

N- Fractional Calculus of Some Irrational Functions.

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Abstract

In this article N- fractional calculus of the functions (which have multiple root signs)

$$f(z) = \sqrt{\sqrt{\sqrt{z-b}-c}-d}$$

are discussed .

Theorem 1. We have

$$(i) \quad (f(z))_\gamma = e^{-i\pi\gamma} (z-b)^{(1/8)-\gamma} \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k \Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{m! \cdot k! \Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})} \\ \times \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m, \quad \left(\left| \frac{\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})} \right| < \infty \right)$$

and

$$(ii) \quad (f(z))_n = (-1)^n (z-b)^{(1/8)-n} \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k [\frac{k}{2}-\frac{1}{8}+\frac{m}{4}]_n}{m! \cdot k!} \\ \times \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m, \quad (n \in \mathbb{Z}_0^+)$$

where

$$\left| \frac{c}{\sqrt{z-b}} \right| < 1, \quad \left| \frac{d}{\sqrt[4]{z-b}} \right| < 1.$$

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with
$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in C$.
(vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group) [3]

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^b)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-b)}{\Gamma(-b)} (z-c)^{b-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-b)}{\Gamma(-b)} \right| < \infty \right), \quad (7)$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty), \quad (8)$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty), \quad (9)$$

where $z-c \neq 0$ for (i) and $z-c \neq 0, 1$ for (ii), (iii),

§ 1. Preliminary

[I] The theorem below is reported by K. Nishimoto already (cf. J. Frac. Calc. Vol. 29, May (2006), p.37). [12]

Theorem D. We have

$$(i) \quad (((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta} \right)^k \quad \left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right) \quad (1)$$

and

$$(ii) \quad (((z-b)^\beta - c)^\alpha)_n = (-1)^n (z-b)^{\alpha\beta-n} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (2)$$

where

$$|c/(z-b)^\beta| < 1,$$

and

$$[\lambda]_k = \lambda(\lambda+1) \cdots (\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad \text{with} \quad [\lambda]_0 = 1.$$

(Notation of Pochhammer).

[11] The theorem below is reported by K. Nishimoto already (cf. J. Frac. Calc. Vol. 31, May (2007), p.13). [13]

Theorem E. We have

$$(i) \quad (((z-b)^\beta - c)^\alpha - d)^\delta)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta\delta-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{m! \cdot k! \Gamma(\beta k - \alpha\beta(\delta-m))} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{\Gamma(\beta k - \alpha\beta(\delta-m))} \right| < \infty \right) \quad (3)$$

and

$$(ii) \quad (((z-b)^\beta - c)^\alpha - d)^\delta)_n = (-1)^n (z-b)^{\alpha\beta\delta-n}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k [\beta k - \alpha\beta(\delta-m)]_n}{m! \cdot k!} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m$$

$$(n \in \mathbb{Z}_0^+) \quad (4)$$

where

$$((z-b)^\beta - c)^\alpha - d \neq 0, \quad |c/(z-b)^\beta| < 1, \quad |d/(z-b)^{\alpha\beta}| < 1.$$

§ 2. N- Fractional Calculus of Functions

$$\sqrt{\sqrt{\sqrt{z-b-c-d}}}$$

Theorem 1. We have

$$(i) \quad \left(\sqrt{\sqrt{\sqrt{z-b-c-d}}} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{(1/8)-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2} - \frac{1}{4}]_k \Gamma(\frac{k}{2} - \frac{1}{8} + \frac{m}{4} + \gamma)}{m! \cdot k! \Gamma(\frac{k}{2} - \frac{1}{8} + \frac{m}{4})} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \quad (1)$$

$$\left(\left| \frac{\Gamma(\frac{k}{2} - \frac{1}{8} + \frac{m}{4} + \gamma)}{\Gamma(\frac{k}{2} - \frac{1}{8} + \frac{m}{4})} \right| < \infty \right)$$

and

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$$\begin{aligned}
 (ii) \quad & \left(\sqrt{\sqrt{\sqrt{z-b-c-d}}} \right)_n = (-1)^n (z-b)^{(1/8)-n} \\
 & \times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k [\frac{k}{2}-\frac{1}{8}+\frac{m}{4}]_n}{m! \cdot k!} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \quad (2) \\
 & (n \in \mathbb{Z}_0^+)
 \end{aligned}$$

where

$$\left| \frac{c}{\sqrt{z-b}} \right| < 1, \quad \left| \frac{d}{\sqrt[4]{z-b}} \right| < 1.$$

