

THREE LECTURES ON FEWNOMIALS AND NOETHERIAN INTERSECTIONS

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The goal of these lectures is to give an introduction to a theory of Neotherian functions (such a theory still is very incomplete). First lecture deal with the theory of fewnomials which could be considered as a real global version of a theory of Neotherian functions. In the second lecture we discuss integration over Euler characteristic which turns out to be very useful. In lecture 3 we will present the first solved (by A.Gabrielov and A.Khovanskii) problem about Noetherian intersections.

LECTURE 1. FEWNOMIALS

The ideology of fewnomials implies that real varieties, defined by “simple” (not too complicated) sets of equations, must have a simple topology. Of course, this is not always true. The fewnomial ideology, however, is helpful in finding a number of rigorous results (see [1–5]).

The classical Bezout theorem states that the number of complex solutions of a set of k polynomial equations in k unknowns can be estimated in terms of their degrees (it equals the product of the degrees). In this lecture we will deal with the real and the transcendental analogues of this theorem: for a wide class of real transcendental equations (including all real algebraic ones) the number of solutions of a set of k such equations in k real unknowns is finite and can be explicitly estimated in terms of the “complexity” of the equations. A more general result involves a construction of a class of transcendental real varieties resembling algebraic varieties.

These results provide new information about real polynomial equations (see Section 1) and level sets of real elementary functions (see Section 2). The whole theory based on a very simple Rolle-Khovanskii theorem (see Section 3). In Section 4 we will consider separating solutions of ordered systems of Pffaf equations and will defined characteristic sequences for such systems. In Section 5 we will discuss the main extimations under extra assumptions connected with compactness and transversality. In Section 6 we will get rid of the extra assumptions.

1. Real algebraic geometry. The topology of geometric objects determined by algebraic equations (real algebraic curves, surfaces, singularities, etc.) gets more and more complex as the degree of the equation increase. As recently found complexity of the topology depends only on the number of monomials contained in the equations rather than on their degrees: the following Theorems 1 and 2 assess the complexity of the topology of geometrical objects in terms of the complexity of equations determining the object.

We begin with the following well-known

Descartes rule. *The number of positive roots of a polynomial in a single real variable does not exceed the number of sign alternation in the sequence of its coefficients (null coefficients are deleted from the sequence).*

Corollary (the Descartes estimate). *The number of positive roots a polynomial is less than the number of its terms.*

A. G. Kushnirenko proposed to call polynomials with a small number of terms *fewnomials*. The Descartes estimate shows that independently of the degree of a fewnomial (which may be as large as we wish) the number of its positive roots is small.

The following Theorems 1 and 2 generalize the Descartes estimate to the case of systems of polynomial equations in multidimensional real space.

Denote by q the number of monomials appearing with nonzero coefficients in at least one of the polynomials of the system.

Theorem 1. *The number of non-degenerate solutions of a systems of k polynomial equations in k positive real unknowns is less than $2^{q(q-1)/2}(k+1)^q$.*

Theorem 2. *The sum of Betti numbers of a non-singular algebraic manifold defined in R^k by a non-degenerate system of polynomial equations is not greater than an explicitly expressed function of k and q . The number of connected components of a singular algebraic variety can also be estimated from above in terms of k and q .*

The known estimates of the sum of Betti numbers and of the number of connected components in Theorem 2, as well as the estimate of the number of roots in Theorem 1 contain an unpleasant factor of order $2^{q^2/2}$. Apparently, these estimates are far from being exact.

The arguments proving Theorems 1 and 2 are not only useful in algebra. Let us state a result related to the theory of elementary functions

2. Level surfaces of elementary functions. We begin with definitions. Here is a list of *principal elementary functions*: the exponent, the logarithm, trigonometric functions ($\sin, \cos, \tan, \cot$) and their inverse functions. The function defined in a domain in R^n which can be represented as a composition of a finite number of algebraic functions and principal elementary functions is called *elementary*. An *elementary manifold* is the transversal intersection of non-singular level surfaces of several elementary functions. A map of degree $\leq m$ of an elementary manifold in R^n is the restriction to the manifold of such a map of R^n into R^k that all its components are polynomials of degree $\leq m$.

Choose a compact subset K in some k -dimensional elementary manifold.

Theorem 3. *In any regular value in R^k of a map of degree m of a k -dimensional elementary manifold, the number of points in the inverse image contained in K is less than cm^r . In this estimate the constant r depends only on the elementary manifold, while the constant C depends on the choice of the set K as well.*

One can define also so-called *non-oscilating elementary functions*. The definition is similar to the definition of elementary functions, but from the list of principal elementary functions one should delete all oscilating functions ($\sin, \cos, \tan, \cot$), keeping their inverse functions. *Complexity of non-oscilating elementary function* is a following a sequence of integral numbers: a number of principal non-oscilating elementary functions used in representation of the function, a number of arithmetic

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operations and a list of degrees of algebraic functions used in this representation. *Complexity of non-oscilating elementary manifold* is the list of complexities of non-oscilating elementary functions used in the system of equations which definite this manifold.

Theorem 4. *In any regular value in R^k of a map of degree m of a k -dimensional elementary non-oscilating manifold, the number of points in the inverse image is finite and can be estimated explicitly via the complexity of the manifold and the degree m .*

3. Separating solutions and Rolle-Khovanskii theorem. Let M be a smooth manifold (possibly disconnected, non-oriented and infinite-dimensional) and let α be a 1-form on it. Of great significance for the sequel is the following definition.

Definition. A submanifold of co dimension one in M is said to be a *separating solution of the Pfaff equation $\alpha = 0$* if

- (a) the restriction of the form α to the submanifold is identically zero;
- (b) the submanifold does not pass through the singular points of the equation (i.e., at each point of the submanifold the form α does not vanish on the tangent space);
- (c) the submanifold is a boundary of a domain in M and its coorientation defined by the form coincides with the coorientation of the domain boundary (i.e., on the vectors, applied at the submanifold points and outgoing from the domain, the form α is positive).

Example. The surface $H = c$ of a non-singular level of the function H is a separating solution of the equation $dH = 0$ (it bounds the domain $H < c$).

A *Pfaff hypersurface* in R^n is a separating solution of the equation $\alpha = 0$ where α is a 1-form in R^n with polynomial coefficients. An algebraic hypersurface is a Pfaff hypersurface (see the example). The Pfaff hypersurface resembles an algebraic one in many ways. Suppose β is the restriction of the 1-form with polynomial coefficients to the Pfaff hypersurface. A separating solution of the equation $\beta = 0$ on the Pfaff hypersurface also possesses properties similar to those of an algebraic manifold. This process may be continued. We obtain a wide class of manifolds resembling algebraic ones. The formal definition of this class can be found in [4].

Here we will dwell on a certain property of separating solutions. For such solutions we have the following multidimensional variant of Rolle's theorem.

Rolle-Khovanskii theorem. *Between two intersection points of a connected smooth curve with a separating solution of a Pfaff equation there is a point of contact i.e., a point at which the tangent vector to the curve lies in hyperplane $\alpha = 0$.*

The proof is especially easy in the case where the curve intersects the separating solution transversally. In this case, at the neighbouring points of intersection, the values of the form α on the tangent vectors orienting the curve have different signs. Therefore, the form α vanishes at a certain intermediate point.

To demonstrate the significance of the usual Rolle's theorem, we consider a simple transcendental generalization of the Descartes estimate.

Proposition (Laguerre). *The number of real roots of a linear combination of exponents $\sum_{i=1}^q \lambda_i \exp(a_i t)$ is less than the number of exponents q .*

The Descartes estimate of the number of positive roots of a polynomial follows from the Laguerre proposition by substitution $x = \exp t$. The proposition is proved by induction. Let us divide the linear combination by one of its exponents and differentiate the quotient. The derivative contains fewer exponents. According to Rolle's theorem, the number of zeroes of the function does not exceed the number of zeroes of the derivative plus 1.

Fewnomials theory is something of a multidimensional generalization of this simple argument which instead of the Rolle's theorem uses the Rolle-Khovanskii theorem (unidimensional generalization can be found in [6]).

4. Ordered systems of Pfaff equations, their separating solutions and characteristic sequences. In this section we define separating solutions and characteristic sequences of ordered systems of Pfaff equations.

A cooriented submanifold Γ of codimension 1 in an ambient manifold M is called a separating submanifold if there exists a submanifold with boundary, M_- , of codimension 0 in the ambient manifold M , whose cooriented boundary is the submanifold Γ . Then M_- is called a film spanning the separating submanifold Γ . The closure of the complement to the spanning film M_- is a manifold with boundary, M_+ . The boundary of M_+ coincides with Γ , and its coorientation, as the coorientation of a boundary, is opposite to the coorientation of the submanifold Γ . The intersection of M_- with M_+ is the submanifold Γ , and their union is M .

Proposition 1. *A cooriented submanifold separates a manifold M if and only if there is a function $f: M \rightarrow \mathbb{R}^1$ for which:*

- (a) *the zero level set is nonsingular (i.e., if $f(a) = 0$ then $df(a) \neq 0$), and*
- (b) *the cooriented submanifold coincides with the zero level set $f = 0$ of the function f , cooriented by the form df .*

In one direction this proposition is obvious: the nonsingular zero level set $f = 0$, cooriented by the form df , can be spanned by the film $f \leq 0$. The converse statement also is simple.

A chain of inclusions of submanifolds

$$M = \Gamma^0 \supset \dots \supset \Gamma^k$$

is called a *separating chain* provided for each $i > 0$, the manifold Γ^i is a separating submanifold in the preceding manifold Γ^{i-1} . Each submanifold in a separating chain inherits, in the original manifold M , the composite coorientation.

Proposition 2. *Let $f: N \rightarrow M$ be a map from the manifold N into the manifold M , which is transversal to each submanifold of the separating chain of submanifolds*

$$M = \Gamma^0 \supset \dots \supset \Gamma^k.$$

Then the chain of preimages

$$N = f^{-1}(\Gamma^0) \supset \dots \supset f^{-1}(\Gamma^k),$$

in which each of the manifolds has the induced coorientation in the preceding, is a separating chain in N .

Proof. The set $f^{-1}(\Gamma^i)$ is a submanifold of codimension i in N . This fact follows from the transversality of f to Γ^i . If $\Gamma_-^{i-1} \subset \Gamma^{i-1}$ is a spanning film for the submanifold Γ^i , then $f^{-1}(\Gamma_-^{i-1})$ is a spanning film for the submanifold $f^{-1}(\Gamma^i)$.

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Example. Consider the chain of inclusions of linear spaces

$$L = L^0 \supset \dots \supset L^k,$$

in which L is a k -dimensional space with coordinate functions $x_1, \dots, x_k$ and each L^i is a subspace of co dimension i in L determined by the equations

$$x_k = \dots = x_{k-i+1} = 0,$$

cooriented in the preceding space L^{i-1} as the boundary of the half-space $x_{k-i+1} \leq 0$. The chain is a separating chain having k separating manifolds.

The following proposition shows that an arbitrary separating chain with k separating sub manifolds is the preimage under a transversal map of the separating chain in the previous example.

