

Entropy Densities for Gibbs States and Algebraic States (A joint work with D. Petz)

茨大理 日合文雄 (Fumio Hiai)

1. Entropy densities for Gibbs states

First we briefly survey Gibbs states of quantum spin systems. See [4] for more details. A quantum spin system over ν -dimensional cubic lattice $\mathbf{Z}^\nu$ is described as the infinite tensor product C^* -algebra $\mathcal{A} = \bigotimes_{x \in \mathbf{Z}^\nu} \mathcal{A}_x$, $\mathcal{A}_x = M_d(\mathbb{C})$ being the $d \times d$ matrix algebra; namely $\mathcal{A}$ is the norm completion of $\bigcup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{A}_x$ for finite regions $\Lambda \subset \mathbf{Z}^\nu$. The space-translation on $\mathcal{A}$ is denoted by γ_x ($x \in \mathbf{Z}^\nu$), which satisfies $\gamma_x(\mathcal{A}_\Lambda) = \mathcal{A}_{\Lambda+x}$ and the asymptotic abelianness property. A translation-invariant interaction Φ is a function from the finite $X \subset \mathbf{Z}^\nu$ into the selfadjoint elements of $\mathcal{A}$ such that $\Phi(X) \in \mathcal{A}_X$ and $\gamma_x(\Phi(X)) = \Phi(X+x)$ for all $X \subset \mathbf{Z}^\nu$ and $x \in \mathbf{Z}^\nu$. Then for each finite region Λ the local Hamiltonian $H(\Lambda)$ and the surface energy $W(\Lambda)$ are given as

$$H(\Lambda) = \sum_{X \subset \Lambda} \Phi(X), \quad W(\Lambda) = \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda^c \neq \emptyset}} \Phi(X).$$

We assume in the following that an interaction Φ is of relatively short range and of finite body; namely $\sum_{\Lambda \ni 0} \|\Phi(\Lambda)\|/|\Lambda| < +\infty$ and there exists N_0 such that $\Phi(\Lambda) = 0$ if $|\Lambda| > N_0$. This is the case if Φ is of finite range, i.e. there exists d_0 such that $\Phi(\Lambda) = 0$ if $d(\Lambda) > d_0$ ($d(\Lambda)$ denotes the diameter of Λ).

Given an interaction Φ and an inverse temperature $\beta (> 0)$ the local Gibbs state φ_Λ^c on a local algebra $\mathcal{A}_\Lambda$ is the canonical state:

$$\varphi_\Lambda^c(a) = \frac{\text{Tr}_\Lambda a e^{-\beta H(\Lambda)}}{\text{Tr}_\Lambda e^{-\beta H(\Lambda)}}, \quad a \in \mathcal{A}_\Lambda,$$

where Tr_Λ is the canonical trace on $\mathcal{A}_\Lambda$. In order to state the Gibbs condition we recall the inner perturbation of a state. For a state φ on $\mathcal{A}$ and $h = h^* \in \mathcal{A}$ the perturbed state $[\varphi^h]$ is defined as the unique minimizer of the weakly* lower semicontinuous strictly convex functional $\omega \mapsto S(\omega, \varphi) + \omega(h)$ on the state space of $\mathcal{A}$, where $S(\omega, \varphi)$ is the relative entropy. Then φ is said to satisfy the Gibbs condition (with respect to Φ and β) if $[\varphi^{-\beta W(\Lambda)}]|_{\mathcal{A}_\Lambda} = \varphi_\Lambda^c$ for any finite $\Lambda \subset \mathbf{Z}^\nu$. This definition of the Gibbs condition is a bit weaker than the usual one, but both are equivalent under the above assumption of Φ .

As thermodynamic limit $\Lambda \rightarrow \infty$ we consider the limit along the parallelepipeds; namely $\Lambda = \{x \in \mathbf{Z}^\nu : 0 \leq x_i < a_i, i = 1, \dots, \nu\}$ tends to infinity with $a_i \rightarrow \infty$ for $i = 1, \dots, \nu$. (Indeed, most results below hold true in a more general limit of van Hove.)

