

An explicit sixth-order pseudo-Runge-kutta formula

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§0 はじめに.

常微分方程式の初期値問題:

$$(1) \quad y' = f(x, y), \quad y(x_0) = y_0.$$

の近似式について述べる. 論文[1]である種の近似式で位数6の公式を導いたが近似式の係数が特別の値をとる($w_2=0$)場合について議論している. ここでは係数が一般の場合について調べる.

§1 近似式の導出.

(1) 式の近似式を次のように与える.

$$(2) \quad \begin{aligned} y_{n+1} &= y_n - v(y_n - y_{n-1}) + h \sum_{i=0}^p w_i k_i, \\ k_0 &= f(x_{n-1}, y_{n-1}), \quad k_1 = f(x_n, y_n), \\ k_i &= f(x_n + a_i h, y_n + b_i(y_n - y_{n-1}) + h \sum_{j=0}^{i-1} b_{ij} k_j) \\ &\quad (i=2, 3, \dots, p) \end{aligned}$$

$$a_i = \sum_{j=0}^{i-1} b_{ij}$$

(2) 式が order 6 になるための係数条件を tree で表現すると次のようになる。

(3) 
$$(-1)^i \frac{v}{i!} + \sum_{l=0}^4 \frac{a_l}{(l-1)!} w_l = \frac{1}{i!}$$

 $(i=1, 2, \dots, 6),$


$$\frac{v}{6!} - \frac{w_0}{5!} + P_{24} w_2 + (P_{34} + P_{23} b_{32}) w_3$$

 $+ (P_{44} + P_{23} b_{43} + (P_{33} + \frac{1}{3!} a_2^3 b_{32}) b_{44}) w_4 = \frac{1}{6!},$


$$\frac{v}{6!} - \frac{w_0}{5!} + P_{24} w_2 + (P_{34} + \frac{1}{4!} a_2^4 b_{32}) w_3$$

 $+ (P_{44} + \frac{1}{4!} a_2^4 b_{43} + \frac{1}{4!} a_3^4 b_{44}) w_4 = \frac{1}{6!},$


$$\frac{6}{6!} v - \frac{5}{5!} w_0 + a_2 P_{23} w_2 + a_3 (P_{33} + \frac{1}{3!} a_2^3 b_{32}) w_3$$

 $+ a_4 (P_{44} + \frac{1}{3!} a_3^3 b_{43} + \frac{1}{3!} a_4^3 b_{44}) w_4 = \frac{5}{6!},$

$$P_{ij} = (-1)^{i+1} \left\{ \frac{1}{(i+2)!} b_j + \frac{1}{(i+1)!} b_{j0} \right\},$$

 $(i=2, 3, 4, j=1, 2, 3, 4).$

$$a_j^{i+2} = (i+2)! \left\{ P_{ij} + \frac{1}{(i+1)!} \sum_{l=2}^{j-1} a_l^{i+1} b_{jl} \right\} \quad (i=0, 1, j=2, 3, 4)$$

ただし $a_0 = -1, a_1 = 0$ とする。

(3) 式を $w_2 = 0$ と $w_2 \neq 0$ の二通りの場合に区けて解くと。

[1] $w_2 = 0$ のとき。

(4) $a_3 = \frac{1}{\sqrt{3}}, v = \frac{139 - 240 a_3}{11}, w_0 = \frac{18 - 31 a_3}{11},$
 $a_4 = 1.$

$$w_1 = \frac{40 - 64a_3}{11}, \quad w_3 = \frac{-18(8a_3 - 5)}{11}, \quad w_4 = \frac{2 - a_3}{11},$$

$$b_2 = -(2a_2^3 + 3a_2^2), \quad b_{20} = a_2^2 + a_2^3,$$

$$b_3 = \frac{2(2+\sqrt{3})}{3(2a_2+1)} - \frac{3\sqrt{3}+2}{3\sqrt{3}}, \quad b_{30} = -\frac{(2+\sqrt{3})(3a_2+2)}{9(2a_2+1)(a_2+1)} + \frac{\sqrt{3}+1}{3\sqrt{3}},$$

$$b_{32} = \frac{2+\sqrt{3}}{9a_2(2a_2+1)(a_2+1)},$$

$$b_4 = -\frac{12}{2a_2+1} + 12\frac{6a_3+4}{7a_3+4} - 5,$$

$$b_{40} = \frac{2(3a_2+2)}{(a_2+1)(2a_2+1)} - 6\frac{5a_3+3}{7a_3+4} + 2,$$

$$b_{42} = \frac{2}{a_2(a_2+1)(2a_2+1)}, \quad b_{43} = 6\frac{3a_3+1}{7a_3+4},$$

$$b_{i1} = a_i - \sum_{\substack{j=0 \\ j \neq 1}}^{i-1} b_{ij}, \quad (i=2, 3, 4).$$

[II] $w_2 \neq 0$ のとき.

