

On Beppo Levi spaces

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Let R^n be the n -dimensional Euclidean space. We write $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n), \dots$ for the elements of R^n . The inner product of $x, y \in R^n$ is the number $(x, y) = \sum_{j=1}^n x_j y_j$; the norm of $x \in R^n$ is the number $|x| = (x, x)^{1/2}$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of nonnegative integers α_j , we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \partial/\partial x_j$ for $1 \leq j \leq n$, then $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$. We shall use the following notations of L.Schwarz [5]: $\mathcal{D}(R^n) = \mathcal{D}$, $\mathcal{D}'(R^n) = \mathcal{D}'$. For a positive integer m and $p > 1$, the Beppo Levi space $L_m^p(R^n) = L_m^p$ is defined as follows:

$$L_m^p = \{u \in \mathcal{D}' ; D^\alpha u \in L^p(R^n) \text{ for any } \alpha \text{ with } |\alpha| = m\}.$$

Furthermore $u_k \rightarrow 0 (k \rightarrow \infty)$ in L_m^p means that $u_k \rightarrow 0 (k \rightarrow \infty)$ in $\mathcal{D}'$ and $|u_k|_{m,p} = \sum_{|\alpha|=m} \|D^\alpha u_k\|_p \rightarrow 0 (k \rightarrow \infty)$. We note that L_m^p is contained in L_{loc}^p ([2]).

Our purpose is to investigate the aspects of the space L_m^p .

First we give a remark about the topology in L_m^p .

Remark. The following three conditions are mutually equivalent:

- i) $u_k \rightarrow 0 (k \rightarrow \infty)$ in $\mathcal{D}'$ and $|u_k|_{m,p} \rightarrow 0 (k \rightarrow \infty)$,
- ii) $u_k \rightarrow 0 (k \rightarrow \infty)$ in L_{loc}^p and $|u_k|_{m,p} \rightarrow 0 (k \rightarrow \infty)$,

iii)

$$\left(\int_{|x| \leq 1} |u_k|^p dx \right)^{1/p} + |u|_{m,p} \rightarrow 0 \quad (k \rightarrow \infty).$$

We also note that L_m^p is a Banach space. We shall give some observations on the space L_m^p in case of $m - (n/p) < 0$. We assume $m - (n/p) < 0$. For $u \in \mathcal{D}$, u has the following integral representation ([2], [4]):

$$\begin{aligned} u(x) &= \sum_{|\alpha|=m} a_\alpha \int_0^\infty \xi^\alpha t^{m-1} D^\alpha u(x-t\xi) dt \\ &= \sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y) dy \end{aligned}$$

where ξ is an arbitrary point on the unit sphere. By the Hardy-Littlewood-Sobolev's inequality, we see

$$\begin{aligned} &\left(\int \left| \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y) dy \right|^{p_m} dx \right)^{1/p_m} \\ &\leq \left(\int \left(\int |x-y|^{m-n} |D^\alpha u(y)| dy \right)^{p_m} dx \right)^{1/p_m} \\ &\leq C \|D^\alpha u\|_p, \end{aligned}$$

where $1/p_m = (1/p) - (m/n)$, and hence we have $\|u\|_{p_m} \leq C |u|_{m,p}$.

Therefore for a sequence $\{u_k\}$ in $\mathcal{D}$ it follows from $|u_k|_{m,p} \rightarrow 0$ ($k \rightarrow \infty$) that u_k converges to 0 in L^{p_m} and hence $u_k \rightarrow 0$ ($k \rightarrow \infty$) in L_m^p .