Proof of (i). We have

$$\sqrt{\sqrt{\sqrt{z-b-c-d}}} = (((z-b)^{1/2} - c)^{1/2} - d)^{1/2}, \quad (3)$$

hence, operating N^γ to the both sides of (3), we obtain

$$\left(\sqrt{\sqrt{\sqrt{z-b-c-d}}} \right)_\gamma = \left((((z-b)^{1/2} - c)^{1/2} - d)^{1/2} \right)_\gamma \quad (4)$$

$$= e^{-i\pi\gamma} (z-b)^{(1/8)-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k \Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{m! \cdot k! \Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \quad (5)$$

setting $\alpha = \beta = \delta = 1/2$ in Theorem E. (i) in preliminary, under the conditions stated before.

Proof of (ii). Set $\gamma = n$ in (i).

Corollary 1. We have

$$\begin{aligned}
 (i) \quad & \left(\sqrt{\sqrt{\sqrt{z-b-c}}} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{(1/8)-\gamma} \\
 & \times \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k \Gamma(\frac{k}{2}-\frac{1}{8}+\gamma)}{k! \Gamma(\frac{k}{2}-\frac{1}{8})} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (6) \\
 & \left(\left| \frac{\Gamma(\frac{k}{2}-\frac{1}{8}+\gamma)}{\Gamma(\frac{k}{2}-\frac{1}{8})} \right| < \infty \right)
 \end{aligned}$$

and

$$(ii) \quad \left(\sqrt{\sqrt{\sqrt{z-b}-c}} \right)_n = (-1)^n (z-b)^{(1/8)-n} \times \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k [\frac{k}{2}-\frac{1}{8}]_n}{k!} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (7)$$

$(n \in \mathbb{Z}_0^+)$

where

$$\left| \frac{c}{\sqrt{z-b}} \right| < 1.$$

Proof of (i). Set $d = 0$ in Theorem 1 (i).

We have (6) from Theorem D, (i), setting $\beta = 1/2$ and $\alpha = 1/4$.

Proof of (ii). Set $\gamma = n$ in (i).

Corollary 2. We have

$$(i) \quad \left(\sqrt{\sqrt{\sqrt{z-b}}} \right)_\gamma = e^{-i\pi\gamma} \frac{\Gamma(-\frac{1}{8}+\gamma)}{\Gamma(-\frac{1}{8})} (z-b)^{(1/8)-\gamma} \quad (8)$$

$(|\Gamma(-\frac{1}{8}+\gamma)| < \infty)$

and

$$(ii) \quad \left(\sqrt{\sqrt{\sqrt{z-b}}} \right)_n = (-1)^n [-\frac{1}{8}]_n (z-b)^{(1/8)-n} \quad (9)$$

$(n \in \mathbb{Z}_0^+)$

where

$$z-b \neq 0.$$

Proof of (i). Set $c = 0$ in Corollary 1 (i).

Or set $c = d = 0$ in Theorem 1. (i).

Proof of (ii). Set $\gamma = n$ in (i).

Note. We have

$$\left(\sqrt{\sqrt{\sqrt{z-b}}} \right)_\gamma = ((z-b)^{1/8})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(-\frac{1}{8}+\gamma)}{\Gamma(-\frac{1}{8})} (z-b)^{(1/8)-\gamma} \quad (10)$$

from Lemma (i), directly.

§ 3. Semi Derivatives and Integrals

Theorem 2. We have

$$\begin{aligned}
 (i) \quad & \left(\sqrt{\sqrt{\sqrt{z-b-c-d}}} \right)_{1/2} = -i(z-b)^{-3/8} \\
 & \times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2} - \frac{1}{4}]_k \Gamma(\frac{k}{2} + \frac{3}{8} + \frac{m}{4})}{m! \cdot k! \Gamma(\frac{k}{2} - \frac{1}{8} + \frac{m}{4})} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \quad (1) \\
 & \text{(semi derivatives)}
 \end{aligned}$$

and

$$\begin{aligned}
 (ii) \quad & \left(\sqrt{\sqrt{\sqrt{z-b-c-d}}} \right)_{-1/2} = i(z-b)^{5/8} \\
 & \times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2} - \frac{1}{4}]_k \Gamma(\frac{k}{2} - \frac{5}{8} + \frac{m}{4})}{m! \cdot k! \Gamma(\frac{k}{2} - \frac{1}{8} + \frac{m}{4})} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \quad (2) \\
 & \text{(semi integrals)}
 \end{aligned}$$

where

$$\left| \frac{c}{\sqrt{z-b}} \right| < 1, \quad \left| \frac{d}{\sqrt[4]{z-b}} \right| < 1.$$