Proposition 3. *A chain consisting of k embedded submanifolds of the manifold M is separating if and only if there is a sequence of k functions $f_k, \dots, f_1$ such that:*

(a) *for each natural number $i \leq k$ the common zero level set of the first i functions is nonsingular (i.e., if*

$$f_k = \dots = f_{k-i+1} = 0,$$

then the differentials $df_k, \dots, df_{k-i+1}$ are linearly independent),

(b) *the i th manifold in the chain is the common zero level set of the first i functions (i.e., it is determined by the equations $f_k = \dots = f_{k-i+1} = 0$), cooriented in the preceding manifold as the boundary of the film $f_{k-i+1} \leq 0$.*

Proof. According to Proposition 1, the first manifold in the chain is a nondegenerate zero level set of some function f_k . By the same proposition, the second manifold in the chain is a nondegenerate zero level set of some function f_{k-1} defined on the first manifold of the chain, and so on. To finish the proof, it remains to extend arbitrarily to the whole manifold the functions defined on the submanifolds.

Let $\alpha_1, \dots, \alpha_k$ be an ordered set of 1-forms on M . We say that a decreasing sequence of submanifolds

$$M = \Gamma^0 \supset \dots \supset \Gamma^k$$

is a *separating chain of integral manifolds* (a *separating chain*, for short) for the system of Pfaff equations

$$\alpha_1 = \dots = \alpha_k = 0,$$

if

(1) for each i the submanifold Γ^i is a nonsingular integral submanifold of the system of Pfaff equations

$$\alpha_1 = \dots = \alpha_i = 0,$$

i.e., the restriction to the manifold Γ^i of the forms $\alpha_1, \dots, \alpha_i$ is identically equal to zero, the manifold Γ^i has codimension i , and the forms $\alpha_1, \dots, \alpha_i$ are linearly independent at each point of the manifold Γ^i ;

(2) the manifold Γ^i , equipped with the coorientation in Γ^{i-1} determined by the 1-form that is the restriction to Γ^{i-1} of the form α_i is a separating submanifold.

In other words, the chain

$$M = \Gamma^0 \supset \dots \supset \Gamma^k$$

of inclusions of submanifolds is a separating chain of integral manifolds for the ordered Pfaff system of equations $\alpha_1 = \dots = \alpha_k = 0$, if Γ^0 is equal to M and, for each $i > 0$, Γ^i is a separating solution of the Pfaff equation $\tilde{\alpha}_i = 0$ on the preceding manifold Γ^{i-1} , where $\tilde{\alpha}_i$ is equal to the restriction to Γ^{i-1} of the form α_i .

Two ordered systems of equations

$$\alpha_1 = \dots = \alpha_k = 0 \quad \text{and} \quad \tilde{\alpha}_1 = \dots = \tilde{\alpha}_k = 0$$

are said to be *equivalent* if there exist lower triangular $k \times k$ matrix functions on the manifold, $\varphi_{i,j}$, $\varphi_{i,j} \equiv 0$ for $i < j$, such that, on the main diagonal, $\varphi > 0$, and such that

$$\tilde{\alpha}_i = \sum \varphi_{i,j} \alpha_j.$$

Proposition 4. *Each separating chain of integral submanifolds (together with the chain of spanning films of these submanifolds) of an ordered system of Pfaff equations is a separating chain of integral manifolds (with chain of spanning films) of any equivalent ordered system of Pfaff equations.*

Proof. The restriction of the first i forms of the system to the i th manifold of the separating chain is identically equal to zero. Therefore, the restriction of the first i forms of any equivalent system are also identically equal to zero on that manifold. The restriction of the $(i+1)$ st form of the equivalent system differs from the $(i+1)$ st form of the given system only by a positive multiplier.

The singular points of an ordered system of Pfaff equations

$$\alpha_1 = \dots = \alpha_k = 0,$$

i.e., the points in the tangent space at which the forms $\alpha_1, \dots, \alpha_k$ are linearly dependent, form a closed set. In the complementary open set in the tangent space to each point, the ordered system

$$\alpha_1 = \dots = \alpha_k = 0$$

determines a flag of linear subspaces

$$TM \supset L^1 \supset \dots \supset L^k,$$

in which each subspace L^i has codimension i and is determined by the equations

$$\alpha_1 = \dots = \alpha_i = 0.$$

For each $i > 0$, the space L^i is cooriented in the preceding space L^{i-1} , the coorientation being given by the restriction of the form α_i to the space L^{i-1} . For each i , the space L^i is equipped with the composite coorientation in the tangent space to the manifold.

The characteristic chain of an ordered system of Pfaff equations

$$\alpha_1 = \dots = \alpha_k = 0$$

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is by definition the system of decomposable forms $\beta_1, \dots, \beta_k$ defined by

$$\begin{aligned}\beta_1 &= \alpha_1, \\ \beta_2 &= \alpha_2 \wedge \alpha_1, \\ &\vdots \\ \beta_k &= \alpha_k \wedge \dots \wedge \alpha_1.\end{aligned}$$

The form β_i gives the composite coorientation on the space L^i in the tangent space to the manifold.

For equivalent Pfaff systems,

- (1) the sets of singular points coincide,
- (2) the coorientations of the spaces of flags and distributions of flags coincide on the open set that is the complement to the set of singular points,
- (3) the characteristic sequences differ only by a positive multiplier.

The characteristic sequence of forms has the following properties: the i th form in the sequence has degree i , each following form is divisible by the preceding. On the other hand, each sequence of forms that has the above properties is the characteristic sequence of some ordered system of Pfaff equations, determined uniquely up to equivalence.

A submanifold Γ of codimension k in the manifold M is called a *separating solution of the ordered system of Pfaff equations*

$$\alpha_1 = \dots = \alpha_k = 0$$

if there is a separating chain of submanifolds for that system,

$$M = \Gamma^0 \supset \dots \supset \Gamma^k,$$

in which the last manifold Γ^k is the submanifold Γ .

How do separating solutions of ordered systems of Pfaff equations behave under maps of manifolds? The answer to this question is given by the following:

Proposition 5. *Let $f: N \rightarrow M$ be a map from the manifold N into the manifold M , which is transversal to each submanifold of the separating chain of integral submanifolds*

$$M = \Gamma^0 \supset \dots \supset \Gamma^k$$

of the ordered system of Pfaff equations

$$\alpha_1 = \dots = \alpha_k = 0.$$

Then the manifolds

$$N = f^{-1}(\Gamma^0) \supset f^{-1}(\Gamma^1) \supset \dots \supset f^{-1}(\Gamma^k)$$

form a separating chain of integral submanifolds for the ordered system of Pfaff equations

$$f^*(\alpha_1) = \dots = f^*(\alpha_k) = 0$$

on the manifold N .

The proof follows from Proposition 2.

5. the main estimations under extra assumptions. The central question is the estimate of the number of points in a zero-dimensional separating solution of an ordered system of n Pfaff equations on an n -dimensional manifold.

Here is one such estimate. We omit the technical details connected with the compactification and bringing into general position, and we will assume that all manifolds that we encounter are compact, all equations are nondegenerate, etc. So, consider the following situation:

Let M be a compact n -dimensional manifold and

$$M = \Gamma^0 \supset \dots \supset \Gamma^n$$

a separating chain of submanifolds of maximal length, i.e., consisting of n separating submanifolds (the manifold Γ^n is zero-dimensional). Let $\beta_1, \dots, \beta_n$ be a chain of forms on the manifold M in which the i th form has degree i and gives the composite coorientation on the submanifold Γ^i in the manifold M . (For example,

$$M = \Gamma^0 \supset \dots \supset \Gamma^n$$

is a separating chain of integral submanifolds of an ordered system of Pfaff equations, and $\beta_1, \dots, \beta_n$ is a characteristic sequence of forms for this system.) Assume that:

(1) the form β_n on M has a nondegenerate set of zeroes, Σ_{n-1} , the restriction of the form β_{n-1} to the manifold Σ_{n-1} has a nondegenerate set of zeroes, Σ_{n-2} , etc., the restriction of the form β_1 to the manifold Σ_1 has a nondegenerate set of zeroes, Σ_0 ;

(2) each submanifold in the chain

$$\Sigma_{n-1} \supset \dots \supset \Sigma_0$$

is transversal to each submanifold in the chain

$$\Gamma^n \supset \dots \supset \Gamma^0 = M.$$

Under these conditions the following holds.

Proposition 6. *The number of points in the zero-dimensional manifold Γ^n does not exceed the number of points in the zero-dimensional manifold Σ_0 .*

One can prove this proposition by induction using at each step Rolle-Khovanskii theorem [4]. One can get rid of the compactness and transversality assumptions.

We comment on versions of the definitions and claims in this section. They concern the following situation: in the separating chain of integral manifolds of an ordered Pfaff system some of the manifolds are level sets of simple functions (e.g. a polynomial). Such is the case, for example, when, for the study of a system of Pfaff equations on a noncompact manifold, a simple compactifying equation is added. Then these simple functions may be used to estimate the number of points in a zero-dimensional separating solution. This is the reason for introducing the following versions of the definitions and claims. We note that in the most general situation the separating solutions are level sets (cf. Proposition 3). But in the general situation, the functions one encounters are complicated, and their use to estimate the number of points in a zero-dimensional separating solution in

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the framework of the claims that follow leads to tautologies (namely, the original problem consisted of finding the complexity of the level sets of such functions and the intersections of such level sets; the estimate uses not the functions themselves, but the Pfaff equations that they satisfy).

Let $q_1, \dots, q_k$ be an ordered set consisting of 1-forms and functions on the manifold M . We say that the chain of inclusions of submanifolds

$$M = \Gamma^0 \supset \dots \supset \Gamma^k$$

is a separating chain of integral manifolds for the ordered system of Pfaff equations if the manifold Γ^0 coincides with the manifold M , and for each $i > 0$:

- (1) if q_i is a function, say f_i , then the manifold Γ^i is the zero nonsingular level submanifold of the restriction of the function f_i to the preceding submanifold Γ^{i-1} ,
- (2) if $q_i = \alpha_i$ is a 1-form, then the manifold Γ^i is a separating solution of the Pfaff equation $\alpha_i = 0$ on the preceding manifold Γ^{i-1} .

A submanifold Γ of codimension k in the manifold M is called a *separating solution* of the ordered system of Pfaff equations and functional equations

$$q_1 = \dots = q_k = 0$$

if, for this system, there exists a separating chain of submanifolds

$$M = \Gamma^0 \supset \dots \supset \Gamma^k,$$

the last submanifold being equal to Γ .

We assign to each sequence $q_1, \dots, q_k$, consisting of 1-forms and functions, a corresponding sequence of 1-forms $\alpha_1, \dots, \alpha_k$ as follows: if q_i is a function, then α_i is defined to be the 1-form dq_i ; if q_i is a 1-form, then α_i is defined to be q_i .

Each separating solution of an ordered system of Pfaff equations and functional equations

$$q_1 = \dots = q_k = 0$$

is clearly a separating solution of the corresponding ordered system of Pfaff equations

$$\alpha_1 = \dots = \alpha_k = 0.$$

(The converse claim is not true.)

The *characteristic sequence* of the ordered system of Pfaff equations and functional equations

$$q_1 = \dots = q_k = 0$$

is defined as the sequence $\omega_1, \dots, \omega_k$ consisting of 1-forms and functions on the manifold as follows: if q_i is a function, then ω_i is equal to q_i and if q_i is a 1-form, then ω_i is the i-form $\alpha_1 \wedge \dots \wedge \alpha_i$ (the forms α_j were defined above).