次の二つの場合に分けて考える.

(1) $a_4 \neq 1$ のとき.

$$(5) \quad v = \frac{\rho_1 - \rho_2 \rho_3}{\rho_4 - \rho_3 \rho_5}, \quad w_3 = \frac{\rho_1 - \rho_4 v}{a_3(1+a_3)(a_2-a_3)(a_4-a_3)\rho_3},$$

$$w_4 = \frac{\frac{9}{20}a_2 - \frac{11}{30} - \{(\frac{1}{20}a_2 + \frac{11}{30})v + a_3^3(a_2-a_3)(1+a_3)w_3\}}{a_4^3(a_2-a_4)(a_4+1)},$$

$$w_2 = \frac{\frac{9}{20} - \{\frac{1}{20}v + a_3^3(a_3+1)w_3 + a_4^3(a_4+1)w_4\}}{a_2^3(a_2+1)},$$

$$w_0 = \frac{1}{2}(v-1) + \sum_{i=2}^4 a_i w_i,$$

$$w_1 = 1 + v - (w_0 + w_2 + w_3 + w_4).$$

$$b_{32} = \frac{62a_4 - 49 + (2a_4 + 1)v}{120(a_4 - a_3)a_2(2a_2 + 1)(a_2 + 1)w_3},$$

$$b_{43} = \frac{160a_2 + 49 - v}{120\{5(2a_2 + 1)^2 a_2(a_2 + 1)b_{32} + a_3(a_3 + 1)(a_2 - a_3)\}w_4},$$

$$b_{42} = \frac{1}{a_2(2a_2 + 1)(a_2 + 1)} \left\{ \frac{49 - 62a_3 - (2a_3 + 1)v}{120(a_4 - a_3)w_4} - \frac{a_3(2a_3 + 1)(a_3 + 1)b_{43}}{120(a_4 - a_3)w_4} \right\},$$

$$b_2 = -(2a_2^3 + 3a_2^2), \quad b_{20} = a_2^2 + a_2^3,$$

$$b_4 = 6\{(a_2 + a_2^2)b_{42} + (a_3 + a_3^2)b_{43} - \frac{5}{6}\},$$

$$b_{40} = -\frac{1}{2}b_4 + a_2b_{42} + a_3b_{43} - \frac{1}{2}a_4^2,$$

$$b_3 = 6a_2(a_2 + 1)b_{32} - a_3^2(2a_3 + 3),$$

$$b_{30} = -\frac{1}{2}b_3 + a_2b_{32} - \frac{1}{2}a_3^2,$$

$$b_{i1} = a_i - \sum_{\substack{j=0 \\ j \neq 1}}^{i-1} b_{ij} \quad (i=2, 3, 4).$$

ただし

$$g_1 = a_4^2 \left(\frac{5}{6}a_2^2 - \frac{9}{20} \right) - \left(\frac{9}{20}a_2 - \frac{11}{30} \right)(a_2 + a_4),$$

$$g_2 = \frac{1}{12} \{ 10a_2a_4 - 7(a_2 + a_4) + \frac{27}{5} \},$$

$$g_3 = a_2a_3 + a_3a_4 + a_4a_2,$$

$$g_4 = a_4^2 \left(\frac{1}{6}a_2^2 - \frac{1}{20} \right) - \left(\frac{1}{20}a_2 + \frac{1}{30} \right)(a_2 + a_4),$$

$$g_5 = \frac{1}{12} (2a_2a_4 + a_2 + a_4 + \frac{3}{5}),$$

である。また定数 a_3 は次の二次方程式:

$$z_1 a_3^2 + z_2 a_3 + z_3 = 0,$$

の根である. ここで z_1, z_2, z_3 は次のように与えられる.

$$z_1 = u_{11} \{ 2u_2 u_7 u_8 - (u_2 u_{12} - u_1)(u_5 u_8 + u_6 u_7) - 2u_1 u_2 u_5 u_6 \\ + u_9(u_3 u_6 u_8 + u_4 u_8) - u_{10}(u_3 u_6^2 + u_4 u_6 u_8) \},$$

$$z_2 = u_{13} \{ 2u_2 u_7 u_8 - (u_2 u_{12} - u_1)(u_5 u_8 + u_6 u_7) \\ - 2u_1 u_5 u_6 u_{12} \} + (u_3 u_9 - u_4 u_{10})(u_5 u_8 + u_6 u_7) \\ + 2(u_4 u_7 u_8 u_9 - u_3 u_5 u_6 u_{10}) \\ + u_{11} \{ u_2 u_7^2 - (u_2 u_{12} - u_1)u_5 u_7 - u_1 u_5^2 u_{12} \},$$