In the above situation we state the central result concerning Gibbs states as follows. For a translation-invariant state φ of $\mathcal{A}$ the following three conditions are equivalent:

- (i) φ satisfies the Gibbs condition with respect to Φ and β ;
- (ii) φ satisfies the KMS condition with respect to σ_t and β where

$$\sigma_t(a) = \lim_{\Lambda} e^{itH(\Lambda)} a e^{-itH(\Lambda)}, \quad a \in \mathcal{A};$$

- (iii) φ satisfies the variational principle:

$$s(\varphi) = \beta\varphi(A_\Phi) + p(\beta, \Phi),$$

where

$$s(\varphi) = \lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} S(\varphi_\Lambda) \quad (\varphi_\Lambda = \varphi|_{\mathcal{A}_\Lambda}),$$

$$A_\Phi = \sum_{\Lambda \ni 0} \frac{\Phi(\Lambda)}{|\Lambda|},$$

$$p(\beta, \Phi) = \lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} \log \text{Tr}_\Lambda e^{-\beta H(\Lambda)}.$$

($s(\varphi)$ is called the mean entropy and $p(\beta, \Phi)$ the pressure or the free energy.)

Via the GNS representation π_φ of $\mathcal{A}$ induced by the Gibbs state φ , we have a von Neumann algebra $\mathcal{M} = \pi_\varphi(\mathcal{A})''$ and a faithful normal state $\bar{\varphi}$ so that $\varphi = \bar{\varphi} \circ \pi_\varphi$. A net $\{x_j\}$ in $\mathcal{M}$ is said to converge almost uniformly to $x \in \mathcal{M}$ if for every $\varepsilon > 0$ there exists a projection $q \in \mathcal{M}$ such that $\bar{\varphi}(q) \geq 1 - \varepsilon$ and $\|(x_j - x)q\| \rightarrow 0$. The following result resembles the McMillan theorem from information theory.

Theorem 1.1. *Assume that φ is an ergodic Gibbs state for Φ .*

(1)

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} \pi_\varphi(-\log D_\Lambda) = s(\varphi)I \text{ strongly.}$$

(2) *If $\nu = 1$ and Φ is of finite range, then*

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} \pi_\varphi(-\log D_\Lambda) = s(\varphi)I \text{ almost uniformly.}$$

(Indeed the almost uniform convergence holds true also when $\nu > 1$ but $\beta < \beta_0$ for some $\beta_0 > 0$ determined by Φ .)

The above (2) was proved in [6] by using the noncommutative ergodic theorem [10] and Araki's deep analysis on Gibbs states [2, 3].

We define for $0 < \varepsilon < 1$

$$\beta_\varepsilon(\varphi_\Lambda) = \min\{\log \text{Tr}_\Lambda q : q \in \mathcal{A}_\Lambda \text{ is a projection with } \varphi(q) \geq 1 - \varepsilon\}.$$

If $\lambda_1 \geq \lambda_2 \geq \dots$ is the eigenvalue list of the density matrix of φ_Λ , then $\beta_\varepsilon(\varphi_\Lambda)$ is given by

$$\beta_\varepsilon(\varphi_\Lambda) = \log \left(\min \left\{ N : \sum_{i=1}^N \lambda_i \geq 1 - \varepsilon \right\} \right).$$

The following illustrates the macroscopic uniformity which is a basic feature of statistical mechanical systems, and it resembles the asymptotic equipartition property of information theory.

Theorem 1.2. Assume that φ is an ergodic Gibbs state for Φ .

(1) For every $0 < \varepsilon < 1$

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} \beta_\varepsilon(\varphi_\Lambda) = s(\varphi).$$

(2) If the surface energies $W(\Lambda)$ are uniformly bounded, then for every $0 < \varepsilon < 1$

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} \beta_\varepsilon(\varphi_\Lambda) = \lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} \beta_\varepsilon(\varphi_\Lambda^c) = s(\varphi).$$

(Note in this case that the Gibbs state is unique, i.e. the phase transition does not occur.)

(3) If $\nu = 1$ and Φ is of finite range, then

$$\lim_{n \rightarrow \infty} \frac{\beta_\varepsilon(\varphi_{[1,n]})}{n} = \lim_{n \rightarrow \infty} \frac{\beta_\varepsilon(\varphi_{[1,n]}^c)}{n} = \lim_{n \rightarrow \infty} \frac{S(\varphi_{[1,n]})}{n} = \lim_{n \rightarrow \infty} \frac{S(\varphi_{[1,n]}^c)}{n}.$$

For a translation-invariant state ω , the limit

$$h(\omega|\beta, \Phi) = \lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} S(\omega_\Lambda, \varphi_\Lambda^c)$$

exists due to the existence of the mean entropy and the pressure. In fact, we have

$$h(\omega|\beta, \Phi) = -s(\omega) + \beta\omega(A_\Phi) + p(\beta, \Phi),$$

and ω is a Gibbs state for Φ if and only of $h(\omega|\beta, \Phi) = 0$.