Let

$$M = \Gamma^0 \supset \dots \supset \Gamma^k$$

be a separating chain of integral manifolds for an ordered system of Pfaff equations and functional equations. If the i th equation in the system is a functional equation, then the i th term of the characteristic sequence of the system is a function on whose zero level set lies the integral manifold Γ^i . If the i th equation is a Pfaff equation,

then the i th term of the characteristic sequence is a 1-form giving the composite coorientation of the submanifold Γ^i in the manifold M .

Let M be a compact n -dimensional manifold and let

$$M = \Gamma^0 \supset \dots \supset \Gamma^n$$

be a separating chain of submanifolds of maximal length n . Let $\omega_1, \dots, \omega_n$ be a chain of forms and functions on M such that for each i either ω_i is an i -form giving the composite coorientation of Γ^i or ω_i is a function and Γ^i is the nonsingular zero level set of the restriction of that function to the preceding submanifold Γ^{i-1} (for example: $M = \Gamma^0 \supset \dots \supset \Gamma^n$ is a separating chain of integral manifolds of an ordered system of Pfaff equations and functional equations, and $\omega_1, \dots, \omega_n$ is the characteristic sequence of that system).

Assume that:

- (1) the n th term of the sequence, ω_n (which is either an n -form or a function on the manifold M), has a nondegenerate set of zeroes, Σ_{n-1} ; the restriction of the $(n-1)$ st term of the sequence, ω_{n-1} (which is either an $(n-1)$ -form or a function on the manifold M), to the sub manifold Σ_{n-1} has a nondegenerate set of zeroes, Σ_{n-2} , etc., the restriction of the first term of the sequence, ω_1 (which is either a 1-form or a function on the manifold M), has a nondegenerate set of zeroes, Σ_0 ;
- (2) each submanifold in the chain

$$\Sigma_{n-1} \supset \dots \supset \Sigma_0$$

is transversal to each manifold in the chain

$$\Gamma^n \subset \dots \subset \Gamma^0.$$

Proposition 7. *The number of points in the zero-dimensional manifold Γ^n does not exceed the number of points in the zero-dimensional manifold Σ_0 .*

The proof of this proposition 7 as well as the proof of proposition 6 use Rolle-Khovanskii theorem [4].

6. Estimate of the number of points in a zero-dimensional separating solution of an ordered system of Pfaff equations, via the generalised number of zeroes of the characteristic sequence of the system. Propositions 6 and 7 in are fundamental for the estimate of the number of points in a zero-dimensional separating solution. In order to formulate this estimate, we need to introduce the notion of generalised number of zeroes of the characteristic sequence of an ordered system of Pfaff equations and functional equations, of generalised number of points in a manifold Σ_0 , which are defined only under some nondegeneracy conditions.

A sequence $g_1, \dots, g_n$ consisting of functions and forms on an n -dimensional manifold is called a *complete divisorial sequence* if its i th term is either a form of degree i or a function. The characteristic sequence of an ordered system of n Pfaff equations and functional equations on an n -dimensional manifold is an example of a complete divisorial system. As each term g_j of the sequence is either a form or a function, we can talk about the restriction of the term g_n to the submanifold and about a neighbourhood of this term in the $\mathbb{C}^\infty$ topology. A term g_j of a complete

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divisorial system is said to be nonsingular provided: in the case that g_n is a function, the zero level set of this function is nonsingular; in the case that g_n is an n -form, the set of zeroes of this form is nonsingular (i.e., if the section of the canonical fibre bundle, determined by this form, is transversal to the zero section). In either case, the set $O(g_n)$ of zeroes of the term g_n is a submanifold of codimension 1 or is empty.

There are various functionals on the collection of complete divisorial sequences whose values we will call generalised number of zeroes of the complete divisorial sequence. Each such functional satisfies, by definition, the Axioms 1–4 stated below. As we will see later, each such functional gives rise to an estimate of the number of points in the separating solutions. In [4] one can find a description of a series of such functionals (among them the "virtual number of zeroes" functional which gives the best estimate).

Axiom 1. *A generalised number of zeroes is defined for each complete divisorial sequence on each manifold and is equal to either a nonnegative integer or the symbol $+\infty$. (The convention is that $+\infty$ is greater than any number.)*

We note that a divisorial sequence with an infinite number of zeroes does not lead to meaningful estimates. (The only reason for introducing them is to shorten the statements of some theorems.)

Axiom 2. *A generalised number of zeroes of a 1-form (function) on a manifold is no smaller than the number of sign changes of that 1-form (function).*

Axiom 3. *Assume that a generalised number of zeroes of a complete divisorial sequence $g_1, \dots, g_n$ on an n -dimensional manifold is equal to N . Then in each neighbourhood in the C^∞ topology of the term g_n there exists a nonsingular term $\tilde{g}_n$ such that if the manifold $O(\tilde{g}_n)$ of zeroes of $\tilde{g}_n$ is nonempty, then any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to this manifold does not exceed N .*

In the Axiom 4 that follows, we state a condition that is more stringent than the one in Axiom 3 (which is included in the list only in order to make the understanding of Axiom 4 easier). In Axiom 4 we require that the term $\tilde{g}_n$ referred to in Axiom 3 can be deformed in such a way that the manifold $O(\tilde{g}_n)$ becomes transversal to a given finite collection of submanifolds.

Axiom 4. *Let a generalised number of zeroes of a complete divisorial sequence $g_1, \dots, g_n$ on an n -dimensional manifold be equal to N . Then for each finite collection of submanifolds and for each neighbourhood in the C^∞ topology of the term g_n there exists a nonsingular term $\tilde{g}_n$ such that if the manifold $O(\tilde{g}_n)$ of its zeroes is nonempty, then $O(\tilde{g}_n)$ is transversal to the given collection of submanifolds, and any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ is at most N .*

Remark 1. The axioms 1–4 can be modified. One can allow the dependence of the generalised number of zeroes on some structure (such as a volume form) on the manifold. Then one should modify Axioms 3 and 4: one should require that the manifold $O(\tilde{g}_n)$ of zeroes have a structure for which the generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ does not exceed the number N . The modified generalised number of zeroes thus obtained is useful

in estimating the number of points in separating solutions of ordered systems of Pfaff equations and functional equations on manifolds equipped with this structure. The proof of this fact coincides with the proof given below (cf. Theorems 5 and 6) of the analogous fact for a generalised number of zeroes. In [4] we define the upper number of zeroes of complete divisorial sequences on manifolds equipped with a volume form. This upper number of zeroes satisfies the modified axioms and is useful for estimating the number of points in separating solutions on manifolds equipped with a volume form.

Theorem 5. *On a compact n -dimensional manifold, the number of points in any separating zero-dimensional solution of an ordered system of n Pfaff equations and functional equations does not exceed a generalised number of zeroes of the characteristic sequence of that system.*

In the statement of Theorem 5 “a generalised number of zeroes” means any generalised number of zeroes that satisfies Axioms 1–4. The statement of the theorem is meaningful provided the generalised number of zeroes of the characteristic sequence of the system is finite.

Proof. Induction over the dimension. For 1-dimensional manifolds the theorem is simple, since according to Axiom 2 any generalised number of zeroes of a 1-form on a 1-dimensional manifold is no smaller than the number of sign changes of this 1-form (the case of one functional equation on a 1-dimensional manifold is obvious: the number of nondegenerate zeroes of a function is no greater than the number of sign change of this function; according to Axiom 2 any generalised number of zeroes of a function is no smaller than the number of sign changes of this function). Assume that Theorem 1 is proved for all compact $(n - 1)$ -dimensional manifolds. Let

$$M = \Gamma^0 \supset \dots \supset \Gamma^n$$

be a separating chain of integral manifolds for the ordered system

$$q_1 = \dots = q_n = 0$$

of Pfaff equations and functional equations on the manifold M . Let a generalised number of zeroes of the characteristic sequence $g_1, \dots, g_n$ of this system be equal to N . According to Axiom 4, in each neighbourhood of the term g_n in the $\mathbb{C}^\infty$ topology there exists a nonsingular term $\tilde{g}_n$ such that the manifold $O(\tilde{g}_n)$ of the zeroes of $\tilde{g}_n$ is transversal to the chain of submanifolds

$$\Gamma^1 \supset \dots \supset \Gamma^{n-1}$$

and any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ does not exceed N . We will show that the number of intersection points of the manifold $O(\tilde{g}_n)$ and the curve Γ^{n-1} does not exceed N . If the manifold $O(\tilde{g}_n)$ is empty then there is nothing to prove. Otherwise, the $(n - 1)$ -dimensional manifold $O(\tilde{g}_n)$ is transversal to the intersection of the chain of submanifolds

$$\Gamma^1 \supset \dots \supset \Gamma^{n-1}.$$

Therefore the set $\Gamma^{n-1} \cap O(\tilde{g}_n)$ is a separating solution of the restriction of the system $q_1 = \dots = q_{n-1} = 0$ to the $(n - 1)$ -dimensional manifold $O(\tilde{g}_n)$. By the

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inductive hypothesis, the number of points in that separating solution does not exceed any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the $(n-1)$ -dimensional manifold $O(\tilde{g}_n)$. We now finish the proof of Theorem 5. Two cases are possible: the last equation $q_n = 0$ may be either a Pfaff equation, or a functional equation. In the first case, the set Γ_n is a separating solution on the compact curve Γ^{n-1} (the curve Γ^{n-1} is compact since it is a submanifold in the compact manifold M), and the form g_n gives the composite coorientation of the set Γ^n in the manifold M . By what was proved, in each neighbourhood in the $\mathbb{C}^\infty$ topology of the form g_n there exists a form $\tilde{g}_n$ whose set of zeroes intersects the compact curve Γ^{n-1} in at most N points. Therefore, according to Rolle theorem, the set Γ^n contains at most N points. In the second case, the set Γ^n is the nonsingular zero level set of the restriction of the function g_n to the curve Γ^{n-1} . By what was proved, in each neighbourhood of the function g_n in the $\mathbb{C}^\infty$ topology there exists a function $\tilde{g}_n$ whose set of zeroes intersects the curve Γ^{n-1} in at most N points. It therefore follows obviously that the set Γ^n contains at most N points (if one could choose an $(N+l)$ st point in the set Γ^n then each sufficiently close function $\tilde{g}_n$ would be equal to zero at a point close to the $(N+l)$ st point, a contradiction). This proves Theorem 5.

Theorem 5 remains valid in the situation of Proposition 7 from which the additional transversality assumptions are omitted. Namely, let M be a compact n -dimensional manifold and let

$$M = \Gamma^0 \supset \dots \supset \Gamma^n$$

be a separating chain of submanifolds of M , of length n . Let $\omega_1, \dots, \omega_n$ be a chain of forms and functions on M such that for each $i > 0$ either ω_i is an i -form giving the composite coorientation of the manifold Γ^i , or ω_i is a function and the submanifold Γ^i is a nonsingular level set of the restriction of this function to the preceding submanifold Γ^{i-1} .

Theorem 5'. *The number of points in the zero-dimensional manifold Γ does not exceed any generalised number of zeroes of the sequence $\omega_1, \dots, \omega_n$.*

The proof repeats the proof of Theorem 5. Here is one more equivalent formulation of this theorem.