$$z_3 = u_4 u_7^2 u_9 + u_5 u_7 (u_3 u_9 - u_4 u_{10}) - u_3 u_5^2 u_{10} \\ + u_{13} \{ u_2 u_7^2 - (u_2 u_{12} - u_1)u_5 u_7 - u_1 u_5^2 u_{12} \},$$

$$u_1 = 2 \left\{ 10a_2 a_4 - 7(a_2 + a_4) + \frac{27}{5} \right\},$$

$$u_2 = -2 \left\{ 2a_2 a_4 + a_2 + a_4 + \frac{3}{5} \right\},$$

$$u_3 = (2a_2 + 1)(62a_4 - 49) + u_1, \quad u_4 = -\frac{1}{5},$$

$$u_5 = 24 \left\{ a_4^2 \left(\frac{1}{6} a_2^2 - \frac{1}{20} \right) - \left(\frac{1}{20} a_2 + \frac{1}{30} \right) (a_2 + a_4) + u_2 a_2 a_4 \right\},$$

$$u_6 = (a_2 + a_4) u_2,$$

$$u_7 = 24 \left\{ a_4^2 \left(\frac{5}{6} a_2^2 - \frac{9}{20} \right) - \left(\frac{9}{20} a_2 - \frac{11}{30} \right) (a_2 + a_4) \right\} \\ - u_1 a_2 a_4,$$

$$u_8 = -(a_2 + a_4) u_1, \quad u_9 = 10a_2^2 + 10a_2 + 2,$$

$$u_{10} = -310a_2^2 + 10a_2 + 98, \quad u_{11} = 10a_2 + 5,$$

$$u_{12} = 160a_2 + 49, u_{13} = 5a_2 + 3.$$

(6) (ロ) $a_4 = 1$ のとき.

(5) 式において

$$z_1 = z_2 = z_3 = 0,$$

となり. 定数 v, w_i, b_i, b_{ij} は (5) 式で与えられる. この場合, a_2, a_3 は free parameter となる.

§2 局所打ち切り誤差.

(2) 式において 4 段 6 位 のときの打ち切り誤差を調べることにする.

まず

$$d_{ij} = (-1)^j \left\{ \frac{b_{i0}}{(j+1)!} + \frac{b_{i1}}{j!} \right\} \quad (i=2,3,4, j=1,2,\dots,5)$$

$$e_{30} = 3d_{23} + \frac{1}{6}a_2^4,$$

$$e_{31} = d_{33} + \frac{1}{6}a_3^3 b_{32},$$

$$e_{41} = d_{34} + \frac{1}{4!}a_2^4 b_{32},$$

$$e_{42} = d_{34} + d_{22} b_{32},$$

$$e_{43} = 4(d_{34} + \frac{1}{4!}a_2^4 b_{32}) + 3(d_{34} + d_{22} b_{32}),$$

$$e_{51} = d_{35} + \frac{1}{5!}a_2^5 b_{32},$$

$$e_{52} = d_{35} + \frac{1}{5} a_2 d_{23} b_{32},$$

$$e_{53} = 13 d_{35} + (3 d_{24} + \frac{1}{12} a_2^5) b_{32},$$

$$e_{54} = 16 d_{35} + (6 d_{24} + \frac{1}{12} a_2^5) b_{32},$$

$$e_{55} = 12 d_{35} + (7 d_{24} + a_2 d_{23}) b_{32},$$

$$e_{56} = 9 d_{35} + (4 d_{24} + a_2 d_{23}) b_{32},$$

$$e_{57} = d_{35} + d_{24} b_{32},$$

$$p_{31} = d_{43} + \frac{1}{6} a_2^3 b_{42} + \frac{1}{6} a_3^3 b_{43},$$

$$p_{41} = d_{44} + \frac{1}{24} a_2^4 b_{42} + \frac{1}{24} a_3^4 b_{43},$$

$$p_{43} = d_{44} + d_{23} b_{42} + e_{32} b_{43},$$

$$p_{51} = d_{45} + \frac{1}{5!} a_2^5 b_{42} + \frac{1}{5!} a_3^5 b_{43},$$

$$p_{52} = d_{45} + \frac{1}{5} a_2 d_{23} b_{42} + \frac{1}{5} a_3 (d_{33} + \frac{1}{3!} a_2^3 b_{32}) b_{43},$$

$$p_{53} = 13 d_{45} + (3 d_{24} + \frac{1}{12} a_2^5) b_{42} \\ + (3 d_{34} + \frac{1}{8} a_2^4 b_{32} + \frac{1}{12} a_3^5) b_{43},$$

