

Approximate innerness of positive linear maps of factors of type II

東北大学 教授 武元英夫 (Hideo Takemoto)

We in this paper shall discuss the properties of positive linear maps which continue from the former work by the author [7].

Let M be a σ -finite, semi-finite von Neumann algebra, then there exists a faithful, normal semi-finite trace Tr and we can define a norm $||\cdot||_2$ on the ideal $S = \{x \in M; \text{Tr}(x^*x) < +\infty\}$. In particular, if M is a finite von Neumann algebra, then $S = M$.

Let A and B be C^* -algebras. A linear map ρ of A to B is said to be n -positive if the multiplicity map ρ_n from the matrix algebra $M_n(A)$ over A to the algebra $M_n(B)$ over B defined by $\rho_n([a_{ij}]) = [\rho(a_{ij})]$ is a positive map. If ρ is n -positive for every positive integer n , we call ρ completely positive. Many authors (for example, [1], [5], [7], [8] and [9]) studied the completely positive linear maps of C^* -algebras. In particular, we have the following Stinespring's theorem [5]: Let A be a C^* -algebra and ρ a completely positive linear map of A to $B(H)$ where $B(H)$ is the von Neumann algebra of all bounded operators on a Hilbert space H . Then, there exists a representation π of A to a Hilbert space K and a bounded operator v of H to K such that $\rho(x) = v^*\pi(x)v$ for every

$x \in A$. In particular, if ρ is unital (ie. $\rho(1) = 1$), v is an isometry. Furthermore, A is a von Neumann algebra and ρ is normal, then π is a normal representation. We in general can not take the above operator v in A . For this problem, we have the following result by Haagerup [3; Proposition 2.1]: Let N be a properly infinite von Neumann algebra and let F be a finite dimensional subfactor. Let ρ be a completely positive map from F to N . Then there exists an element $a \in N$ such that $\rho(x) = a^*xa$ for every $x \in F$. In this report, we shall consider the above problem for finite von Neumann algebras by using the approximate innerness and extend the obtained results to the semi-finite von Neumann algebras. Thus, we here introduce the notation of the approximate innerness.

Definition 1. Let M be a σ -finite, finite von Neumann algebra with a fixed faithful, normalized normal trace Tr and A a C^* -subalgebra of M . A positive linear map ρ of A into M is approximate inner if there exists a net $\{a_\lambda\}$ (not necessarily bounded) in M satisfying $\lim ||\rho(x) - a_\lambda^*xa_\lambda||_2 = 0$ for every $x \in A$.

If we consider the approximate innerness for positive linear maps, we can show that those positive linear maps are closely related to the $*$ -homomorphisms. Before we denote the theorems, we shall mention the following lemma by Choi [1] (and also see [9]).

Lemma 2. Let A and B be unital C^* -algebra and ρ a unital completely positive map of A to B . If ρ is a C^* -homomorphism (ie., $\rho(a^2) = \rho(a)^2$ for every self-adjoint element a of A), then ρ is a $*$ -homomorphism of A to B .

Consider Lemma 2, we have the following theorem that a positive linear map with the approximate innerness is closely related to $*$ -homomorphism. The following theorem is in a sense a generalization of Theorem 3 in [7].

Theorem 3. Let M be a σ -finite, finite von Neumann algebra and A a C^* -subalgebra with the unit in M . Let ρ be a positive linear map of A to M and approximate inner with respect to a net $\{a_\lambda\}$ such that $\|a_\lambda * a_\lambda - e\|_2 \rightarrow 0$ and $\|a_\lambda a_\lambda^* - f\|_2 \rightarrow 0$ for a projection e of M and a projection f in A . Then, ρ is a $*$ -homomorphism of fAf to eMe .