Theorem 5''. *Let*

$$M = \Gamma^0 \supset \dots \supset \Gamma^n$$

be a separating chain of integral manifolds of an ordered system of Pfaff equations and functional equations on a compact manifold M . Let $\omega_1, \dots, \omega_n$ be a sequence of forms and functions on M such that, at the points of the manifold Γ^i , the form or function ω_i differs from the i th term of the characteristic sequence of this system only by a positive multiplier. Then the number of points in the separating solution Γ^n does not exceed any generalised number of zeroes of the sequence $\omega_1, \dots, \omega_n$.

Remark. Assume, under the conditions of the theorem, that ω_i is an i -form. The form ω_i is automatically decomposable at the points of the manifold Γ^i . At the other points of M the form need not be decomposable.

Let an ordered system of k Pfaff equations and functional equations

$$q_1 = \dots = q_k = 0$$

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be given on a compact n -dimensional manifold. Let the submanifold Γ^k be a separating solution of this system. How can one estimate the number of nondegenerate roots on Γ^k of the system of equations

$$f_1 = \dots = f_{n-k} = 0,$$

where $f_1, \dots, f_{n-k}$ are functions defined on the manifold M ? To answer this question consider the extended system of n Pfaff equations and functional equations

$$q_1 = \dots = q_k = q_{k+1} = \dots = q_n = 0$$

on the given n -dimensional manifold M , where

$$q_1 = \dots = q_k = 0$$

is the old system of equations, and for each $i = 1, \dots, n - k$, $q_{k+i} = f_i$

Theorem 6. *The number of nondegenerate roots of the system of equations*

$$f_1 = \dots = f_{n-k} = 0$$

on each separating solution of a system of k Pfaff equations and functional equations defined on an n -dimensional manifold M does not exceed any generalised number of zeroes on M of the characteristic sequence of the extended system.

Proof. For $k = n$ Theorem 6 coincides with Theorem 5. We will use induction on the number $(n - k)$. Let

$$M = \Gamma^0 \supset \dots \supset \Gamma^k$$

be a separating chain of integral manifolds for the ordered system of Pfaff equations and functional equations

$$q_1 = \dots = q_k = 0.$$

Let $\tilde{f}_{n-k}$ be any function with nonempty nonsingular zero level set, whose zero level set $O(\tilde{f}_{n-k})$ intersects transversely the chain of submanifolds $\Gamma^0 \supset \dots \supset \Gamma^k$. Then the number of nondegenerate roots of the system

$$f_1 = \dots = f_{n-k-1} = 0$$

on the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$, by the inductive hypothesis, does not exceed any generalised number of zeroes of the restriction to the manifold $O(\tilde{f}_{n-k})$ of the characteristic sequence of the system

$$q_1 = \dots = q_{n-1} = 0.$$

Indeed, the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$ is a separating solution of the restriction to the manifold $O(\tilde{f}_{n-k})$ of the system

$$q_1 = \dots = q_k = 0,$$

and the system of functional equations

$$f_1 = \dots = f_{n-k-1} = 0$$

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contains $n - k - 1$ equations. We now finish the proof of the theorem. Assume that the generalised number of zeroes of the characteristic sequence of the extended system is equal to N , and that the number of nondegenerate roots of the system

$$f_1 = \cdots = f_{n-k} = 0$$

on the manifold Γ^k is greater than N . Choose $(N + 1)$ nondegenerate roots of the system

$$f = \cdots = f_{n-k} = 0$$

on the manifold Γ^k . There exists a neighbourhood of the function $\tilde{f}_{n-k}$ in the $\mathbb{C}^\infty$ topology such that for each function $\tilde{f}_{n-k}$ in that neighbourhood, whose restriction to the manifold Γ^k has a nonsingular zero level set, the restriction of the system

$$f_1 = \cdots = f_{n-k-1} = 0$$

to the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$ of this zero level set has at least $(N + 1)$ nondegenerate roots. This fact follows from the implicit function theorem—for each root of the old system there is a nearby root of the new system. On the other hand, by Axiom 4, there exists a function $\tilde{f}_{n-k}$ in this neighbourhood such that

(1) the zero level set of $\tilde{f}_{n-k}$ is nonsingular and the zero level manifold $O(\tilde{f}_{n-k})$ is transversal to each submanifold in the chain

$$\Gamma^0 \supset \cdots \supset \Gamma^k$$

(2) if the zero level manifold $O(\tilde{f}_{n-k})$ is nonempty, then the generalised number of zeroes of the restriction to that manifold of the characteristic sequence of the system

$$q_1 = \cdots = q_{n-1} = 0$$

does not exceed N .

The manifold $O(\tilde{f}_{n-k})$ cannot be empty because the system

$$f_1 = \cdots = f_{n-k-1} = 0$$

has at least $(N + 1)$ nondegenerate roots on the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$ (and $N + 1 > 0$). Further, by the inductive hypothesis, this system has at most N nondegenerate roots on the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$. This contradiction proves the theorem.

Theorems 5 and 6 apply to compact manifolds only. The estimate of the number of separating solutions on a noncompact manifold reduces to the estimate on compact manifolds. We describe this fact.

Fix a proper positive function p on the manifold M . Consider the Cartesian product $M \times \mathbb{R}^1$ of M with the real line $\mathbb{R}^1$ together with the function $\tilde{p}$ on $M \times \mathbb{R}^1$ defined by $\tilde{p}(x, y) = p(x) + y^2$. Denote by π the projection of $M \times \mathbb{R}^1$ onto the first coordinate: $\pi(x, y) = x$.

Claim. *Let A denote the number of nondegenerate roots of the system*

$$f_1 = \cdots = f_{n-k} = 0$$

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on each separating solution of the ordered system of Pfaff equations and functional equations

$$q_1 = \dots = q_k = 0$$

defined on the (noncompact) n -dimensional manifold M . Then A does not exceed half of the maximum over the set of regular values a of the function p of any generalised number of zeroes of the restriction of the characteristic sequence of the system

$$\pi^* q_1 = \dots = \pi^* q_k = \pi^* f_1 = \dots = \pi^* f_k = 0$$

to the manifold defined by the equation $\tilde{\rho} = a$ in the Cartesian product $M \times \mathbb{R}^1$.

Proof. Denote by M_a the submanifold in $M \times \mathbb{R}^1$ defined by the equation $\tilde{\rho} = a$, where a is a regular value of the function ρ . Denote by π_a the restriction of the projection π to the manifold M_a . Each point of the set $\rho < a$ in M has exactly two inverse images in M_a under the map $\pi_a : M_a \rightarrow M$. For almost all values a the map $\pi_a : M_a \rightarrow M$ is transversal to a fixed separating chain of integral manifolds

$$M = \Gamma^0 \supset \dots \Gamma^k$$

of the system

$$q_1 = \dots = q_k = 0$$

on M . For such values of the parameter a the manifold $\pi^{-1}(\Gamma^k)$ is a separating solution of the restriction of the system

$$\pi^* q_1 = \dots = \pi^* q_k = 0$$

to the manifold M_a . To each nondegenerate root of the system

$$f_1 = \dots = f_{n-k} = 0$$

in the region $\rho < a$ on the solution Γ^k there correspond exactly two nondegenerate roots of the system

$$\pi^* f_1 = \dots = \pi^* f_{n-k} = 0$$

on the manifold $\pi_a^{-1}(\Gamma^k)$. Each finite set $Z \subset \Gamma^k$, for a sufficiently large value of the parameter a , lies in the region $\rho < a$, as the function ρ is positive and proper. Therefore, if there were more non-degenerate roots of the system

$$f_1 = \dots = f_{n-k} = 0$$

on the manifold Γ^k than stated in the proposition this would contradict Theorem 6 applied to the restriction of the system

$$\pi^* q_1 = \dots = \pi^* q_k = \pi^* f_1 = \dots = \pi^* f_{n-k} = 0$$

to the manifold M_a .

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LECTURE 2. INTEGRATION OVER EULER CHARACTERISTIC

The integral over the Euler characteristic turns out to be very useful. For our purposes the most important application is a Complex Variant of the Rolle theorem (see Example 2.1 below). The modern formalization of the theory and a lot of it's of applications to classical problems were found by O.Va. Viro [1] (see also [2, 3] for numerous applications of this technique to the theory of convex polytopes and [4] for applications to the Neotherian intersections). The following presentation based on [5].

The Euler characteristic has some properties of a measure. Let us consider two typical situations.

- (1) Let X_1 , X_2 and $X_1 \cap X_2$ be compact finite subcomplexes of a CW -complex X . Then $X_1 \cup X_2$ is also a compact subcomplex and there is the Mayer-Vietor is exact sequence $\cdots \rightarrow H^i(X_1 \cup X_2) \rightarrow H^i(X_1) \oplus H^i(X_2) \rightarrow H^i(X_1 \cap X_2) \rightarrow H^{i+1}(X_1 \cup X_2) \rightarrow \cdots$ (for example, with integral coefficients). Thus for the Euler characteristic

$$\chi(\cdot) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rk} H^i(\cdot)$$

we get the additivity $\chi(X_1 \cup X_2) + \chi(X_1 \cap X_2) = \chi(X_1) + \chi(X_2)$. Besides, for finite CW -complexes Y and Z , $\chi(Y \times Z) = \chi(Y)\chi(Z)$.

- (2) For another example let X be a smooth connected manifold, X_1 , X_2 and $X_1 \cap X_2$ its open submanifolds of finite type [6]. Then for cohomology with compact support there is the Mayer-Vietoris sequence $\cdots \rightarrow H_c^i(X_1 \cap X_2) \rightarrow H_c^i(X_1) \oplus H_c^i(X_2) \rightarrow H_c^i(X_1 \cup X_2) \rightarrow H_c^{i+1}(X_1 \cap X_2)$ whereof we get the additivity of the Euler characteristic again. The Euler characteristic with respect to cohomology with compact support also multiplies when the direct product is taken.

These examples show that the Euler characteristic looks very much like a measure, and the corresponding integration theory is going to be a full-fledged one in the sense of Fubini-type theorems — because of the multiplicativity of the Euler characteristic. But examples given above immediately point out the main obstacles to construction of the theory of integration over the Euler characteristic. There are two points of trouble: first, in both situations considered above the “measure” $\chi(\cdot)$ is defined for a certain system of subsets of the ambient space closed with respect to finite intersections and unions, whereas for an integration theory this system should be extended to an algebra of subsets. So the question is whether the topological Euler characteristic can be extended to a measure on this algebra. Then, in both situations there are certain conditions of finiteness which make sure

that the Euler characteristic does exist, and this is a very strong restriction for an algebra of measurable sets. A situation when two algebras of χ -measurable sets are not both contained in any χ -measurable algebra, is typical (in particular there can be no universal algebra of χ -measurable sets). At the same time, the χ -measures of a set belonging to both algebras coincide.

The first of the two difficulties can be easily overcome. But the second is a non-avoidable feature of the theory of integral over the Euler characteristic: every application of the theory should be preceded by pointing out a system of sets and verification of its "permissibility". In practice, though, this verification is usually trivial.

1. Construction of integral.

Definition 1. Let X be a topological space.

A) If X is compact then a structure of a finite CW -complex

$$X = \bigcup_{q \in \mathbb{Z}_+} \bigcup_{i \in I_q} e_i^q$$

is said to be a regular CW -structure if the following conditions are satisfied:

- (i) the characteristic mapping of any cell is a homeomorphism of the closed ball onto its closure,
- (ii) the boundary of any cell falls apart into a union of cells of smaller dimension.

The sets representable as a union of cells are said to be cellular.