$$p_{54} = 16 d_{45} + (6 d_{24} + \frac{1}{12} a_2^5) b_{42} \\ + (6 d_{34} + \frac{1}{4} a_2^4 b_{32} + \frac{1}{12} a_3^5) b_{43},$$

$$p_{55} = 12 d_{45} + (7 d_{24} + a_2 d_{23}) b_{32} \\ + (7 d_{34} + \frac{1}{6} a_2^4 b_{32} + \frac{7}{6} d_{23} b_{32} + a_3 (d_{33} + \frac{1}{3!} a_2^3 b_{32})) b_{43}$$

$$p_{56} = 9 d_{45} + (4 d_{24} + a_2 d_{23}) b_{42} \\ + (4 d_{34} + \frac{1}{3!} a_2^4 b_{32} + a_3 (d_{33} + \frac{1}{3!} a_2^3 b_{32})) b_{43},$$

$$p_{57} = d_{45} + d_{24} b_{42} + (d_{34} + d_{23} b_{32}) b_{43},$$

$$p_{58} = d_{45} + d_{24} b_{42} + (d_{34} + \frac{1}{4!} a_2^4 b_{32}) b_{43},$$

とある。 θ_i ($i=0 \sim 28$) は次の様に定める。

$$\theta_0 = \frac{1}{720} w_0 - \frac{1}{5040} - \frac{v}{5040},$$

$$\theta_1 = (3a_2 d_{24} + \frac{3}{2} a_2^2 d_{23}) w_2 + (3a_3 e_{41} + \frac{3}{2} a_3^2 e_{31}) w_3 \\ + (3a_4 p_{41} + \frac{3}{2} a_4^2 p_{31}) w_4,$$

$$\theta_2 = (15d_{25} + \frac{1}{12} a_2^6) w_2 + (15e_{51} + \frac{1}{12} a_3^6) w_3 \\ + (15p_{51} + \frac{1}{12} a_4^6) w_4,$$

$$\theta_3 = d_{25} w_2 + e_{51} w_3 + p_{51} w_4,$$

$$\theta_4 = d_{25} w_2 + e_{51} w_3 + p_{51} w_4,$$

$$\theta_5 = \frac{1}{48} (a_2^6 w_2 + a_3^6 w_3 + a_4^6 w_4),$$

$$\theta_6 = \frac{1}{2} (a_2^2 d_{23} w_2 + a_3^2 e_{31} w_3 + a_4^2 p_{31} w_4),$$

$$\theta_7 = (\frac{1}{36} a_2^6 + 10d_{25}) w_2 + (\frac{1}{36} a_3^6 + 10e_{51}) w_3 \\ + (\frac{1}{36} a_4^6 + 10p_{51}) w_4,$$

$$\theta_8 = \frac{1}{36} \sum_{i=2}^4 a_i^6 w_i, \theta_9 = \frac{1}{16} \sum_{i=2}^4 a_i^6 w_i, \theta_{10} = \frac{1}{12} \sum_{i=2}^4 a_i^6 w_i,$$

$$\theta_{11} = \frac{1}{48} \sum_{i=2}^4 a_i^6 w_i, \theta_{12} = \frac{1}{72} \sum_{i=2}^4 a_i^6 w_i, \theta_{13} = \frac{1}{720} \sum_{i=2}^4 a_i^6 w_i,$$

$$\theta_{14} = (\frac{1}{2} a_2^2 d_{23} + \frac{1}{36} a_2^6 + 10d_{25}) w_2 \\ + (\frac{1}{2} a_3^2 e_{31} + \frac{1}{36} a_3^6 + 10e_{51}) w_3 \\ + (\frac{1}{2} a_4^2 p_{31} + \frac{1}{36} a_4^6 + 10p_{51}) w_4,$$

$$\theta_{15} = (7a_2 d_{24} + 15d_{25}) w_2 + (a_3 e_{43} + 15e_{52}) w_3 \\ + (a_4 p_{42} + 15p_{52}) w_4,$$

$$\theta_{16} = (16d_{25} + \frac{1}{2} a_2^2 d_{23}) w_2 + (e_{54} + \frac{1}{2} a_3^2 e_{31}) w_3 \\ + (p_{54} + \frac{1}{2} a_4^2 p_{31}) w_4,$$

$$Q_{17} = \left(\frac{3}{2}a_2^2 d_{23} + 6a_2 d_{24}\right)w_2 + \left(\frac{3}{2}a_3^2 e_{31} + 6a_3 e_{41}\right)w_3 \\ + \left(\frac{3}{2}a_4^2 p_{31} + 6a_4 p_{41}\right)w_4,$$