Proof. Set $\gamma = 1/2$ and $-1/2$ in Theorem 1(i) we have then (i) and (ii) respectively, clearly.

Corollary 3. We have

$$\begin{aligned}
 (i) \quad & \left(\sqrt{\sqrt{\sqrt{z-b-c}}} \right)_{1/2} = -i(z-b)^{-3/8} \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k \Gamma(\frac{k}{2} + \frac{3}{8})}{k! \Gamma(\frac{k}{2} - \frac{1}{8})} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (3) \\
 & \text{(semi derivatives)}
 \end{aligned}$$

and

$$\begin{aligned}
 (ii) \quad & \left(\sqrt{\sqrt{\sqrt{z-b-c}}} \right)_{-1/2} = i(z-b)^{5/8} \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k \Gamma(\frac{k}{2} - \frac{5}{8})}{k! \Gamma(\frac{k}{2} - \frac{1}{8})} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (4) \\
 & \text{(semi integrals)}
 \end{aligned}$$

where

$$\left| \frac{c}{\sqrt{z-b}} \right| < 1.$$

Proof. Set $d = 0$ in Theorem 2.

Corollary 4. We have

$$(i) \quad \left(\sqrt{\sqrt{\sqrt{z-b}} - d} \right)_{1/2} = -i(z-b)^{-3/8} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m \Gamma(\frac{3}{8} + \frac{m}{4})}{m!! \Gamma(-\frac{1}{8} + \frac{m}{4})} \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \quad (5)$$

(semi derivatives)

and

$$(ii) \quad \left(\sqrt{\sqrt{\sqrt{z-b}} - d} \right)_{-1/2} = i(z-b)^{5/8} \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m \Gamma(-\frac{5}{8} + \frac{m}{4})}{m! \Gamma(-\frac{1}{8} + \frac{m}{4})} \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \quad (2)$$

(semi integrals)

where

$$\left| \frac{d}{\sqrt[4]{z-b}} \right| < 1.$$

Proof. Set $c = 0$ in Theorem 2.

§ 4. Some Special Cases

[I] When $n = 0$, we have

$$\begin{aligned} \left(\sqrt{\sqrt{\sqrt{z-b-c-d}}} \right)_0 &= (z-b)^{1/8} \\ &\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2} - \frac{1}{4}]_k}{m! \cdot k!} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{\sqrt[4]{z-b}} \right)^m \end{aligned} \quad (1)$$

from § 2. (2).

Now we have

$$\text{RHS of (1)} = (z-b)^{1/8} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} T^m \sum_{k=0}^{\infty} \frac{[\frac{m}{2} - \frac{1}{4}]_k}{k!} S^k \quad (2)$$

$$\left(S = \frac{c}{\sqrt{z-b}}, \quad T = \frac{d}{\sqrt[4]{z-b}} \right)$$

$$= (z-b)^{1/8} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} T^m (1-S)^{(1/4)-(m/2)} \quad (3)$$

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$$= (z-b)^{1/8} (1-S)^{1/4} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m T^m}{m! (1-S)^{m/2}} \quad (4)$$

$$= (z-b)^{1/8} (1-S)^{1/4} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} \left(\frac{d}{\sqrt{\sqrt{z-b}-c}} \right)^m \quad (5)$$

$$= (z-b)^{1/8} \left(1 - \frac{c}{\sqrt{z-b}} \right)^{1/4} \left(1 - \frac{d}{\sqrt{\sqrt{z-b}-c}} \right)^{1/2} \quad (6)$$

$$= \sqrt{\sqrt{\sqrt{z-b}-c}-d} \quad (\text{LHS of (1)}) \quad (7)$$

[II] When $n=1$, we have

$$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d} \right)_1 = -(z-b)^{-7/8} \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k [\frac{k}{2}+\frac{m}{4}-\frac{1}{8}]_1}{m! \cdot k!} S^k T^m \quad (8)$$

$$= -(z-b)^{-7/8} \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k (\frac{k}{2}+\frac{m}{4}-\frac{1}{8})}{m! \cdot k!} S^k T^m \quad (9)$$

from § 2. (2).