B) If X is arbitrary then the following data is said to be a regular CW -structure on X :

- (i) a dense injection $X \subset \tilde{X}$, where $\tilde{X}$ is compact,
- (ii) a regular CW -structure on $\tilde{X}$ such that X is an open cellular subset. The definition of cellular subsets of X is evident.

C) We define a finitely-additive measure χ on the algebra of cellular subsets of a space X with a regular CW -structure setting for an open cell $e \subset X$ $\chi(e) = (-1)^{\dim e}$. This measure is said to be the Euler characteristic.

Proposition 1. Let $Z \subset X$ be a subset with compact closure $\bar{Z}$ and suppose that there exists at least one regular CW -structure on X such that Z is cellular. Then the Euler characteristic $\chi(Z)$ does not depend on the choice of a regular CW -structure.

Proof. Suppose at first that Z is compact. If Z is cellular it is easy to see that $\chi(Z)$ in the sense of Definition 1 is the same as $\sum_{i=0}^{\infty} (-1)^i \dim H^i(Z, \mathbb{R})$ and thus does not depend on the choice of a regular CW -structure. If Z is not necessarily compact, let us construct the following series of cellular (with respect to a fixed regular structure) sets $Z^{(n)}$, $n \in \mathbb{Z}_+$: $Z^{(0)} = Z$, $Z^{(i+1)} = \bar{Z}^{(i)} \setminus Z^{(i)}$. Evidently, $\bar{Z}^{(i)}$ are compact and $Z^{(i)} = \emptyset$ for $i \gg 0$, because $Z^{(i+1)}$ consists of cells of dimension smaller than the maximum of dimensions of cells in $Z^{(i)}$. Thus $\chi(Z) = \sum_{i=0}^{\infty} (-1)^i \chi(\bar{Z}^{(i)})$, but for $\bar{Z}^{(i)}$ the invariance of χ has been already proved.

Definition 2. Let X be a topological space with a regular CW -structure, A be an Abelian group. A function $f: X \rightarrow A$ is said to be cellular if $f^{-1}(a)$ is a cellular

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set for any $a \in A$. In particular, f is finite-valued. The integral of f over the Euler characteristic is set to be equal to

$$\int_X f d\chi = \sum_{a \in A} \chi(f^{-1}(a)), \quad a \in A.$$

Definition 3. A function $f: X \rightarrow A$ where X is a topological space, A is an Abelian group, is said to be permissible, if it is cellular with respect to some regular CW -structure on X .

Corollary 1 (from Proposition 1). *For a permissible function $f: X \rightarrow A$ with a compact support its integral over the Euler characteristic $\int_X f d\chi$ does not depend on the choice of a regular CW -structure on X*

Example 1. Let $V \cong \mathbb{R}^n$ be a real vector space, $\mathcal{P}(V)$ be the set of (compact) convex polytopes in V . A function $\alpha: V \rightarrow \mathbb{Z}$ representable as $\alpha = \sum_{i \in I} n_i \mathbb{I}_{A_i}$ where $\#I < \infty$, $n_i \in \mathbb{Z}$, $A_i \in \mathcal{P}(V)$, $\mathbb{I}_A$ denotes the indicator of the set A , is said to be a (convex) chain. The additive group of chains is denoted by $Z(V)$. Evidently chains are permissible and for a written out above

$$\int_V \alpha d\chi = \sum_{i \in I} n_i.$$

In particular the latter integer does not depend on the representation of the chain. As in [2] we call it the degree and denote by $\deg \alpha$.

2. Fubini theorem. In the notations of Sec. 1 let $f: X \rightarrow A$ be a permissible function, $\Phi: X \rightarrow Y$ be a continuous mapping of topological spaces. Suppose that for each fiber $X_y = \Phi^{-1}(y)$, $y \in Y$, a regular CW -structure is given, such that $f_y = f|_{X_y}$ is cellular. Then the "direct image" of f is defined as follows:

$$\Phi_* f: Y \rightarrow A, \quad \Phi_* f: y \rightarrow \int_{X_y} f d\chi.$$

This operation makes sense if $\Phi_* f$ is permissible too.

Definition 1.

- A) A continuous mapping $\Phi: X \rightarrow Y$ of spaces with regular CW -structures is said to be cellular if Φ maps each cell $e \subset X$ surjectively onto a cell $h \subset Y$.
- B) The direct product of spaces with regular CW -structure is defined by taking direct product of cells.
- C) Let $\Phi: X \rightarrow Y$ be a continuous mapping of spaces with regular CW -structures. The following data is said to be a fibration structure for Φ : for each cell $e \subset Y$ a space F_e with a regular CW -structure and a homeomorphism $\Phi_e: \Phi^{-1}(e) \rightarrow F_e \times e$ such that Φ_e and Φ_e^{-1} are cellular.
- D) A cellular function $f: X \rightarrow A$ is said to be compatible with the fibration structure for Φ , if for each cell $e \subset Y$ there is a cellular function $f_e: F_e \rightarrow A$ such that $f|_{\Phi^{-1}(e)} = f_e \circ pr_1 \circ \Phi_e$.

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Theorem 1 (Fubini theorem). *In the situation described in C) and D) of the last definition the function $\Phi_*f: Y \rightarrow A$, $\Phi_*f: y \rightarrow \int_{\Phi^{-1}(y)} fd\chi = \int_{F_e} f_e d\chi$, $y \in e \subset Y$, is cellular and*

$$\int_Y \Phi_*f d\chi = \int_X f d\chi.$$

In other words, integration of a function over the Euler characteristic can be represented as the composition of two operations: first, its integration over the fibers of the mapping, second, integration of the resulting function over the base.

The proof of the theorem is evident.

Clearly, the “direct image” operation is connected with a definite fibration structure in the existence aspect only, while its result does not depend on the fibration structure (Proposition 1).

Definition 2. A permissible function is said to be compatible with the map $\Phi: X \rightarrow Y$ if there exists a fibration structure for Φ such that f is compatible with it.

Example 1. *The Riemann–Hurwitz theorem.*

Let $C, \tilde{C}$ be compact Riemann surfaces of genera g and $\tilde{g}$, respectively, $\pi: \tilde{C} \rightarrow C$ be a holomorphic covering of the degree m , $B \subset C$ be the ramification divisor (a point $b \in \tilde{C}$ comes into B with multiplicity k if $k + 1$ sheets of the covering meet in b). The classic Riemann–Hurwitz theorem asserts that

$$2\tilde{g} - 2 = m(2g - 2) + \deg B.$$

The standard topological proof [7] of the theorem can be interpreted as a computation of certain integral over the Euler characteristic [1]: let $f: \tilde{C} \rightarrow \mathbb{Z}$ be equal to 1 identically then by the Fubini theorem

$$\chi(\tilde{C}) = \int_{\tilde{C}} f d\chi = \int_C \pi_* f d\chi.$$

But $\pi_* f(c) = \#\pi^{-1}(c)$ for $c \in C$, whereof we get the theorem.

Example 2. *Complex Rolle theorem (or Multidimensional Riemann–Hurwitz theorem).*

Complex Rolle Theorem. *Let Z be an analytical n -dimensional space, let B be an open ball in $\mathbb{C}^n$, let $\pi: Z \rightarrow B$ be an analytical map. Assume that π is a finite μ -fold ramified covering (counting the multiplicities). Denote by Z_q a subset in Z which contains all points p such that multiplicity of π at p is $\geq q$. Then*

$$\mu = \sum_{q \geq 1} \chi(Z_q).$$

On one hand this theorem looks like usual Rolle theorem because it estimate the number of preimages of a point under a map in terms of topology of a source space and topology of sets of critical points. On the other hand for $n = 1$ this theorem is a particular case of the Riemann–Hurwitz theorem (see example 1).

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Proof. Let $f_q : Z \rightarrow \mathbb{Z}$ be a function which value at a point x is equal to the number of points $p \in Z$ such that $\pi(p) = x$ and multiplicity of π at p is $\geq q$. By definition $f_1 + \dots + f_\mu = \mu$. So

$$\int_B f_1 d\chi + \dots + \int_B f_\mu d\chi = \int_B \mu d\chi = \mu \cdot 1 = \mu.$$

But by Fubini theorem

$$\int_B f_q d\chi = \chi(\mathbb{Z}_q).$$

3. Radon transform. Let $X = \mathbb{R}P^n$, $X^* = \mathbb{R}P^{n*}$ be the dual projective space, so that points of X^* correspond to hyperplanes in X , and vice versa, $Z \subset X \times X^*$ be the graph of the incidence correspondence: $\{(x, h) | x \in h\}$. We say that a permissible function $f : X \rightarrow A$ permits the Radon transform if the function $\text{res}_Z \circ \text{pr}_1^*(f) : Z \rightarrow A$ is compatible with the fibration $\text{pr}_2 : Z \rightarrow X^*$. If that is the case, the function $f^* : X^* \rightarrow A$,

$$f^* = (\text{pr}_2)_* \circ \text{res}_Z \circ \text{pr}_1^*(f),$$

(so that $f^*(h) = \int_h f d\chi$ for a hyperplane $h \subset X$) is said to be the Radon transform of f (with respect to the integration over the Euler characteristic).

Theorem 1. *If f, f^* permit the Radon transform, then the following identity holds: $f^{**} + f = \int_X f d\chi = \int_{X^*} f^* d\chi$ for even $n = \dim X$ and $f^{**} = f$ for odd n .*

Proof. For $x \in X$ set

$$W_x = \{(y, h) \in X \times X^* | y \in h, x \in h\}.$$

Evidently,

$$f^{**}(x) = \int_{\{h \in X^* | x \in h\}} f^*(h) d\chi(h) = \int_{\{h \in X^* | x \in h\}} \int_{\{y \in X | y \in h\}} f(y) d\chi(y) d\chi(h).$$

By the Fubini theorem, $f^{**}(x) = \int_{W_x} \text{pr}_1^*(f) d\chi$. On the other hand, the projection onto the first factor $\text{pr}_1 : W_x \rightarrow X$ is a fibration over $X \setminus \{x\}$ with the fiber $\mathbb{R}P^{n-2}$ and $\text{pr}_1^*(f)$ is evidently constant on the fibers of pr_1 . Finally, $\text{pr}_1^{-1}(x) \cong \mathbb{R}P^{n-1}$. Applying Fubini theorem again, now to the map $\text{pr}_1 : W_x \rightarrow X$, we get

$$\begin{aligned} f^{**}(x) &= \int_{W_x \setminus \text{pr}_1^{-1}(x)} \text{pr}_1^*(f) d\chi + \int_{\text{pr}_1^{-1}(x)} \text{pr}_1^*(f) d\chi = \\ &= \chi(\mathbb{R}P^{n-2}) \int_{X \setminus \{x\}} f(y) d\chi(y) + \chi(\mathbb{R}P^{n-1}) f(x) = \\ &= \chi(\mathbb{R}P^{n-2}) \int_X f d\chi + (\chi(\mathbb{R}P^{n-1}) - \chi(\mathbb{R}P^{n-2})) f(x). \end{aligned}$$

For even m $\chi(\mathbb{R}P^m) = 1$, for odd m — zero, Q.E.D.