$$Q_{18} = (12d_{25} + a_2 d_{24})w_2 + (e_{55} + a_3 e_{42})w_3 \\ + (p_{55} + a_4 p_{43})w_4,$$

$$Q_{19} = d_{25}w_2 + e_{57}w_3 + p_{58}w_4,$$

$$Q_{20} = (9d_{25} + a_2 d_{24})w_2 + (e_{56} + a_3 e_{42})w_3 \\ + (p_{56} + a_4 p_{43})w_4,$$

$$Q_{21} = 4a_2 d_{24}w_2 + 4a_3 e_{41}w_3 + 4a_4 p_{41}w_4,$$

$$Q_{22} = \left(\frac{1}{2}a_2^2 d_{23} + 10d_{25}\right)w_2 + \left(\frac{1}{2}a_3^2 e_{31} + 10e_{51}\right)w_3 \\ + \left(\frac{1}{2}a_4^2 p_{31} + 10p_{51}\right)w_4,$$

$$Q_{23} = d_{25}w_2 + e_{57}w_3 + p_{57}w_4,$$

$$Q_{24} = (a_2 d_{24} + 5d_{25})w_2 + (a_3 e_{41} + 5e_{52})w_3 \\ + (a_4 p_{43} + 5p_{52})w_4,$$

$$Q_{25} = Q_{23},$$

$$Q_{26} = \left(\frac{1}{12}a_2^6 + 13d_{35} + \frac{1}{2}a_2^2 d_{23}\right)w_2 \\ + \left(\frac{1}{12}a_3^6 + e_{53} + \frac{1}{2}a_3^2 e_{31}\right)w_3 \\ + \left(\frac{1}{12}a_4^6 + p_{53} + \frac{1}{2}a_4^2 p_{31}\right)w_4,$$

$$Q_{27} = \frac{1}{2}a_2^2 d_{23}w_2 + \frac{1}{2}a_3^2 e_{31}w_3 + \frac{1}{2}a_4^2 p_{31}w_4,$$

$$Q_{28} = a_2 d_{24}w_2 + a_3 e_{41}w_3 + a_4 p_{41}w_4$$

このとき (2) 式の打ち切り誤差は次のように表わせる。

$$\begin{aligned}
 (7) \quad T(x_n, y_n; h) = & \theta_1 f_{yy} \cdot S_1 \cdot G_2 + \theta_2 f_y \cdot G_2 \cdot R_1 \\
 & + \theta_3 f_y \cdot T_5 + \theta_4 f_y^3 \cdot T_3 + \theta_5 T_1 \cdot S_4 + \theta_6 f_{yy} \cdot T_1 \cdot T_2 \\
 & + \theta_7 f_y \cdot T_1 \cdot S_3 + \theta_8 T_2 \cdot S_3 + \theta_9 G_2 \cdot D^2 f_{yy} + \\
 & \theta_{10} T_1 \cdot T_2 \cdot D f_{yy} + \theta_{11} f_{yy} G_3 + \theta_{12} f_{yy} (T_2)^2 + \theta_{13} T_6 \\
 & + \theta_{14} f_y \cdot f_{yy} T_1 \cdot T_2 + \theta_{15} f_y T_1 \cdot (S_1)^2 + \theta_{16} f_y^2 \cdot T_1 \cdot S_2 \\
 & + \theta_{17} T_1 \cdot S_1 \cdot S_2 + \theta_{18} f_y^3 T_1 \cdot S_1 + \theta_{19} f_y^5 \cdot T_1 \\
 & + \theta_{20} f_y^2 \cdot T_2 \cdot S_1 + \theta_{21} \cdot T_2 \cdot (S_1)^2 + \theta_{22} f_y \cdot T_2 \cdot S_2 \\
 & + \theta_{23} f_y^4 \cdot T_2 + \theta_{24} f_y T_3 \cdot S_1 + \theta_{25} f_y^2 \cdot T_4 \\
 & + \theta_{26} f_y^2 \cdot f_{yy} \cdot G_2 + \theta_{27} T_3 \cdot S_2 + \theta_{28} S_1 \cdot T_4.
 \end{aligned}$$

$$\begin{aligned}
 \text{ここで } S_j = D^j f_y, \quad G_j = (Df)^j, \quad R_1 = D f_{yy}, \\
 T_j = D^j f, \quad D = \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right), \quad f_y = \frac{\partial f}{\partial y}
 \end{aligned}$$

を表わす.

関数 f の偏導関数に次のような評価を仮定する.

$$|f(x, y)| \leq L_1, \quad \left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right| \leq L_2 \frac{L_1^{i+j}}{L_1^{j-1}} \quad (i+j \leq 7).$$

このとき打ち切り誤差式 (7) は $(L_1, L_2: \text{定数})$

$$(8) \quad |T(x_n, y_n; h)| \leq \sum_{i=1}^{118} \theta_i L_1 L_2^6 h^7 (= T_0 L_1 L_2^6 h^7)$$

のようになる.