Now we have the identities;

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}, \quad (10)$$

$$[\lambda]_0 = 1, \quad [\lambda]_1 = \lambda, \quad (11)$$

$$[\lambda]_{k+1} = \lambda [\lambda+1]_k, \quad (12)$$

hence

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{[\lambda]_k k}{k!} z^k &= \sum_{k=1}^{\infty} \frac{[\lambda]_k}{(k-1)!} z^k = z \sum_{k=0}^{\infty} \frac{[\lambda]_{k+1}}{k!} z^k \\ &= \lambda z (1-z)^{-\lambda-1}. \end{aligned} \quad (13)$$

Therefore, we have

$$\sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k (\frac{k}{2})}{m! \cdot k!} S^k T^m = \frac{1}{2} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} T^m \sum_{k=0}^{\infty} \frac{[\frac{m}{2}-\frac{1}{4}]_k k}{k!} S^k \quad (14)$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} T^m \left(\frac{m}{2} - \frac{1}{4} \right) S \sum_{k=0}^{\infty} \frac{[\frac{m}{2} + \frac{3}{4}]_k}{k!} S^k \quad (15)$$

$$= \frac{1}{2} S(1-S)^{-3/4} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m (\frac{m}{2} - \frac{1}{4})}{m!} U^m \quad \left(U = \frac{d}{\sqrt{\sqrt{z-b}-c}} \right) \quad (16)$$

$$= \frac{1}{2} S(1-S)^{-3/4} \left\{ -\frac{1}{4} U(1-U)^{-1/2} - \frac{1}{4} (1-U)^{1/2} \right\} \quad (17)$$

$$= -\frac{1}{8} S(1-S)^{-3/4} (1-U)^{-1/2}, \quad (18)$$

$$\sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2} - \frac{1}{4}]_k (\frac{m}{4})}{m! k!} S^k T^m = \frac{1}{4} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m m}{m!} T^m \sum_{k=0}^{\infty} \frac{[\frac{m}{2} - \frac{1}{4}]_k}{k!} S^k \quad (19)$$

$$= \frac{1}{4} (1-S)^{1/4} \sum_{m=1}^{\infty} \frac{[-\frac{1}{2}]_m}{(m-1)!} U^m \quad (20)$$

$$= \frac{1}{4} (1-S)^{1/4} U \left(-\frac{1}{2} \right) \sum_{m=0}^{\infty} \frac{[\frac{1}{2}]_m}{m!} U^m \quad (21)$$

$$= -\frac{1}{8} (1-S)^{1/4} U(1-U)^{-1/2}. \quad (22)$$

and

$$\sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2} - \frac{1}{4}]_k (-\frac{1}{8})}{m! k!} S^k T^m = -\frac{1}{8} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} T^m \sum_{k=0}^{\infty} \frac{[\frac{m}{2} - \frac{1}{4}]_k}{k!} S^k \quad (23)$$

$$= -\frac{1}{8} (1-S)^{1/4} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} U^m \quad (24)$$

$$= -\frac{1}{8} (1-S)^{1/4} (1-U)^{1/2}. \quad (25)$$

Then applying (18), (22) and (25) to (9) we obtain

$$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d} \right)_1 = \frac{1}{8} (z-b)^{-7/8} (1-S)^{1/4} (1-U)^{-1/2} \{ S(1-S)^{-1} + 1 \} \quad (26)$$

$$= \frac{1}{8}(z-b)^{-7/8}(1-S)^{1/4}(1-U)^{-1/2}(1-S)^{-1} \quad (27)$$

$$= \frac{1}{8}(z-b)^{-7/8}(1-S)^{-3/4}(1-U)^{-1/2} \quad (28)$$

$$= \frac{1}{8}(z-b)^{-1/2}(\sqrt{z-b}-c)^{-1/2}(\sqrt{\sqrt{z-b}-c}-d)^{-1/2} \quad (29)$$

This result (29) coincides with the one obtained by classical calculus.

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