal structure.

We define the *dimension* of a nonempty definable set $X \subset M^m$ by

$$\dim X = \max\{i_1 + \cdots + i_m : X \text{ contains an } (i_1, \cdots, i_m)\text{-cell}\}.$$

We assign to each cell C of dimension d the integer $E(C) = (-1)^d$, and given a finite partition $\mathcal{P}$ of a definable set $S \subset M^m$ into cells, we put

$$E_{\mathcal{P}}(S) = \sum_{C \in \mathcal{P}} E(C) = k_0 - k_1 + \cdots + (-1)^d k_d + \cdots + (-1)^m k_m$$

where k_d is the number of d -dimensional cells in $\mathcal{P}$.

We call $E_{\mathcal{P}}(S)$ ($E(S)$) *Euler characteristic of S* .

Theorem 10. [3] *For an ultra-o-minimal structure $\prod_{i \in I} M_i / \mathcal{U}$ given as the ultraproduct of o-minimal structures M_i , a necessary and sufficient condition to be o-minimal is that for each formula φ without parameters, there exists an $N_{\varphi} \in \mathbb{N}$ such that $|E_{M_i}(\varphi)| \leq N_{\varphi}$ for almost all i .*

(In the above, we take absolute values for cells).

Next example by A.Fornasiero is known as that of locally o-minimal structure which has the independence property. This structure is an ultraproducts of o-minimal fields expanded by a binary relation.

Example 11. [8]

There is an ultraproduct of o-minimal structures which has the independence property.

In the summer meeting of model theory this year, I referred to an example of ultraproduct of o-minimal structures that has the tree property of the second kind. After that, A.Tsuboi suggested a more applicable example. We verify his proof here.

Definition 12. A formula $\phi(\bar{x}, \bar{y})$ has TP_2 , that is, *the tree property of the second kind* if there is an array $(\bar{a}_{t,i})_{t,i < \omega}$ such that ;

$$\{\phi(\bar{x}, \bar{a}_{t,i})\}_{i < \omega} \text{ is } k\text{-inconsistent for every } t < \omega \text{ and,}$$

$$\{\phi(\bar{x}, \bar{a}_{t,f(t)})\}_{t < \omega} \text{ is consistent for any } f : \omega \longrightarrow \omega.$$

Lemma 13. Let $\mathcal{M} = (M, <, \cdots)$ be o-minimal and let $X \subset M^n$ be a finite set.

Then $\mathcal{M}_X = (M, <, \cdots, X)$ is o-minimal where X is an interpretation of a predicate symbol.

Proof ;

For $X \subset M^n$, choose $a_1 < \dots < a_k$ such that $X \subset \{a_1, \dots, a_k\}^n$. Then X is $\{a_1, \dots, a_k\}$ -definable. The structure $(M, <, \dots, a_1, \dots, a_k)$ is clearly o-minimal. So $\mathcal{M}_X$ is also o-minimal. ■

Proposition 14. *There is a structure M which is an ultraproduct of o-minimal structures that has the TP_2 .*

Proof ;

We prepare a predicate symbol $E(x, y, z)$ and constant symbols $\{c_n\} (n \in \omega)$. Let $R = (R, <, \dots)$ be an o-minimal structure where the language $L = \{<, \dots\}$. For each $n \in \omega$, we define the expansion R_n of R whose language is the $L_n = (L \cup \{E(x, y, z), c_0, c_1, \dots\})$.

Construct R_n in the following :

- (1) First, let $D = \{d_\eta \mid \eta : n \rightarrow n\} \subset R$ where $d_\eta \neq d_\nu$ if $\eta \neq \nu$.
- (2) $c_i^{R_n} = i$ ($i \leq n$), $c_i^{R_n} = n + 1$ ($i > n$) (for some enumeration of a subset in R_n).
- (3) $E(x, y, c_i)^{R_n}$ is an equivalence relation on D such that $\models E(d_\eta, d_\nu, c_i)$ if and only if $\eta(i) = \nu(i)$.
- (4) $R_n := (R, E, c_0, c_1, \dots)$.

By the proposition above, each R_n is o-minimal, since E is satisfied by finite elements.

Now let $R^* = \prod_{n \in \omega} R_n / \mathcal{U}$ where $\mathcal{U}$ is a non-principal ultrafilter.

Then

- (A) R^* is a definably complete locally o-minimal structure.
- (B) $E^{R^*}(x, y, c_i) (i < \omega)$ are cross-cutting equivalence relations.

In R_n , we can take an array of formulas $\{E(x, d_{\eta(i,j)}, c_i) : i, j < n\}$ satisfying that ;

η is $\eta : n \rightarrow n$ and,

in the i -th row, $\eta(i, j)(i) = j$ for $j < n$.