Moreover from Theorem B* in [6] we obtain

$$\begin{aligned} &\left(\int |x|^{-mp} \left| \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y) dy \right|^{p_m} dx \right)^{1/p} \\ &\leq \left(\int |x|^{-mp} \left(\int |x-y|^{m-n} |D^\alpha u(y)| dy \right)^{p_m} dx \right)^{1/p} \end{aligned}$$

$$\leq C \|D^\alpha u\|_p,$$

so that

$$\left(\int |x|^{-mp} |u(x)|^p dx \right)^{1/p} \leq C |u|_{m,p}. \quad (*)$$

Remark. In case of $m - (n/p) < 0$, from (*) we have the following estimate:

$$\left(\int (1+|x|)^{-mp} |u(x)|^p dx \right)^{1/p} \leq C |u|_{m,p} \quad \text{for } u \in \mathcal{S}.$$

In case of $m - (n/p) \geq 0$, the above estimate is not valid. However, in case of $m - (n/p) > 0$ and k integer, we have the following estimates:

$$i) \quad \left(\int (1+|x|)^{-mp} |u(x)|^p dx \right)^{1/p} \leq C(1+r^{m-(n/p)}) |u|_{m,p}$$

for $u \in \mathcal{S}$ and $\text{supp } u \subset \{|x| \leq r\}$.

$$ii) \quad |D^\beta u(0)| \leq C r^{m-(n/p)-|\beta|} |u|_{m,p} \quad \text{for } u \in \mathcal{S}, \text{ supp } u \subset \{|x| \leq r\} \\ \text{and } |\beta| \leq [m-(n/p)].$$

Next we state the following proposition which has independent interest. We denote by e_j the multi-index $(0, \dots, \overset{j}{1}, \dots, 0)$.

Proposition 1. We assume $m - (n/p) < 0$. Let $\{f_\alpha\}_{|\alpha|=m}$ be a family of functions in L^p and assume $D_i f_\alpha = D_j f_\beta$ for any α, β with $|\alpha| = |\beta| = m$ and $\alpha + e_i = \beta + e_j$. Then $D^\alpha F = f_\alpha$ for any α with $|\alpha| = m$ if we put

$$F(x) = \sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha}{|x-y|^n} f_\alpha(y) dy.$$

For $u \in L_m^p$, $D^\alpha v = D^\alpha u$ for any α with $|\alpha| = m$ if we put

$$v(x) = \sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y) dy$$

from Proposition 1. Hence there exists a polynomial P of degree $\leq m-1$ such that $u = v + P$. We note that

$$\left(\int |x|^{-mp} |v(x)|^p dx \right)^{1/p} \leq C |u|_{m,p}.$$

When $m - (n/p) \geq 0$, the aspects of the space L_m^p are rather different

Remark. Let $m - (n/p) \geq 0$ and $[m - (n/p)] = d$.

- i) There exists a sequence $\{\psi_k\}$ in $\mathcal{D}$ such that $\psi_k(x) \rightarrow \infty$ ($k \rightarrow \infty$) for all $x \in \mathbb{R}^n$ and $|\psi_k|_{m,p} \rightarrow 0$ ($k \rightarrow \infty$).
- ii) (see [3]) For any polynomial P of degree $\leq d$, there exists a sequence $\{\phi_k\}$ in $\mathcal{D}$ such that $\phi_k \rightarrow P$ ($k \rightarrow \infty$) in $\mathcal{D}'$ and $|\phi_k|_{m,p} \rightarrow 0$ ($k \rightarrow \infty$).

A general proposition may be formulated as follows.

Proposition 2. (cf [1]) Let m be a positive integer and $p > 1$.

Suppose that u belongs to L_m^p . Then

$$\int |u(x)|^p (1+|x|)^{-mp} (\log(e+|x|))^{-p} dx < \infty$$

if and only if there exists a sequence $\{u_k\}$ in $\mathcal{D}$ such that $u_k \rightarrow u$ ($k \rightarrow \infty$) in L_m^p .

Now we shall study the space L_m^p in case of $m - (n/p) > 0$ and $1, 2, \dots, m-1$. Let $[m - (n/p)] = d$. For $u \in C^\infty(\mathbb{R}^n)$ from the Taylor's formula we have

$$\begin{aligned}
u(x) &= \sum_{|\beta| \leq d} (D^\beta u(0)/\beta!) x^\beta \\
&= \sum_{|\gamma|=d+1} ((d+1)/\gamma!) \int_0^{|x|} (|x|-t)^d x^\gamma D^\gamma u(tx') dt
\end{aligned}$$

where $x' = (x/|x|)$. In order to estimate the right side we establish the following integral inequalities.