Example 1. *Finite covers of $\mathbb{R}P^2$.*

Let S be a smooth connected compact (real) surface, $\pi: S \rightarrow \mathbb{R}P^2$ be a finite map unramified over $\mathbb{R}P^2 \setminus C$, where C is a smooth (possibly non-connected) curve, having the simple fold over C . In other words, for the branch curve $B \subset S$ $\pi: B \rightarrow C$ is an isomorphism and for any point $b \in B$ there are local parameters x, y in b and u, v in $p = \pi(b) \in C$ such that π can be written locally as $u = x, v = y^2$. For a point $p \in C$ which is not a point of inflexion, we define the index $i(p)$, setting it to be equal to $+1$, if (in the notations above) the tangent vector $\frac{\partial}{\partial v}$ and the curve C lie on the same side of the tangent line $T_p C$ near p , and to (-1) , in the other case.

Theorem 2. *Suppose that $x \in \mathbb{R}P^2$ does not lie on the union of tangent lines to C in all the points of inflexion. Then*

$$\chi(S) = \#\pi^{-1}(x) + \sum_{x \in T_p C} i(p).$$

Proof. Define a function $f: \mathbb{R}P^2 \rightarrow \mathbb{Z}$, setting $f(p) = \#\pi^{-1}(p)$. If the line L is not tangent to C , then $\pi^{-1}(L) \subset S$ is a smooth compact one-dimensional variety, i.e., a disjoint union of loops; consequently, $\chi(\pi(L)) = 0$. Thus f^* vanishes outside the curve dual to C . If L is tangent to C in the points $p_j, j \in J$, which are not points of inflexion, then it checks easily that $\chi(\pi^{-1}(L)) = \sum_{j \in J} i(p_j)$. Now apply Theorem 1.

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LECTURE 3. NOETHERIAN FUNCTIONS

A differential ring of analytic functions in several complex variables is called a ring of Noetherian functions if it is finitely generated as a ring and contains the ring of all polynomials. In this lecture, we will discuss an effective bound on the multiplicity of an isolated solution of a system of n equations $f_i = 0$ where f_i belong to a ring of Noetherian functions in n complex variables. In the one-dimensional case, such an estimate is known and has applications in number theory and in control theory. First, we will present a very simple proof of the one-dimensional estimate. Multi-dimensional case provides a solution of a rather old problem concerning finiteness properties of transcendental functions defined by algebraic partial differential equations. Second, we will discuss the proof in the multi-dimensional case. A presentation of the material is based on [GKh].

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1. The main result. A ring K of analytic functions in an open domain $U \subset \mathbb{C}^n$ is called a ring of Noetherian functions in U if

- 1) K contains the ring $\mathbb{C}[x_1, \dots, x_n]$ of polynomials and is finitely generated over that ring;
- 2) K is closed under differentiation. In other words, for each $f \in K$, all partial derivatives $\partial f / \partial x_i$ belong to K . In particular, K is a Noetherian ring, which is the origin of the notation "Noetherian function" introduced by Tougeron [T].

A set of m functions $\psi = \{\psi_1, \dots, \psi_m\}$ is called a *Noetherian chain of order m* if these functions generate K over $\mathbb{C}[x_1, \dots, x_n]$. A function ϕ in K is called a *Noetherian function of degree β* relative to a Noetherian chain ψ if there exists a polynomial P of degree not exceeding β in $n + m$ variables such that $\phi = P(x, \psi(x))$. The Noetherian chain ψ has *degree not exceeding α* if each partial derivative $\partial \psi_i / \partial x_j$ is a Noetherian function of degree α relative to ψ .

Standard Noetherian arguments allow one to prove that the multiplicity of an isolated intersection of Noetherian functions is bounded by a certain function of discrete parameters n , m , α , and β . As usual, these arguments do not provide any effective method of computation of such function. Such computation was done in the paper [GKh]. The main result of this paper is the following

Theorem 1. *Let $\phi_1, \dots, \phi_n$ belong to a ring K of Noetherian in $U \subset \mathbb{C}^n$. Suppose that all ϕ_i have degree not exceeding β relative to a common Noetherian chain ψ of order m and degree $\alpha \geq 1$. Then the multiplicity of any isolated solution of the system of equations $\phi_1 = \dots = \phi_n = 0$ does not exceed maximum of the following two numbers:*

$$\frac{1}{2}Q((m+1)(\alpha-1)[2\alpha(n+m+2)-2m-2]^{2m+2} + 2\alpha(n+2)-2)^{2(m+n)},$$

$$\frac{1}{2}Q(2(Q+n)^n(\beta+Q(\alpha-1)))^{2(m+n)}, \quad \text{where } Q = en \left(\frac{e(n+m)}{\sqrt{n}} \right)^{\ln n + 1} \left(\frac{n}{e^2} \right)^n.$$

2. Pfaff systems with polynomial coefficients. Rings of Noetherian functions and Noetherian chains can be defined in terms of systems of Pfaff equations. This approach is more geometric, and will be used in the proof of the main result.

Definition 1. An analytic n -dimensional distribution in an open domain $U \subset \mathbb{C}^{n+m}$ is defined by

$$(1) \quad dz_i = \sum_{j=1}^n g_{ij}(x, z) dx_j, \quad \text{for } i = 1, \dots, m$$

where $x \in \mathbb{C}^n$, $z \in \mathbb{C}^m$, and g_{ij} are analytic functions in U . An *integral manifold* of an analytic distribution (1) is a n -dimensional submanifold $\Lambda \subset U$ tangent to (1), i.e.,

$$\left(dz_i - \sum_{j=1}^n g_{ij}(x, z) dx_j \right) \Big|_{\Lambda} \equiv 0, \quad \text{for } i = 1, \dots, m.$$

Locally, an integral manifold can be represented as a graph of an analytic vector-function $z = \psi(x) = (\psi_1(x), \dots, \psi_m(x))$ satisfying

$$(2) \quad \frac{\partial \psi_i}{\partial x_j} = g_{ij}(x, \psi(x)), \quad \text{for } i = 1, \dots, m \quad \text{and} \quad j = 1, \dots, n.$$

A Noetherian chain of order m and degree α is an analytic function $\psi(x)$ satisfying (2) where $g_{ij}(x, z)$ are polynomials in $(x, z) \in \mathbb{C}^{n+m}$ of degree not exceeding $\alpha \geq 1$. A Noetherian function of degree β relative to a Noetherian chain ψ is an analytic function $\phi(x) = P(x, \psi(x))$, where $P(x, z)$ is a polynomial in (x, z) of degree not exceeding β .

3. Theory of fewnomials and local conjectures. Suppose that the system (2) of Pfaff equations is triangular, i.e., that functions $g_{ij}(x, \psi)$ depend only on x and $\psi_1, \dots, \psi_i$ (but do not depend on $\psi_{i+1}, \dots, \psi_m$). Suppose, in addition, that all functions ψ_i and polynomials g_{ij} are real. In this case, ψ is called a Pfaffian chain, and Noetherian functions $P(x, \psi(x))$ where P is a real polynomial are called Pfaffian functions relative to ψ . Real solutions of systems of equations with Pfaffian functions have global finiteness properties resembling the finiteness properties of real algebraic sets (see Lecture 1). These systems, studied in [Kh1, Kh2, Kh3], have many applications in real algebraic geometry, computational complexity, control theory, and model theory. For these finiteness properties, both requirements (that system (2) is triangular and all functions are real) are essential. For example, the simplest Noetherian functions $\sin$ and $\cos$ have infinite number of real zeros, and a simple Pfaffian function $\exp x - 1$ has infinite number of complex zeros.

In the beginning of the 80-ies, I conjectured that *local* finiteness properties remain valid for non-triangular systems, also in the complex domain. Here are three variants of this conjecture, in decreasing order of generality.

Let $\phi_1(x, a), \dots, \phi_l(x, a)$ be a set of l analytic functions in a vicinity of a point $(x_0, a_0) \in \mathbb{C}^{n+k}$. Suppose that, for each value of parameters a , functions $\phi_i(x, a)$, considered as analytic functions in x , belong to a ring K of Noetherian functions. Let us fix a Noetherian chain of order m and degree α in K , and let $\phi_1(\cdot, a), \dots, \phi_l(\cdot, a)$ be Noetherian functions of degree not exceeding β relative to this Noetherian chain, for all values of a .

Conjecture 1. *There exists an explicit function $F(n, m, \alpha, \beta, l)$ with the following property. For any small positive ϵ , there exists a positive δ such that, for any fixed value of a with $|a - a_0| < \delta$, the sum of Betti numbers of the set*

$$\phi_1(x, a) = \dots = \phi_l(x, a) = 0, |x - x_0| < \epsilon$$

does not exceed $F(n, m, \alpha, \beta, l)$.

Conjecture 2. *Let $\phi_1(\cdot, a) = \dots, \phi_n(\cdot, a) = 0$ be a system of n equations in n variables depending on parameters a . Suppose that, for a fixed value of a , functions ϕ_i are Noetherian of degree not exceeding β relative to a Noetherian chain of order m and degree α . The number of isolated solutions of this system converging to x_0 as $a \rightarrow a_0$ can be effectively estimated from above in terms of n, m, α, β . Note that x_0 can be a non-isolated solution of $\phi_1(\cdot, a_0) = \dots = \phi_n(\cdot, a_0) = 0$.*

Conjecture 3. *Multiplicity of an isolated solution of a system of equations with Noetherian functions with a common Noetherian chain can be effectively estimated from above in terms of the number of variables, order and degree of the Noetherian chain, and degrees of the Noetherian functions.*

Theorem 1 below proves Conjecture 3. Conjectures 1 and 2 remain open problems. In fact these two conjectures are equivalent. For Pfaffian functions (triangular

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systems (2), also in the complex domain) proof of Conjecture 2 was given in [G1]. It is important for us that Conjectures 2 and 3 are true in one-dimensional case.

To prove Theorem 1, we need four preliminary steps:

- (1) **One-dimensional case.** For $n = 1$, a Noetherian function is a restriction of a polynomial on a trajectory of a vector field $\partial/\partial x + \sum_i g_i(x, z)\partial/\partial z_i$ with polynomial coefficients. This case is interesting by itself and relatively simple.
- (2) **Gabrielov's reduction to an integrable system.** Union of all solutions of (2) for given polynomials g_{ij} is an algebraic set. Complexity of this set is estimated. Here we will just present Gabrielov's results without proofs.
- (3) **The Milnor fibers.** For a one-parametric deformation of a (possibly non-isolated) intersection, we define the "Milnor fiber" Z_q as follows: For a small nonzero value of parameter of the deformation, Z_q consists of those points where the intersection is isolated and its multiplicity is at least q . Multiplicity of an isolated intersection equals the sum over q of the Euler characteristics of Z_q ("Complex Rolle theorem")
- (4) **Maximal multiplicity in a generic family.** We present an estimate for the maximal value of q such that Z_q is nonempty in a generic family. This estimate allows to bound the number of nonzero terms in the formula for the multiplicity in terms of Euler characteristics, and the values of these nonzero terms.

4. One-dimensional case. The problem in one-dimensional case was first formulated and solved by Nesterenko [N]. His motivation came from number theory, and his results have important applications in this area (see [W]). Later this problem was re-discovered by Risler [R], in connection with non-holonomic dynamics and control theory. He was interested in degree of nonholonomy of control systems. Solution of this problem was given by Gabrielov [G2]. Later, Gabrielov found a new solution [G3], more simple and with a better estimate. Then I strongly simplified Gabrielov's solution, using integration over Euler characteristics. This made solution so simple that it could be generalized to several variables.