Proof. By the assumption for the net $\{a_\lambda\}$ and the approximate innerness of ρ with respect to $\{a_\lambda\}$, $\rho(1) = e$ and $\rho(1 - f) = 0$. Thus, we can assume that $fa_\lambda e = a_\lambda$ for every $\lambda \in \Lambda$. By the remark before Definition 1, ρ is completely positive map. So ρ is a unital completely positive map of C^* -algebra fAf to von Neumann algebra eMe . To show that ρ is a $*$ -homomorphism of fAf to eMe , we must show by Lemma 2 that

$\rho(x^2) = \rho(x)^2$ for every self-adjoint element $x \in fAf$. Given an arbitrary self-adjoint element $x \in fAf$. Then,

$$\|\rho(x) - a_\lambda * x a_\lambda\|_2^2 = \text{Tr}(\rho(x)^2) - 2\text{Tr}(\rho(x)a_\lambda * x a_\lambda) + \text{Tr}(a_\lambda * x a_\lambda a_\lambda * x a_\lambda).$$

Now, since

$$\begin{aligned} & |\text{Tr}(a_\lambda * x a_\lambda a_\lambda * x a_\lambda - a_\lambda * x^2 a_\lambda)| \\ &= |\text{Tr}(a_\lambda * x(a_\lambda a_\lambda^* - f)x a_\lambda)| = |\text{Tr}((a_\lambda a_\lambda^* - f)x a_\lambda a_\lambda^* x)| \\ &\leq \text{Tr}((a_\lambda a_\lambda^* - f)^2)^{1/2} \text{Tr}(x a_\lambda a_\lambda^* x^2 a_\lambda a_\lambda^* x)^{1/2} \\ &\leq \|a_\lambda a_\lambda^* - f\|_2 \cdot \|x\| \cdot \text{Tr}(x a_\lambda a_\lambda^* a_\lambda a_\lambda^* x)^{1/2} \\ &\leq \|x\|^2 \cdot \|a_\lambda a_\lambda^* - f\|_2 \cdot \text{Tr}(a_\lambda a_\lambda^* a_\lambda a_\lambda^*)^{1/2} \\ &= \|x\|^2 \cdot \|a_\lambda a_\lambda^*\|_2 \cdot \|a_\lambda a_\lambda^* - f\|_2, \end{aligned}$$

$\{\|a_\lambda a_\lambda^*\|_2\}$ is bounded and $\lim \|a_\lambda a_\lambda^* - f\|_2 = 0$, we have the relation

$$\lim \{\text{Tr}(a_\lambda * x a_\lambda a_\lambda * x a_\lambda) - \text{Tr}(a_\lambda * x^2 a_\lambda)\} = 0.$$

Thus, since $\lim \text{Tr}(a_\lambda * x^2 a_\lambda) = \text{Tr}(\rho(x^2))$ by the assumption,

$\lim \text{Tr}(a_\lambda * x a_\lambda a_\lambda * x a_\lambda) = \text{Tr}(\rho(x^2))$. Furthermore, since

$$\begin{aligned} |\text{Tr}(\rho(x) a_\lambda * x a_\lambda) - \text{Tr}(\rho(x)^2)| &= |\text{Tr}(\rho(x)(a_\lambda * x a_\lambda - \rho(x)))| \\ &\leq \|\rho(x)\|_2 \cdot \|\rho(x) - a_\lambda * x a_\lambda\|_2, \end{aligned}$$

$\lim \text{Tr}(\rho(x) a_\lambda * x a_\lambda) = \text{Tr}(\rho(x)^2)$. By the above considerations and the relation $\lim \|\rho(x) - a_\lambda * x a_\lambda\|_2 = 0$,

$$\text{Tr}(\rho(x)^2) - 2\text{Tr}(\rho(x)^2) + \text{Tr}(\rho(x^2)) = 0.$$

So, $\text{Tr}(\rho(x^2) - \rho(x)^2) = 0$. Now, since ρ is a completely positive map, $\rho(x)^2 \leq \rho(x^2)$. Therefore, $\rho(x^2) = \rho(x)^2$ and so, by Lemma 2, ρ is a $*$ -homomorphism of fAf to eMe . q.e.d.