Proposition 3. i) Let $m - (n/p) > d$ and h be a nonnegative measurable function on $(0, \infty)$. Then we have

$$\begin{aligned}
& \left(\int_0^\infty r^{-mp+n-1} \left(\int_0^r (r-t)^d t^{m-d-1-((n-1)/p)} h(t) dt \right)^p dr \right)^{1/p} \\
& \leq C \left(\int_0^\infty h(r)^p dr \right)^{1/p}.
\end{aligned}$$

ii) Let $m - (n/p) > d$ and w be a nonnegative continuous function on $\mathbb{R}^n$. Then we obtain

$$\begin{aligned}
& \left(\int |x|^{-mp} \left(\int_0^{|x|} (|x|-t)^d w(tx') dt \right)^p dx \right)^{1/p} \\
& \leq C \left(\int |x|^{-(m-d-1)p} w(x)^p dx \right)^{1/p}.
\end{aligned}$$

It follows from Proposition 3 that

$$\begin{aligned}
& \left(\int |x|^{-mp} \left| u(x) - \sum_{|\beta| \leq d} (D^\beta u(0)/\beta!) x^\beta \right|^p dx \right)^{1/p} \\
& \leq C \sum_{|\gamma|=d+1} \left(\int |x|^{-(m-d-1)p} |D^\gamma u(x)|^p dx \right)^{1/p}
\end{aligned}$$

for $u \in C^\infty(\mathbb{R}^n)$. When in particular u belongs to $\mathcal{D}$, we get

$$\left(\int |x|^{-(m-d-1)p} |D^\gamma u(x)|^p dx \right)^{1/p}$$

$$\leq C \sum_{|\delta|=m-d-1} \left(\int |D^\delta D^\gamma u(x)|^p dx \right)^{1/p}$$

since $(m-d-1)p < n$, and hence we have

$$\left(\int |x|^{-mp} |u(x) - \sum_{|\beta| \leq d} (D^\beta u(0)/\beta!) x^\beta|^p dx \right)^{1/p} \leq C |u|_{m,p}.$$

Thus, if we put $P(x) = \sum_{|\beta| \leq d} (D^\beta u(0)/\beta!) x^\beta$ and $v(x) = u(x) - p(x)$

for $u \in \mathcal{D}$, P and v satisfy the following conditions:

(i) P can be approximated by a sequence in $\mathcal{D}$,

(ii) $D^\alpha v = D^\alpha u$ for any α with $|\alpha| = m$,

(iii) $D^\beta v(0) = 0$ for any β with $|\beta| \leq d$,

$$(iv) \quad \left(\int |x|^{-mp} |v(x)|^p dx \right)^{1/p} \leq C |u|_{m,p}.$$

Next, in order to give the decomposition of u in L_m^p we state

the following proposition about a primitive of functions. We require

several notations. For $x \in \mathbb{R}^n$, Let $L = \{\xi \in \mathbb{R}^n; (\xi, x) = 0\}$ and

$M = L \cap B$, where B is the unit ball, centered at the origin.

Furthermore we put $M_1^x = M + (x/|x|)$, $M_2^x = M + x - (x/|x|)$,

$$D_1^x = \{t(\xi + (x/|x|)); \xi \in M_1^x, 0 \leq t \leq |x|/2\},$$

and

$$D_2^x = \{x - (|x| - t)((x/|x|) - \xi); \xi \in M_2^x, |x|/2 \leq t \leq |x|\}.$$

Proposition 4. We assume that $p > n$. Let $\{f_j\}_{j=1, \dots, n}$ be

a family of functions in L^p and assume $D_i f_j = D_j f_i$ for any i, j .

If we put

$$G(x) = a \sum_{j=1}^n \int_{D_1^x} \frac{y_j}{|y|^n \cos^n \theta_1} f_j(y) dy + \int_{D_2^x} \frac{x_j - y_j}{|x-y|^n \cos^n \theta_2} f_j(y) dy,$$

then $G(x)$ is continuous, $G(0) = 0$, $D_j G = f_j$ for $1 \leq j \leq n$ and

$$\left(\int |x|^{-\sigma p} |G(x)|^p dx \right)^{1/p} \leq C \sum_{j=1}^n \left(\int |x|^{-(\sigma-1)p} |f_j(x)|^p dx \right)^{1/p}$$

for $\sigma > (n/p)$, where θ_1 (resp. θ_2) is the angle between x and y (resp. $-x$ and $y-x$).