Noetherian chains in one-dimensional case are exactly trajectories of vector fields with polynomial coefficients, and Noetherian functions are polynomials restricted to trajectories of such vector fields.

Let $x \in \mathbb{C}$, $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, and let $\gamma = \{z = \psi(x)\}$ be a germ of a trajectory through $0 \in \mathbb{C}^{m+1}$ of a vector field $\xi = \partial/\partial x + \sum_i g_i(x, z)\partial/\partial z_i$, where g_i are germs of analytic functions at $0 \in \mathbb{C}^{m+1}$. Let $P(t, z)$ be a germ of an analytic function at $0 \in \mathbb{C}^{m+1}$, and let $\phi(t) = P(t, \psi(t))$ be a restriction of $P(t, z)$ to γ . Suppose that $\phi(t) \not\equiv 0$, and let μ be the order of a zero of ϕ at $t = 0$. Let $S(t, z, \epsilon)$ be a one-parametric deformation of P , i.e., a germ of an analytic function at $0 \in \mathbb{C}^{m+2}$ such that $S(t, z, 0) = P(t, z)$. We write $S_\epsilon(t, z)$ for $S(t, z, \epsilon)$ considered as a function in $\mathbb{C}^{m+1}$, with a fixed value of ϵ .

Definition 2. For a positive integer q , the Milnor fiber $Z_q(\xi, S)$ of the deformation S relative to the vector field ξ is the intersection of a ball $\|(t, z)\| \leq \delta$ in $\mathbb{C}^{m+1}$ with a set $S_\epsilon = \xi S_\epsilon = \dots = \xi^{q-1} S_\epsilon = 0$, for a small positive δ and a complex nonzero ϵ much smaller than δ . According to [Le], the homotopy type of Z_q depends only on the deformation S and on the vector field ξ . Let $\chi(Z_q)$ be the Euler characteristics

Theorem 2. *Let S be a one-parametric deformation of an analytic function P , and let $Z_q = Z_q(\xi, S)$ be the Milnor fibers of S relative to an analytic vector field ξ . Suppose that P restricted to a trajectory of ξ through 0 has a zero of order $\mu < \infty$ at 0. Let $Q = \max\{q : Z_q \neq \emptyset\}$. Then*

$$(3) \quad \mu = \sum_{q=1}^Q \chi(Z_q).$$

Proof. (See also [G3, Theorem 1].) Let $(t, y_1, \dots, y_m)$ be a system of coordinates in $\mathbb{C}^{m+1}$ where $\xi = \partial/\partial t$, and let π be projection $\mathbb{C}^{m+1} \rightarrow \mathbb{C}^m$ along the t -axis. Let B_r be a closed ball of radius r in $\mathbb{C}^m$ centered at the origin. We can choose a norm $\|\cdot\|$ in $\mathbb{C}^{m+1}$ so that $\{\|(t, y)\| \leq \delta\} = \{y \in B_r, |t| \leq \delta\}$, where $r = r(\delta)$, and projection $\pi : \{S_\epsilon = 0\} \rightarrow B_{r(\delta)}$ is a finite μ -fold ramified covering (counting the multiplicities).

For a small $\delta > 0$ and a small nonzero ϵ much smaller than δ , projection $\pi : Z_q \rightarrow B_r$ is finite. For $y \in B_r$, a set $\pi^{-1}y \cap Z_q$ is finite. Hence its Euler characteristics $\zeta_q(y) = \chi(\pi^{-1}y \cap Z_q)$ equals the number of points in it (not counting multiplicities). Then $\sum_{q=1}^Q \zeta_q(y) \equiv \mu$ does not depend on y . Standard "integration over Euler characteristic" arguments [V] show that

$$\int_{B_r} \zeta_q(y) d\chi = \chi(Z_q), \quad \text{and} \quad \int_{B_r} \sum_{q=1}^Q \zeta_q(y) d\chi = \int_{B_r} \mu d\chi = \mu.$$

This proves (3).

Lemma 1. *Let $c_0, \dots, c_m$ be a generic set of $m+1$ complex numbers. For a deformation $S(t, z, \epsilon) = P(t, z) + \epsilon \sum_{i=0}^m c_i t^i$, the sets Z_q are nonsingular, for $q = 1, \dots, m+1$, and empty for $q > m+1$.*

Proof. This is a special case of Thom's transversality theorem. See also [G3, Lemma 1].

Corollary. *For the deformation in Lemma 1,*

$$(4) \quad \mu = \chi(Z_1) + \dots + \chi(Z_{m+1}).$$

Theorem 3. *Let ξ be a vector field defined by $dz_i/dt = g_i(t, z)$ where g_i are polynomials of degree not exceeding $\alpha \geq 1$, and let P be a polynomial of degree not exceeding $\beta \geq m$. Suppose that P does not vanish identically on the trajectory γ of ξ through the origin. Then the multiplicity μ of a zero of $P|_\gamma$ does not exceed*

$$\frac{1}{2} \sum_{k=0}^m [2\beta + 2k(\alpha - 1)]^{2m+2}.$$

Proof. This follows from (4) and from an estimate [M] of the Euler characteristics of the set Z_q defined by polynomial equations of degree not exceeding $\beta + (q-1)(\alpha-1)$.

5. Gabrielov's reduction to an integrable system. Using arguments from control theory Gabrielov found two following results [GKh].

Lemma 2. Let $x \in \mathbb{C}^n$, $z \in \mathbb{C}^m$, and let $g_{ij}(x, z)$ be analytic functions in $U \subset \mathbb{C}^{n+m}$. The union of all integral manifolds of (1) is an analytic subset of U .

Theorem 4. Let g_{ij} in (1) be polynomials of degree not exceeding $\alpha \geq 1$, and let Y be the union of all integral manifolds of (1). Then Y can be defined by a system of algebraic equations of degree not exceeding

$$(7) \quad d_Y(m, n, \alpha) = \frac{(m+1)(\alpha-1)}{2} [2\alpha(n+m+2) - 2m - 2]^{2m+2} + \alpha(n+2) - 1.$$

6. The Milnor fibers. Let $g_{ij}(x, z)$ be germs of analytic functions at $0 \in \mathbb{C}^{n+m}$, and let $\{z = \psi(x)\}$ be a germ of an integral manifold of (1) through 0. Let $P(x, z) = (P_1(x, z), \dots, P_n(x, z))$ be a germ of an analytic vector-function at 0, and let $\phi_i(x) = P_i(x, \psi(x))$. Let $S(x, z, \epsilon) = (S_1(x, z, \epsilon), \dots, S_m(x, z, \epsilon))$ be a one-parametric deformation of $P(x, z)$, i.e., a germ of an analytic vector-function at 0 $\in \mathbb{C}^{n+m+1}$ such that $S(x, z, 0) = P(x, z)$. We write $S_\epsilon(x, z)$ for $S(x, z, \epsilon)$ considered as a function in $\mathbb{C}^{n+m}$, with a fixed value of ϵ .

Let Y be the union of all integral manifolds of (1). Due to Lemma 2, Y is a germ of an analytic set. For $(x, z) \in Y$, let $\mu_\epsilon(x, z)$ be the multiplicity of the intersection $S_{1,\epsilon}|_\Lambda = \dots = S_{n,\epsilon}|_\Lambda = 0$ at (x, z) , where Λ is an integral manifold of (1) through (x, z) .

Definition 3. For a positive integer q , the *Milnor fiber* Z_q of the deformation S relative to the distribution (1) is the intersection of a closed ball $B_\delta = \{\|(x, z)\| \leq \delta\}$ with a set of those points $(x, z) \in Y$ where $\mu_\epsilon(x, z) \geq q$, for a small positive δ and a complex nonzero ϵ much smaller than δ . According to [Le] and Lemma 3 below, the homotopy type of Z_q depends only on the deformation S , and on the coefficients g_{ij} in (1). Let $\chi(Z_q)$ be the Euler characteristics of Z_q .

Lemma 3. Let W_q be the set of small $(x, z, \epsilon) \in \mathbb{C}^{n+m+1}$ such that $(x, z) \in Y$, and $\mu_\epsilon(x, z) \geq q$. Then W_q is a germ of an analytic set.

Proof. One can choose a system of coordinates (x, y) in the neighborhood of 0 $\in \mathbb{C}^{n+m}$ so that each integral manifold of (1) through a point $(x_0, y_0) \in Y$ is defined by $y = y_0$.

Due to [AGV, Lemma 5.5], the condition $\mu_\epsilon(x_0, y_0) \geq q$ depends only on the Taylor expansion $\check{S}_i$ in x of S_i at (x_0, y_0, ϵ) of order $q-1$. The coefficients of $\check{S}_i$ are

$$(8) \quad \frac{\partial^{|\nu|} S_i}{\partial x^\nu}(x_0, y_0, \epsilon),$$

which are analytic in x_0, y_0, ϵ .

Let $K = \binom{q+n-1}{n}$ be the number of monomials in n variables of degree less than q . Consider $\check{S}_i$ as a vector in $\mathbb{C}^K$. For any multi-index $\nu = (\nu_1, \dots, \nu_n)$, with $|\nu| = \nu_1 + \dots + \nu_n < q$, consider $(x - x_0)^\nu \check{S}_i$ as a vector in $\mathbb{C}^K$, disregarding terms of order q and higher in $x - x_0$.

Condition $\mu_\epsilon(x_0, y_0) \geq q$ means that rank of the set of Kn vectors $x^\nu \check{S}_i$ in $\mathbb{C}^K$ is at most $K - q$. This means vanishing of all $(K - q + 1)$ -minors of a $(K \times Kn)$ -matrix composed of these vectors. As the elements of this matrix are the partial derivatives (8), which are analytic in (x_0, y_0, ϵ) , this, in combination with equations for Y , provides a system of analytic equations for W_q .

Theorem 5. Let $P = (P_1(x, z), \dots, P_n(x, z))$ be a germ of an analytic function at 0 in $\mathbb{C}^{n+m}$. Let $\psi = (\psi_1(x), \dots, \psi_m(x))$ be a germ of an analytic function at 0 in $\mathbb{C}^n$ satisfying (2), and let $\phi_i(x) = P_i(x, \psi(x))$. Suppose that the intersection $\phi_1(x) = \dots = \phi_n(x) = 0$ is isolated at $x = 0$, with the multiplicity μ . Let $S(x, z, \epsilon)$ be a one-parametric deformation of P , let Z_q be the Milnor fibers of S relative to the distribution (1), and let $Q = \max\{q : Z_q \neq \emptyset\}$. Then

$$(9) \quad \mu = \sum_{q=1}^Q \chi(Z_q).$$

Proof. The arguments essentially repeat the arguments in the proof of Theorem 2, except an additional restriction to the set Y of all integral manifolds of (1).

Let a system of coordinates (x, y) in $\mathbb{C}^{n+m}$ be chosen, as in the proof of Lemma 3, so that each integral manifold of (1) through $(x_0, y_0) \in Y$ is defined by $y = y_0$. Let π be projection $\mathbb{C}^{n+m} \rightarrow \mathbb{C}^n$. Then $\pi : \{S_\epsilon = 0\} \cap Y \rightarrow \pi Y$ is a finite μ -fold ramified covering (counting the multiplicities). Let $\zeta_q(y) = \chi(\pi^{-1}y \cap Z_q)$ be the number of preimages of a point y in Z_q , not counting multiplicities. Then $\sum_{q=1}^Q \zeta_q(y) \equiv \mu$ does not depend on y . We have

$$\int_{\pi Y} \zeta_q(y) d\chi = \chi(Z_q), \quad \text{and} \quad \int_{\pi Y} \sum_{q=1}^Q \zeta_q(y) d\chi = \int_{\pi Y} \mu d\chi = \mu.$$

The last equality holds because πY is a closed contractible set, hence its Euler characteristics equals 1 (compare with the Complex Rolle theorem, Example 2, Lecture 2).