Under the definition of approximate innerness, if ρ is approximate inner, then ρ is completely positive like as [7]. Furthermore, we can replace the conditions in Theorem 3 as the following by the remark in [7]. That is, if ρ is approximately inner with respect to $\{a_\lambda\}$ and $\rho(1) = e$ is a projection, then the conditions in Theorem 3 is equivalent that A has a projection f satisfying $\rho(1 - f) = 0$ and $\text{Tr}(e) = \text{Tr}(f)$.

By considering Theorem 3 and a Sakai's result [4], we have the following theorem.

Theorem 4. Let M be an approximately finite dimensional factor of type II_1 . Let ρ be a positive linear map of M into M such that $\rho(1) = e$ is a projection, $\rho(1 - f) = 0$ and $\text{Tr}(e) = \text{Tr}(f)$ for a projection f of M . Then ρ is approximately inner with respect to a net $\{a_\lambda\}$ if and only if ρ is a $*$ -isomorphism of fMf to eMe .

Proof. Necessity: By Theorem 3, ρ is a $*$ -homomorphism of fMf to eMe , and so the kernel of ρ in fMf is a closed two-sided ideal. Since M is a finite factor, the kernel of $\rho = \{0\}$ and so ρ is a $*$ -isomorphism of fMf to eMe .

Sufficiency: Since M is an approximately finite dimensional factor of type II_1 , both fMf and eMe are so. Let $fMf =$

$\bigcup A_n$ ($\bigcup$ means the weak closure of $\cdot$) where A_n is a subfactor of type I_{2^n} of fMf satisfying $A_n \subset A_{n+1}$ ($n = 1, 2, \dots$). Let $\{f_{ij}^{(n)}\}_{i,j=1}^{2^n}$ be the matrix units of A_n . Put $B_n = \rho(A_n)$, then $\rho(fMf) = N = \bigcup B_n$ eMe and B_n is a factor of type I_{2^n} . Furthermore, put $e_{ij}^{(n)} = \rho(f_{ij}^{(n)})$, then $\{e_{ij}^{(n)}\}$ is

the matrix units for B_n . It is sufficient for us to show that,

for an arbitrary finite set $\{a_1, \dots, a_k\}$ in fMf and each $\epsilon > 0$, there exists an element $u \in M$ such that $\|\rho(a_j) -$

$u^*a_ju\|_2 < \epsilon$ ($j = 1, 2, \dots, k$). Given any finite set $\{a_1, \dots,$

$\dots, a_k\}$ in fMf and $\epsilon > 0$, then there exist a positive integer

m and $\{b_1, \dots, b_k\}$ in A_m such that $\|a_j - b_j\|_2 < \varepsilon/2$ ($j = 1, 2, \dots, k$). Since $\text{Tr}(e) = \text{Tr}(f)$,

$$\sum_{i=1}^{2^m} e_{ii}^{(m)} = e \quad \text{and} \quad \sum_{i=1}^{2^m} f_{ii}^{(m)} = f,$$

$\text{Tr}(f_{ii}^{(m)}) = \text{Tr}(e_{ii}^{(m)})$. And so, there exists a partial isometry v in M such that $vv^* = f_{ii}^{(m)}$ and $v^*v = e_{ii}^{(m)}$. Put $u = \sum_{i=1}^{2^m} f_{ii}^{(m)} v e_{ii}^{(m)}$, then u is an element of M and $u^*u = e$.

Furthermore, we have the following;

$$\begin{aligned} u^* f_{ij}^{(m)} u &= \left(\sum_{s=1}^{2^m} e_{s1}^{(m)} v^* f_{is}^{(m)} \right) f_{ij}^{(m)} \left(\sum_{t=1}^{2^m} f_{t1}^{(m)} v e_{it}^{(m)} \right) \\ &= \sum_{s,t=1}^{2^m} e_{s1}^{(m)} v^* f_{is}^{(m)} f_{ij}^{(m)} f_{t1}^{(m)} v e_{it}^{(m)} = \sum_{s,t=1}^{2^m} e_{s1}^{(m)} v^* (\delta_{si} \delta_{tj} f_{11}^{(m)}) v e_{it}^{(m)} \\ &= e_{i1}^{(m)} v^* f_{11}^{(m)} v e_{1j}^{(m)} = e_{i1}^{(m)} v^* v e_{1j}^{(m)} = e_{i1}^{(m)} e e_{1j}^{(m)} = e_{ij}^{(m)}. \end{aligned}$$