For states ψ_1, ψ_2 on a matrix algebra $\mathcal{B}$, a variant of the relative entropy is defined as

$$\begin{aligned} S_{\text{co}}(\psi_1, \psi_2) &= \max \left\{ \sum_i \psi_1(q_i) \log \frac{\psi_1(q_i)}{\psi_2(q_i)} : q_i \text{ are projections in } \mathcal{B}, \sum_i q_i = I \right\} \\ &= \max \{ S(\psi_1|\mathcal{C}, \psi_2|\mathcal{C}) : \mathcal{C} \text{ is a commutative } C^*\text{-subalgebra of } \mathcal{B} \}. \end{aligned}$$

The monotonicity of relative entropy implies $S_{\text{co}}(\psi_1, \psi_2) \leq S(\psi_1, \psi_2)$ and this inequality is strict except for the case of ψ_1, ψ_2 having the commuting densities. S_{co} may be related to measurements described by projection-valued measures.

Theorem 1.3. *Let ω be a translation-invariant state on $\mathcal{A}$.*

(1)

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} S_{\text{co}}(\omega_\Lambda, \varphi_\Lambda^c) = h(\omega|\beta, \Phi).$$

(2) *If $\nu = 1$, Φ is of finite range, and φ is the Gibbs state for Φ , then*

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} S_{\text{co}}(\omega_\Lambda, \varphi_\Lambda^c) = \lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} S_{\text{co}}(\omega_\Lambda, \varphi_\Lambda) = \lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} S(\omega_\Lambda, \varphi_\Lambda) = h(\omega|\beta, \Phi).$$

(Indeed this is true also when $\nu > 1$ but $\beta < \beta_0$ for some $\beta_0 > 0$.)

In the case (2), the mean relative entropy

$$S_M(\omega, \varphi) = \lim_{\Lambda \rightarrow \infty} \frac{1}{|\Lambda|} S(\omega_\Lambda, \varphi_\Lambda)$$

exists, and the relative entropy S and its variant S_{co} have the same asymptotic limit. The particular case when φ is a product state was shown in [7].

The results in Section 1 were given in [8] except Theorem 1.1(2) in [6].

2. Entropy densities for algebraic states

An algebraic state is a translation-invariant state on one-dimensional quantum spin system $\mathcal{A} = \bigotimes_{x \in \mathbb{Z}} \mathcal{A}_x$, $\mathcal{A}_x = M_d(\mathbb{C})$, which is a slight generalization of “quantum Markov states” [1] and identical to “ C^* -finitely correlated states” introduced in [5]. The definition is as follows: Let $\mathcal{B}$ be a finite-dimensional C^* -algebra, and let a completely positive unital map $\mathcal{E} : M_d(\mathbb{C}) \otimes \mathcal{B} \rightarrow \mathcal{B}$ and a state ρ on $\mathcal{B}$ be given so that

$$\rho(\mathcal{E}(I \otimes b)) = \rho(b), \quad b \in \mathcal{B}.$$

For any $a \in M_d(\mathbb{C})$ define $\mathcal{E}_a : \mathcal{B} \rightarrow \mathcal{B}$ by $\mathcal{E}_a(b) = \mathcal{E}(a \otimes b)$, $b \in \mathcal{B}$. Then the algebraic state φ generated by $(\mathcal{B}, \mathcal{E}, \rho)$ is given as

$$\varphi(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \rho(\mathcal{E}_{a_0} \circ \mathcal{E}_{a_1} \circ \cdots \circ \mathcal{E}_{a_n}(I))$$

for $a_0, \dots, a_n \in M_d(\mathbb{C})$. Here the complete positivity of $\mathcal{E}$ ensures the positivity of φ .

The ergodicity and the strong mixing for algebraic states are characterized in several ways as follows. (1) was shown in [5].

Proposition 2.1. *Let φ be an algebraic state generated by $(\mathcal{B}, \mathcal{E}, \rho)$.*

(1) *The following conditions are equivalent:*

- (i) φ is ergodic;
- (ii) $\mathcal{E}_I$ is irreducible, i.e. I is the only eigenvector of $\mathcal{E}_I$ with respect to the eigenvalue 1.