Theorem 6. Let g_{ij} in (1) be polynomials of degree not exceeding α , and let S_i be polynomials in (x, z) of degree not exceeding β . Then the Milnor fiber Z_q of S can be defined by polynomial equations of degree not exceeding maximum of (7) and

$$(10) \quad d(n, q, \alpha, \beta) = (K - q + 1)[\beta + (q - 1)(\alpha - 1)], \quad \text{where} \quad K = \binom{q + n - 1}{n}.$$

Proof. Let us fix a small nonzero ϵ . According to the arguments in the proof of Lemma 3, condition $(x_0, z_0) \in Z_q$ is equivalent to vanishing of all $(K - q + 1)$ -minors of a matrix composed of the partial derivatives (8) of S_ϵ of order $\nu < q$, in a system of coordinates (x, y) where integral manifolds of (1) are rectified.

Let us define a germ of the integral manifold Λ of (1) through (x_0, z_0) by a function $\psi(x)$ satisfying (2). Equations (2) allow one to represent a partial derivative (8) as a polynomial in (x_0, z_0) of degree not exceeding $\beta + \nu(\alpha - 1)$. Hence the elements of our matrix are polynomials in (x_0, z_0) of degree not exceeding $\beta + (q - 1)(\alpha - 1)$, and its $(K - q + 1)$ -minors are polynomials in (x_0, z_0) of degree not exceeding $(K - q + 1)[\beta + (q - 1)(\alpha - 1)]$. These polynomials, in combination with equations for Y , provide a system of equations for Z_q .

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Corollary. Under conditions of Theorem 6, the absolute value of the Euler characteristics of Z_q does not exceed

$$(11) \quad \frac{1}{2} \max(2d_Y(m, n, \alpha), 2d(n, q, \alpha, \beta))^{2(m+n)},$$

where $d_Y(m, n, \alpha)$ and $d(m, n, q)$ are defined in (7) and (10), respectively.

Proof. This follows from an estimate [M] of the Euler characteristics of the set Z_q defined by polynomial equations of degree not exceeding maximum of (7) and (10).

7. Maximal multiplicity in a generic family.

Theorem 7. For an analytic mapping $P = (P_1, \dots, P_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and a non-negative integer r , let

$$(12) \quad P^c = (P_1^c, \dots, P_n^c) \quad \text{where} \quad P_j^c(x) = P_j(x) + \sum_{i: |i| \leq r} c_{i,j} x^i.$$

Here $i = (i_1, \dots, i_n)$ is a sequence of nonnegative integers, $|i| = i_1 + \dots + i_n$, and $x^i = x_1^{i_1} \dots x_n^{i_n}$.

For $0 \leq m \leq r$, the set of those (x, c) where the multiplicity of P^c at x exceeds

$$(13) \quad Q(m, n) = \left(\frac{n+m}{1+\dots+1/n} \right)^{1+\dots+1/n} \prod_{k=1}^n \left(\frac{(k-1)!}{k} \right)^{1/k}.$$

has codimension greater than $n+m$.

Corollary. Let $P_z(x) = (P_{z,1}(x), \dots, P_{z,n}(x))$ be a generic family of analytic mappings $\mathbb{C}^n \rightarrow \mathbb{C}^n$ depending on parameters $z \in \mathbb{C}^m$. Then the multiplicity of P_z , at any point $x \in \mathbb{C}^n$ and for any z , is less than (13).

To prove Theorem 7 we need the following

Definition 4. For an analytic set V of codimension r , we define the order $\text{ord}_{x_0} P|_V$ of an analytic function P on the set V at $x_0 \in V$ as a maximal integer ν such that, for a generic $(r+1)$ -dimensional plane L , $P(x) = o(|x - x_0|^{\nu-1})$, for all x in at least one of irreducible components of $V \cap L$. If such a number does not exist (i.e., when P vanishes identically on an irreducible component of V through x_0) we define $\text{ord}_{x_0} P|_V = \infty$. For $V = \mathbb{C}^n$, $\text{ord}_{x_0} P|_V = \text{ord}_{x_0} f$ is the usual vanishing order of P at x_0 .

For example, let $V = \{x_1^2 = x_2^3\}$ and $x_0 = (0, 0)$. Then $\text{ord}_0 x_1|_V = 2$ and $\text{ord}_0 x_2|_V = 1$.

Lemma 4. Let $r \geq 0$, and let P^c be defined as in (12). For a sequence $\nu_1 \leq \dots \leq \nu_n \leq r$ of nonnegative integers, let $X_{\nu_1, \dots, \nu_n}$ be the set of (x, c) where

$$\begin{aligned} \text{ord}_x P_1^c &= \nu_1, \text{ord}_x P_2^c|_{P_1^c=0} = \nu_2, \dots, \text{ord}_x P_n^c|_{P_1^c=\dots=P_{n-1}^c=0} = \nu_n; \\ \text{ord}_x P_j^c &\geq \nu_1, \text{ for } j > 1, \text{ord}_x P_j^c|_{P_1^c=0} \geq \nu_2, \text{ for } j > 2, \dots, \\ \text{ord}_x P_n^c|_{P_1^c=\dots=P_{n-2}^c=0} &\geq \nu_{n-1}. \end{aligned}$$

The codimension of $X_{\nu_1, \dots, \nu_n}$ is not less than

$$(14) \quad \nu_1 \binom{\nu_1 + n - 2}{n - 1} + \nu_2 \binom{\nu_2 + n - 3}{n - 2} + \dots + \nu_n,$$

and the multiplicity of P^c at any point of $X_{\nu_1, \dots, \nu_n}$ is less than $\nu_1 \cdots \nu_n$.

Proof. The condition $\text{ord}_x P_j^c \geq \nu_1$, for all j , means that the values of $-c_{i,j}$, for $|i| < \nu_1$, coincide with the coefficients of the Taylor expansion of P_j at x of the order $\nu_1 - 1$. This gives $n \binom{\nu_1 + n - 1}{n}$ independent conditions on $c_{i,j}$.

Let us fix P_1^c and consider a one-dimensional linear subspace l outside the tangent cone to $\{P_1^c = 0\}$ at x . We can choose coordinates so that l is the x_n -axis. Let us fix all the terms in P_j^c , for $j > 1$, except those that do not contain x_n . The condition $\text{ord}_x P_j^c|_{P_1^c=0} \geq \nu_2$, for $j > 1$, defines at most one possible value for each of the remaining terms. This gives us at least $(n - 1) \left[\binom{\nu_2 + n - 2}{n - 1} - \binom{\nu_1 + n - 2}{n - 1} \right]$ additional independent conditions on $c_{i,j}$. The same arguments allow one to prove that the codimension of $X_{\nu_1, \dots, \nu_d}$ is not less than

$$n \binom{\nu_1 + n - 1}{n} + (n - 1) \left[\binom{\nu_2 + n - 2}{n - 1} - \binom{\nu_1 + n - 2}{n - 1} \right] + \dots + \left[\binom{\nu_n}{1} - \binom{\nu_{n-1}}{1} \right].$$

Applying the identity $k \binom{\nu_{n-k+1} + k - 1}{k} - (k - 1) \binom{\nu_{n-k+1} + k - 2}{k - 1} = \nu_{n-k+1} \binom{\nu_{n-k+1} + k - 2}{k - 1}$, we obtain (14).

To estimate the multiplicity μ of f_c at a point $x \in X_{\nu_1, \dots, \nu_n}$, we have to count the number of zeros (with their multiplicities) of a system of equations $P_1^c = \dots = P_{n-1}^c = P_n^c - \epsilon = 0$ converging to x as $\epsilon \rightarrow 0$. Due to the condition $\text{ord}_x P_n^c|_{P_1^c=\dots=P_{n-1}^c=0} = \nu_n$, this number is less than ν_n multiplied by the multiplicity μ' of $\{P_1^c = \dots = P_{n-1}^c = s = 0\}$ at x , where s is a generic linear function. The same arguments show that μ' is less than ν_{n-1} multiplied by the multiplicity of $\{P_1^c = \dots = P_{n-2}^c = s_1 = s_2 = 0\}$ at x , where s_1 and s_2 are generic linear functions. Repeating these arguments, we obtain the necessary estimate $\mu < \nu_1 \cdots \nu_n$.

Proof of Theorem 7. Due to Lemma 4, we have to estimate maximal possible value of $\nu_1 \cdots \nu_n$ over the sequences $(\nu_1, \dots, \nu_n)$ such that (14) does not exceed $n + r$. Replacing $\binom{\nu_{n-k+1} + k - 2}{k - 1}$ by $\nu_{n-k+1}^{k-1}/(k - 1)!$, we see that (14) is not less than $\sum_{k=1}^n \nu_{n-k+1}^k / (k - 1)!$. Thus it is enough to maximize $\nu_1 \cdots \nu_n$ when $\sum_{k=1}^n \nu_{n-k+1}^k / (k - 1)! = n + r$. Substituting $u_k = \nu_{n-k+1}^k$, we have to maximize $\prod_{k=1}^n u_k^{1/k}$ when $\sum_{k=1}^n u_k / (k - 1)! = n + r$. The maximum (13) is achieved when

$$u_k = \frac{(k - 1)!(n + r)}{k(1 + \dots + 1/n)}.$$

8. Proof of Theorem 1. Consider a deformation $S(x, z, \epsilon)$ of the polynomial $P(x, z)$ defined by

$$S_j(x, z, \epsilon) = P_j(x, z) + \epsilon \sum_{i: |i| \leq m} c_{i,j} x^i,$$

where $c_{i,j}$ are generic complex numbers. From Theorem 7, the Milnor fibers Z_q of this deformation are empty for $q \geq Q(m, n)$, where $Q(m, n)$ is defined in (13).

THREE LECTURES ON FEWNOMIALS AND NOETHERIAN INTERSECTIONS

According to (9),

$$(15) \quad \mu = \sum_{q=1}^{Q(m,n)} \chi(Z_q) \leq Q(m,n) \max_{q \leq Q(m,n)} |\chi(Z_q)|.$$

From (11), the right side of (15) does not exceed

$$\frac{1}{2} Q(m,n) \max(2d_Y(m,n,\alpha), 2d(n, Q(m,n), \alpha, \beta))^{2(m+n)},$$

where d_Y and d are defined in (7) and (10), respectively. The value of $d(n, q, \alpha, \beta)$ in (10) does not exceed

$$(q+n-1)^n(\beta + (q-1)(\alpha-1)) < (q+n)^n(\beta + q(\alpha-1)).$$

The statement of Theorem 1 follows now from the following estimate for $Q(m,n)$ which one can obtain from the theorem 7:

Proposition 1. *The value of $Q(m,n)$ in (13) does not exceed*

$$(16) \quad en \left(\frac{e(n+m)}{\sqrt{n}} \right)^{\ln n+1} \left(\frac{n}{e^2} \right)^n.$$

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