Thus in every row, $\{E(x, d_{\eta(i,j)}, c_i) : j < n\}$ is 2-inconsistent and,

for any $\nu : n \rightarrow n$, $\{E(x, d_{\eta(i,\nu(i))}, c_i) : i < n\}$ is consistent.

- (C) Since the equivalence relation are uniformly defined, R^* has the TP_2 . ■

The proof above suggests that we can construct ultraproducts of o-minimal structures having other properties, in particular, properties which have finite approximation.

3. Independence in ultraproducts of o-minimal structures

We recall some definitions at first.

Definition 15. Let M be a densely linearly ordered structure and $p(x) \in S_1(M)$.

We say that $p(x)$ is *cut (irrational) over M* if for any $a \in M$, if $a < x \in p(x)$, then there is $b \in M$ such that $a < b < x \in p(x)$, and similarly, if $x < a \in p(x)$, then there is $c \in M$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1(M)$ is *noncut (rational) over M* if $q(x)$ is not a cut type.

Here we consider nonisolated types only.

Definition 16. Let M be locally o-minimal and $p(x) \in S_1(M)$ be noncut.

There are four kinds of noncut types ;

$p(x) \supset \{m < x < a : m < a \in M\}$ or $\{a < x < m : a < m \in M\}$ for some fixed $a \in M$.

Here we call these types *bounded noncut types of a over M* .

$p(x) \supset \{m < x : m \in M\}$ or $\{x < m : m \in M\}$.

We call these types *unbounded noncut types*.

Definition 17. A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with $tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k -inconsistent for some $k \in \omega$.

A formula $\varphi(\bar{x}, \bar{a})$ *forks* over A if $\varphi(\bar{x}, \bar{a}) \vdash \bigvee_{i < n} \psi_i(\bar{x}, \bar{b}_i)$ and each $i < n$, $\psi_i(\bar{x}, \bar{b}_i)$ divides over A .

We argue about forking of 1-variable types in ultraproducts of o-minimal structures.

Fact 18. Let $M_i (i \in I)$ be an o-minimal structures and let $M_U = \prod_{i \in I} M_i/\mathcal{U}$ be an ultraproduct and $A \subset M_U$.

And assume that M_U is $|A|^+$ -saturated.

Then any unbounded noncut type over M_U and bounded noncut type of a over M_U for some $a \in A$, does not fork over A .

Fact 19. Let $M_i (i \in I)$ be an o-minimal structures and let $M_U = \prod_{i \in I} M_i/\mathcal{U}$ be an ultraproduct and $A \subset M_U$.

Assume that for almost all $i \in I$, there are $c_i, d_i \in M_i \setminus A_i$ such that $c_i \equiv_{A_i} d_i$.

Then $c_U < x < d_U$ (or $d_U < x < c_U$) divides over A .

And we can show the next Lemma.

Lemma 20. Let $M_U = \prod_{i \in I} M_i/\mathcal{U}$ where $M_i (i \in I)$ is o-minimal and let $A \subset B \subset M_U$. And assume that M_U is $|A|^+$ -saturated.

If a B -definable set $X \subset M_U$ is cofinal in an A -definable set $Y \subset M_U$, then X does not fork over A .

I can show a few results about forking in ultraproducts of o-minimal structures at present.

Problem 21. *There are many results about forking in o-minimal structures, and more generally, in dp-minimal or VC-minimal structures.*

Under what conditions, and what extent do these properties reflect to forking in ultraproducts of o-minimal structures ?

4. Ultraproducts of expanded fields

We recalled a result by H.Schoutens about the necessary and sufficient condition for ultraproducts to be o-minimal. However, it is difficult to confirm that each structure satisfies that condition.

Some people investigated ultraproducts of expanded real closed fields.

Definition 22. [5]

Let $R \models RCF$ and $(f_i)_{i \in I}$ be an I -indexed sequence of polynomials $f_i \in R[x]$ (x could be a tuple).

We consider the $L_{RCF}(f)$ -structure consisting of $\mathcal{R} = (R^{\mathcal{U}}, (f_n)_{n \in \mathbb{N}}/\mathcal{U})$ where $R^{\mathcal{U}} = \prod_{n \in \mathbb{N}} R_n/\mathcal{U}$ and R_n is expanded R by f_n .

We denote $\tilde{f}(x) = (f_n)_{n \in \mathbb{N}}/\mathcal{U}$ and call $\tilde{f}(x)$ a *pseudopolynomial over $R^{\mathcal{U}}$* .

And let $f(x)$ be a function.

We say that $\tilde{f}(x)$ is a *pseudopolynomial approximation of f* if for all $x \in \text{dom}(f) \subset R$, we have $\tilde{f}(x) = f(x) + \epsilon_f$ for some infinitesimal function ϵ_f .

For example, let $f(x)$ be a real analytic function with Taylor polynomials $(T_n)_{n \in \mathbb{N}}$ defined on the set $\text{dom}(f) \subset \mathbb{R}$. Then $\tilde{f}(x) = (T_n)_{n \in \mathbb{N}}/\mathcal{U}$.