(2) *The following conditions are equivalent:*

- (i) φ is strongly mixing;
- (ii) φ is weakly mixing;
- (iii) φ is completely ergodic;
- (iv) $\mathcal{E}_I$ is primitive, i.e. $\mathcal{E}_I^n$ is irreducible for all $n \in \mathbb{N}$;
- (v) $\lim_{n \rightarrow \infty} \mathcal{E}_I^n(x) = \rho(x)I$ for all $x \in \mathcal{B}$.

Let ψ_1, ψ_2 be states on a matrix algebra $\mathcal{B}$. For $0 < \varepsilon < 1$ we define another type of relative entropy quantity as follows:

$$\beta_\varepsilon(\psi_1, \psi_2) = \inf\{\log \psi_2(q) : q \text{ is a projection in } \mathcal{B}, \psi_1(q) \geq 1 - \varepsilon\}.$$

This quantity has a natural meaning from the viewpoint of quantum hypothesis testing (see [7]).

The following are our main results in [9].

Theorem 2.2. *Assume that φ be a strongly mixing algebraic state on $\mathcal{A}$. Let $\varphi_n = \varphi|_{\mathcal{A}_{[1,n]}}$.*

(1) *For every $0 < \varepsilon < 1$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \beta_\varepsilon(\varphi_n) = s(\varphi).$$

(2) *For every translation-invariant state ω on $\mathcal{A}$, the mean relative entropy $S_M(\omega, \varphi)$ exists and*

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_{\text{co}}(\omega_n, \varphi_n) = S_M(\omega, \varphi).$$

(3) For every completely ergodic state ω on $\mathcal{A}$ and $0 < \varepsilon < 1$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \beta_\varepsilon(\omega_n, \varphi_n) \leq -S_M(\omega, \varphi),$$

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \beta_\varepsilon(\omega_n, \varphi_n) \geq -\frac{1}{1-\varepsilon} S_M(\omega, \varphi).$$

The above (3) shows that we obtain $\exp\{\frac{1}{n} \beta_\varepsilon(\omega_n, \varphi_n)\} \approx \exp\{-S_M(\omega, \varphi)\}$ for large n and small ε . To prove the theorem, we need the next result which says that strongly mixing algebraic states have a certain property of approximately product type.

Proposition 2.3. *Let φ be the algebraic state generated by $(\mathcal{B}, \mathcal{E}, \rho)$. Then φ is strongly mixing if and only if for any $\alpha > 1$ there exists $l \in \mathbb{N}$ such that for all $m \in \mathbb{N}$*

$$\begin{aligned} \alpha^{-1}(\varphi|_{\mathcal{A}_{(-\infty, m]}}) \otimes (\varphi|_{\mathcal{A}_{[m+l+1, \infty)}}) &\leq \varphi|_{\mathcal{A}_{(-\infty, m] \cup [m+l+1, \infty)}} \\ &\leq \alpha(\varphi|_{\mathcal{A}_{(-\infty, m]}}) \otimes (\varphi|_{\mathcal{A}_{[m+l+1, \infty)}}). \end{aligned}$$

References

- [1] L. Accardi and A. Frigerio, Markovian cocycles, *Proc. Roy. Irish Acad.* **83A**(2) (1983), 251–263.
- [2] H. Araki, Gibbs states of a one dimensional quantum lattice, *Commun. Math. Phys.* **14** (1969), 120–157.
- [3] H. Araki, Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of the Tomita-Takesaki theory, in *Proc. Internat. School of Physics (Enrico Fermi)*, pp. 64–100 (1976).
- [4] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, Springer, New York-Heidelberg-Berlin, 1981.
- [5] M. Fannes, B. Nachtergaele, and R.F. Werner, Finitely correlated states on quantum spin chains, *Comm. Math. Phys.* **144** (1992), 443–490.
- [6] F. Hiai, M. Ohya and D. Petz, McMillan type convergence for quantum Gibbs states, preprint.
- [7] F. Hiai and D. Petz, The proper formula for relative entropy and its asymptotics in quantum probability, *Comm. Math. Phys.* **143** (1991), 99–114.
- [8] F. Hiai and D. Petz, Entropy densities for Gibbs states of quantum spin systems, *Rev. Math. Phys.* (to appear).
- [9] F. Hiai and D. Petz, Entropy densities for algebraic states, preprint.
- [10] E.C. Lance, Ergodic theorem for convex sets and operator algebras, *Invent. Math.* **37** (1976), 201–214.