定数 $\theta_i (= \theta_i(\theta_1, \theta_2, \dots, \theta_{28}))$ は次式で与えられる.

$QQ1=ABS(2*Q1+2*Q14):QQ2=ABS(Q2):QQ3=ABS(3*Q7+2*Q9)$
 $QQ4=ABS(5*Q3+2*Q5):QQ5=ABS(Q6)$
 $QQ6=ABS(Q2):QQ7=ABS(Q7):QQ8=ABS(Q5)$
 $QQ9=ABS(Q3):QQ10=ABS(3*Q6)$
 $QQ11=ABS(3*Q7+4*Q9):QQ12=ABS(10*Q3+4*Q5)$
 $QQ13=ABS(2*Q1+Q15):QQ14=ABS(3*Q6+Q10+2*Q17+3*Q24)$
 $QQ15=ABS(Q7+2*Q9):QQ16=ABS(10*Q3+6*Q5)$
 $QQ17=ABS(Q6+Q10+Q24+Q17):QQ18=ABS(5*Q3+4*Q5)$
 $QQ19=ABS(Q3+Q5):QQ20=ABS(2*Q8+4*Q28)$
 $QQ21=ABS(Q1):QQ22=ABS(6*Q8+6*Q28)$
 $QQ23=ABS(6*Q8+4*Q28):QQ24=ABS(2*Q8+Q28)$
 $QQ25=ABS(3*Q6):QQ26=ABS(2*Q9)$
 $QQ27=ABS(3*Q8):QQ28=ABS(4*Q5)$
 $QQ29=ABS(Q1):QQ30=ABS(Q9):QQ31=ABS(Q8)$
 $QQ32=ABS(Q6):QQ33=ABS(Q1+2*Q14+2*Q15+3*Q20)$
 $QQ34=ABS(2*Q2+2*Q16):QQ35=ABS(Q7+4*Q25)$
 $QQ36=ABS(Q5):QQ37=ABS(2*Q2+3*Q11+Q16)$
 $QQ38=ABS(3*Q7+Q9+6*Q25):QQ39=ABS(3*Q7+2*Q9+4*Q25)$
 $QQ40=ABS(Q1+Q14+Q15+Q20):QQ41=ABS(Q7+Q9+Q25)$
 $QQ42=ABS(3*Q4+Q16):QQ43=ABS(Q4)$
 $QQ44=ABS(Q2+3*Q4+2*Q16):QQ45=ABS(Q2+Q4+Q11+Q16)$
 $QQ46=ABS(3*Q6+Q10):QQ47=ABS(3*Q8)$
 $QQ48=ABS(Q8+4*Q28):QQ49=ABS(Q9):QQ50=ABS(6*Q5)$
 $QQ51=ABS(Q6+Q17):QQ52=ABS(3*Q8+6*Q28)$
 $QQ53=ABS(Q8):QQ54=ABS(4*Q5):QQ55=ABS(3*Q8+4*Q28)$
 $QQ56=ABS(Q5):QQ57=ABS(Q8+Q28):QQ58=ABS(2*Q15)$
 $QQ59=ABS(2*Q10+3*Q24+2*Q17):QQ60=ABS(2*Q10+Q24+Q17)$
 $QQ61=ABS(Q10):QQ62=ABS(3*Q11):QQ63=ABS(Q14)$
 $QQ64=ABS(Q15):QQ65=ABS(Q10):QQ66=ABS(Q14)$
 $QQ67=ABS(2*Q10+2*Q17):QQ68=ABS(4*Q12+2*Q21)$
 $QQ69=ABS(2*Q10+Q17):QQ70=ABS(4*Q12+4*Q21)$
 $QQ71=ABS(Q10):QQ72=ABS(6*Q13):QQ73=ABS(Q10)$
 $QQ74=ABS(Q11):QQ75=ABS(Q12):QQ76=ABS(Q13)$
 $QQ77=ABS(Q14+Q20):QQ78=ABS(Q15+2*Q20)$
 $QQ79=ABS(4*Q12):QQ80=ABS(15*Q13)$
 $QQ81=ABS(2*Q12+Q21):QQ82=ABS(20*Q13):QQ83=ABS(15*Q13)$
 $QQ84=ABS(Q12+Q21):QQ85=ABS(6*Q13):QQ86=ABS(Q13)$
 $QQ87=ABS(3*Q24+Q17):QQ88=ABS(Q24):QQ89=ABS(3*Q24+Q17)$
 $QQ90=ABS(Q25):QQ91=ABS(2*Q22+2*Q23):QQ92=ABS(Q22+Q23)$
 $QQ93=ABS(Q22+Q23+Q26+Q18):QQ94=ABS(Q21)$
 $QQ95=ABS(5*Q21):QQ96=ABS(2*Q21):QQ97=ABS(Q17)$
 $QQ98=ABS(Q17):QQ99=ABS(Q20):QQ100=ABS(Q16)$
 $QQ101=ABS(Q18):QQ102=ABS(Q18+2*Q26)$
 $QQ103=ABS(Q18):QQ104=ABS(Q19):QQ105=ABS(Q19)$
 $QQ106=ABS(2*Q17):QQ107=ABS(Q28):QQ108=ABS(2*Q27)$
 $QQ109=ABS(Q28):QQ110=ABS(3*Q27)$
 $QQ111=ABS(Q27):QQ112=ABS(Q26):QQ113=ABS(9*Q27)$
 $QQ114=ABS(Q27):QQ115=ABS(4*Q27):QQ116=ABS(6*Q27)$
 $QQ117=ABS(5*Q27):QQ118=ABS(Q27)$