Thus, $u^* f_{ij}^{(m)} u = e_{ij}^{(m)}$ for $i, j = 1, 2, \dots, 2^m$. And so, $u^* x u = \rho(x)$ for every $x \in A_m$. In particular, $\rho(b_j) = u^* b_j u$ ($j = 1, 2, \dots, k$). Furthermore, we have the following relations;

$$\begin{aligned} \|\rho(a_j) - \rho(b_j)\|_2 &= \text{Tr}((\rho(a_j) - \rho(b_j))^* (\rho(a_j) - \rho(b_j)))^{1/2} \\ &= \text{Tr}(f)^{1/2} \text{Tr}((a_j - b_j)^* (a_j - b_j))^{1/2} = \text{Tr}(f)^{1/2} \|a_j - b_j\|_2 \end{aligned}$$

$$\leq \|a_j - b_j\|_2 < \varepsilon/2 \quad \text{and}$$

$$\begin{aligned} \|u^*a_ju - u^*b_ju\|_2 &= \text{Tr}(u^*(a_j - b_j)^*(a_j - b_j)u)^{1/2} \\ &= \text{Tr}(uu^*(a_j - b_j)^*(a_j - b_j))^{1/2} = \text{Tr}((a_j - b_j)^*(a_j - b_j))^{1/2} \\ &= \|a_j - b_j\|_2 < \varepsilon/2. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\|\rho(a_j) - u^*a_ju\|_2 \\ &\leq \|\rho(a_j) - \rho(b_j)\|_2 + \|\rho(b_j) - u^*b_ju\|_2 + \|u^*b_ju - u^*a_ju\|_2 \\ &< \varepsilon/2 + \varepsilon/2 < \varepsilon \quad \text{for } j = 1, 2, \dots, k. \end{aligned}$$

Therefore, we have the complete proof of Theorem 4. q.e.d.

Remark. In the former work [6] by the author, we have the error in the proof of Theorem 1 in [6] and so we must replace that. Because the results in this report are closely related to the results in [6]. Consider the results in this report and [2] and [3] in the references we have the following considerations for [6]. We replace Theorem 1 in [6] as Theorem 4 in this report and Proposition 2 in [6] as Theorem 3 in this report. Further-

more, if we consider a Haagerup's result [3], the C^* -subalgebra A appeared in Theorem 2 in [6] was an MAF- C^* -subalgebra but we must replace the algebra A as an AF- C^* -subalgebra. The last result (Corollary 4) in [6] is right by [2].

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References

- [1] M.D. Choi; A Schwartz inequality for positive linear maps C^* -algebras, Illinois J. Math., 18(1974), 565 - 574.
- [2] G.A. Elliott and H. Takemoto; On C^* -algebra extensions relative to a factor of type II, Math. Scand., 53(1983), 245 - 255.
- [3] U. Haagerup; A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space, preprint.
- [4] S. Sakai; On automorphism groups of II_1 -factor, Tohoku Math. Journ., 26(1974), 423 - 430.
- [5] W.F. Stinespring; Positive maps on C^* -algebras, Proc. Amer. Math. Soc., 6(1955), 211 - 216.
- [6] H. Takemoto; Approximate innerness of positive linear maps of factors of type II, Proc. Japan Acad., 56(1980), 417 - 421.

- [7] H. Takemoto; Approximate innerness of positive linear maps of finite von Neumann algebras, to appear in Proc. Amer. Math. Soc., vol. 94, No. 3 (1985).
- [8] M. Takesaki; Theory of operator algebras I, Springer-Verlag, 1979.
- [9] J. Tomiyama; Complete positivity in operator algebras (Japanese), Lecture Note No. 4, RIMS Kyoto Univ., 1978.