Let $u \in L_m^p$. For γ with $|\gamma| = d+1$, we put

$$u^\gamma(x) = \sum_{|\delta|=m-d-1} a_\delta \int \frac{(x-y)^\delta}{|x-y|^n} D^\delta D^\gamma u(y) dy.$$

From $(m-d-1)p < n$, we have $D^\delta u^\gamma = D^\delta D^\gamma u$ for δ with $|\delta| = m-d-1$ by Proposition 1. Furthermore we see

$$\left(\int |x|^{-(m-d-1)p} |u^\gamma(x)|^p dx \right)^{1/p} \leq C \sum_{|\delta|=m-d-1} \|D^\delta D^\gamma u\|_p$$

and $u^\gamma \in L^{p_{m-d-1}}$, where $1/p_{m-d-1} = (1/p) + ((m-d-1)/n)$. We note that $p_{m-d-1} > n$ from $m - (n/p) > d$. For γ with $|\gamma| = d$, we put

$$u^\gamma(x) = a \sum_{j=1}^n \int_{D_1^x} \frac{y_j}{|y|^n \cos^n \theta_1} u^{\gamma+e_j}(y) dy + \int_{D_2^x} \frac{x_j - y_j}{|x-y|^n \cos \theta_2} u^{\gamma+e_j}(y) dy.$$

On account of Proposition 4 it follows that $u^\gamma \in C^0$, $u^\gamma(0) = 0$, $D_j u^\gamma = u^{\gamma+e_j}$ and

$$\left(\int |x|^{-(m-d)p} |u^\gamma(x)|^p dx \right)^{1/p} \leq C \sum_{j=1}^n \left(\int |x|^{-(m-d-1)p} |u^{\gamma+e_j}(x)|^p dx \right)^{1/p}.$$

Repeating this argument, we get the function v satisfying the following conditions:

- (i) $v \in C^d(\mathbb{R}^n)$,
- (ii) $D^\beta v(0) = 0$ for any β with $|\beta| \leq d$,
- (iii) $D^\gamma v = u^\gamma$ for any γ with $|\gamma| = d+1$,

$$(iv) \quad \left(\int |x|^{-mp} |v(x)|^p dx \right)^{1/p} \leq C \sum_{|\gamma|=d+1} \left(\int |x|^{-(m-d-1)p} |u^\gamma(x)|^p dx \right)^{1/p}$$

so that

$$(v) \quad D^\alpha v = D^\alpha u \text{ for any } \alpha \text{ with } |\alpha| = m,$$

$$(vi) \quad \left(\int |x|^{-mp} |v(x)|^p dx \right)^{1/p} \leq C |u|_{m,p}.$$

If we put $P = u - v$, then $P(x) = \sum_{|\gamma| \leq m-1} a_\gamma x^\gamma$. From $D^\beta v(0) = 0$ for any β with $|\beta| \leq d$, it follows that $a_\beta = (D^\beta u(0)/\beta!)$ for any β with $|\beta| \leq d$. Thus we have the following theorem.

Theorem. Let $m - (n/p) \in 0, 1, \dots, m-1$ and $d = [m - (n/p)]$.

Then every function $u \in L_m^p$ has the following unique decomposition:

$$u = P_1 + P_2 + v$$

where $P_1(x) = \sum_{d+1 \leq |\gamma| \leq m-1} a_\gamma x^\gamma$, $P_2(x) = \sum_{|\beta| \leq d} (D^\beta u(0)/\beta!) x^\beta$, $v \in C^d(\mathbb{R}^n)$, $D^\beta v(0) = 0$ for any β with $|\beta| \leq d$,

$$\left(\int (1+|x|)^{-mp} |P_2(x)|^p dx \right)^{1/p} \leq C \left(\left(\int_B |u(x)|^p dx \right)^{1/p} + |u|_{m,p} \right),$$

and

$$\left(\int |x|^{-mp} |v(x)|^p dx \right)^{1/p} \leq C |u|_{m,p}.$$

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