局所打ち切り誤差 T_n を最小にするように
free parameter を決定しよう.

(4) 式 のとき a_2 が free parameter で最適値は

$$a_2 = 0.1$$

でこのとき

$$|T(x_n, y_n; h)| \leq 1.31 h^7 L_1 L_2^6,$$

(5) 式 での free parameter a_2, a_3 の最適値は

$$a_2 = 0.05, \quad a_4 = 0.285.$$

またこのとき

$$|T(x_n, y_n; h)| \leq 4.36 h^7 L_1 L_2^6.$$

(6) 式 においては同様に free parameter a_2, a_3 は

$$a_2 = 0.6, \quad a_3 = 0.3$$

のとき最適で打ち切り誤差は

$$|T(x_n, y_n; h)| \leq 1.06 h^7 L_1 L_2^6.$$

となる.

§3 安定性.

微分方程式 $y' = \lambda y, y(0) = 1, (\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) < 0)$

を(2)式で近似すると次の特性多項式が得られる.

$$(7) \quad y_{n+1} = (l_1 + l_2 i) y_n + (l_3 + l_4 i) y_{n-1},$$

l_1, l_2, l_3, l_4 は定数 z 次式 z で与えられる.

$$l_1 = \alpha (v_1 w_2 + v_2 w_3 + v_3 w_4) \\ + \beta (w_1 + u_1 w_2 + u_2 w_3 + u_3 w_4),$$

$$l_2 = 1 - v + \alpha (w_1 + u_1 w_2 + u_2 w_3 + u_3 w_4) \\ - \beta (v_1 w_2 + v_2 w_3 + v_3 w_4),$$

$$l_3 = v + \alpha (w_0 + c_1 w_2 + c_2 w_3 + c_3 w_4) \\ - \beta (d_1 w_2 + d_2 w_3 + d_3 w_4),$$

$$l_4 = \alpha (d_1 w_2 + d_2 w_3 + d_3 w_4) \\ + \beta (w_0 + c_1 w_2 + c_2 w_3 + c_3 w_4),$$

$$c_1 = -b_2 + \alpha b_{20},$$

$$c_2 = -b_3 + \alpha (b_{30} - b_{32} b_2) + (\alpha^2 - \beta^2) b_{32} b_{20},$$

$$c_3 = -b_4 + \alpha (b_{40} - b_2 b_{42} - b_3 b_{43}) \\ + (\alpha^2 - \beta^2) \{ b_{42} b_{20} + b_{43} (b_{30} - b_{32} b_2) \} \\ + \{ (\alpha^2 - \beta^2) \alpha - 2\alpha\beta^2 \} b_{43} b_{32} b_{20},$$

$$d_1 = \beta b_{20},$$

$$d_2 = \beta (b_{30} - b_{32} b_2) + 2\alpha\beta b_{32} b_{20},$$

$$d_3 = \beta (b_{40} - b_2 b_{42} - b_3 b_{43}) \\ + 2\alpha\beta \{ b_{42} b_{20} + b_{43} (b_{30} - b_{32} b_2) \} \\ + \{ 2\alpha^2\beta + \beta(\alpha^2 - \beta^2) \} b_{43} b_{32} b_{20},$$

$$u_1 = 1 + b_2 + \alpha b_{21},$$

$$u_2 = 1 + b_3 + \alpha \{ b_{31} + b_{32}(1 + b_2) \} + (\alpha^2 - \beta^2) b_{32} b_{21},$$