In these argument, Pfaffian functions and Khovanskii's theorem about them are available.

Definition 23. Let R be a definably complete expansion of an ordered field.

And let $f_i : R^n \rightarrow R$ ($i = 1, \dots, s$) be definable function and C^1 .

We say that $(f_1, \dots, f_s)$ is a *Pfaffian chain in R of length s* if $\frac{\partial f_i}{\partial x_j} \in R[\bar{x}, f_1, \dots, f_i]$ for $i = 1, \dots, s$ and $j = 1, \dots, n$.

A definable function $F = (F_1, \dots, F_m) : R^n \rightarrow R^m$ is a *Pfaffian function in R* if $F_l \in R[\bar{x}, f_1, \dots, f_s]$ ($l = 1, \dots, m$) for some Pfaffian chain $(f_1, \dots, f_s)$ in R .

For example, (i) e^x , (ii) e^x, e^{e^x} , (iii) $(x^2 + 1)^{-1}, \arctan x$ are Pfaffian chains.

All Pfaffian functions are analytic.

There is a theorem by A.Fornasiero and T.Servi [7].

Theorem 24. *Let R be a definably complete locally o-minimal expansion (Baire expansion)*

of a field by a family of Pfaffian functions.

Then R is o -minimal.

And there are results by A.Rennet [5].

Theorem 25. Let $\mathcal{R} = \prod_{n \in \mathbb{N}_{>0}} R_n / \mathcal{U}$ where $R_n = \bar{\mathbb{R}}$ and let $\tilde{e}^x$ be a Taylor polynomial approximation.

Then $(\mathcal{R}, \tilde{e}^x)$ is o -minimal.

Remark 26. He argued by means of another approximation of e^x .

By the approximation ; $\lim_{n \rightarrow \infty} (1 + x/n)^n = e^x$, i.e. $f_n(x) = (1 + x/n)^n$ and he proved that $(\mathcal{R}, \tilde{f})$ is o -minimal.

He also considered ultraproducts of expanded fields by iterated functions. And he put a question.

Question 27. If P is a pseudopolynomial approximation of any Pfaffian function, then P is also Pfaffian ? and is $(\mathcal{R}, P)$ o -minimal ?

They paid attention to the fact whether constructed ultraproducts are o -minimal or not.

Problem 28. Can we characterize them ?

Can we characterize ultraproducts of groups, for example, that of expanded $(\mathbb{R}, +, <)$?

References

- [1] C.Toffalori and K.Vozoris, *Note on local o-minimality*, Math.Log.Quart., 55, pp 617–632, 2009.
- [2] T.Kawakami, K.Takeuchi, H.Tanaka and A.Tsuboi, *Locally o-minimal structures*, J.Math.Soc.Japan, vol.64, no.3, pp 783–797, 2012.
- [3] H.Schoutens, *O-minimalism*, J.Symb.Logic, vol.79, no.2, pp 355–409, 2014.
- [4] A.Rennet, *The non-axiomatizability of o-minimality*, 2012, preprint.
- [5] A.Rennet, *Doctor thesis*, 2012.
- [6] D.Marker, *Khovanskii's theorem*, in Algebraic Model Theory, pp 181–193, Kluwer Acad. Publ., 1997.
- [7] A.Fornasiero and T.Servi, *Definably complete Baire structures*, Fund.Math, 209, pp 215–241, 2010.
- [8] A.Fornasiero, *Locally o-minimal structures and structures with locally o-minimal open core*, Annals.P.A.Logic, 164, pp 211–229, 2013.
- [9] S.Shelah, *Strongly dependent theories*, Israel J.Math., vol.204, pp 1–83, 2014.

- [10] P.Simon, *On dp-minimal ordered structures*, J. Symb. Logic, vol.76, no.2, pp 448–460, 2011.
- [11] P.Simon, *Dp-minimality : invariant types and dp-rank*, J. Symb. Logic, vol.79, no.4, pp 1025–1045, 2014.
- [12] S.Cotter and S.Starchenko, *Forking in VC-minimal theories*, J. Symb. Logic, vol.77, no.4, pp 1257–1271, 2012.
- [13] A.Chernikov, *Theories without the tree property of the second kind*, Ann. Pure. and Appl. Logic, vol.165, pp 695–723, 2014.
- [14] P.A.Estevan and I.Kaplan, *Non-forking and preservation of NIP and dp-rank*, Annals.P.A.Logic, 172, pp 1–30, 2021.
- [15] L.van den Dries, *Tame topology and o-minimal structures*, London Math. Soc. Lecture Note Ser, 248, Cambridge University Press, 1998.
- [16] C.C.Chang and H.J.Keisler, *Model theory*, North-Holland, 1990.