$$\begin{aligned}
 U_3 = & 1 + b_4 + \alpha \{ b_{41} + b_{42}(1+b_2) + b_{43}(1+b_3) \} \\
 & + (\alpha^2 - \beta^2) \{ b_{42}b_{21} + b_{43}(b_{31} + b_{32}(1+b_2)) \} \\
 & + \alpha(\alpha^2 - 3\beta^2)b_{43}b_{32}b_{21},
 \end{aligned}$$

$$V_1 = \beta b_{21},$$

$$V_2 = \beta \{ b_{31} + b_{32}(1+b_2) \} + 2\alpha\beta b_{32}b_{21},$$

$$\begin{aligned}
 V_3 = & \beta \{ b_{41} + b_{42}(1+b_2) + b_{43}(1+b_3) \} \\
 & + 2\alpha\beta \{ b_{42}b_{21} + b_{43}(b_{31} + b_{32}(1+b_2)) \} \\
 & + \beta(3\alpha^2 - \beta^2)b_{43}b_{32}b_{21},
 \end{aligned}$$

$$(\lambda = \alpha + \beta i).$$

これより (7) 式の特性格多項式の根 (ξ_1, ξ_2)

$$|\xi_i| \leq 1, \quad (i=1,2)$$

$$|\xi_1, \xi_2| \neq 1,$$

を満足するための条件(絶対安定条件)は

$$(1) \quad l_3^2 + l_4^2 < 1,$$

$$\begin{aligned}
 (2) \quad & \{ l_1 + (l_1 l_3 + l_2 l_4) \}^2 + \{ l_2 + (l_1 l_4 - l_2 l_3) \}^2 \\
 & \leq 1 - (l_3^2 + l_4^2),
 \end{aligned}$$

となる。

(4), (5), (6) 式の安定領域 (1), (2) を

満足する $(\alpha + \beta i)$ の領域) は図のようになる。

§4 数値実験

(4), (5), (6) 式により次の微分方程式:

$$y' = -y + x^2$$

$$y(0) = 3, \quad y(x) = e^{-x} + 2 - 2x + x^2.$$

の近似解を求めよう.

(進み幅 $h = 1/2^4$, $x = 0 \sim 5.00$)

絶対誤差

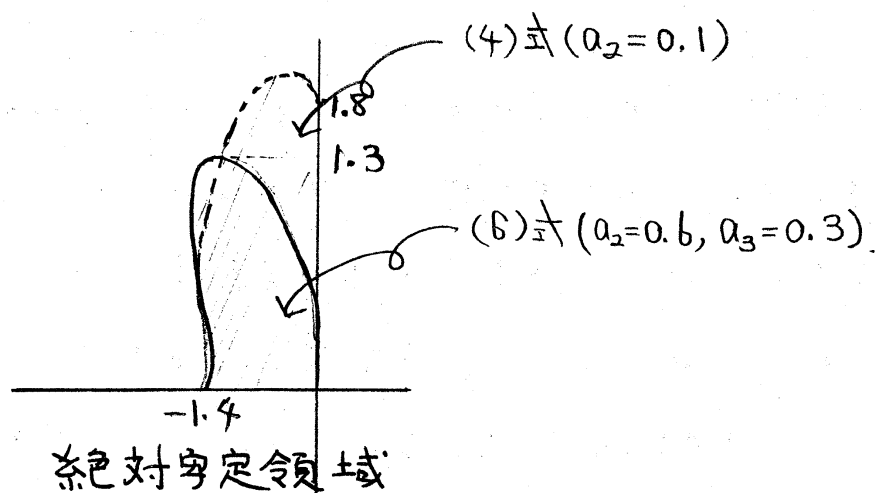
x	(4) 式 $a_2 = 0.1$	(5) 式 $a_2 = 0.05, a_4 = 0.285$	(6) 式 $a_2 = 0.6, a_3 = 0.3$
0.125	-4.67E-12	-1.546E-10	-5.87E-12
0.1875	-8.60E-12	-1.00E-9	-1.08E-11
0.3125	-1.50E-11	-2.36E-8	-1.90E-11
0.5	-2.17E-11	-2.33E-6	-2.74E-11
1.00	-2.81E-11	-4.73E-1	-3.56E-11
1.500	-2.61E-11	-9.62E+4	-3.31E-11
3.00	-1.19E-11	-8.06E+20	-1.50E-11
5.00	-2.71E-12		-3.43E-12

(7) 式で $h\lambda = \alpha + \beta i = 0$ とおいたとき、特性多項式の根は $\xi_1 = 1$, $\xi_2 = -v$ となり.

A_0 -安定条件は

$$|\xi_2| = |-v| < 1$$

となる。しかるに (5) 式ではこの条件を満足するような parameter a_2, a_4 ($0 < a_2, a_4 \leq 1$) は存在しない。よって係数 (5) は A_0 -不安定である。



参考文献

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