

Cluster algebras and higher order generalizations of the q -Painlevé equations of type $A_7^{(1)}$ and $A_6^{(1)}$

By

Tetsu MASUDA*, Naoto OKUBO* and Teruhisa TSUDA**

Abstract

We construct higher order generalizations of the q -Painlevé equations of surface type $A_7^{(1)}$ and $A_6^{(1)}$ based on the cluster algebras corresponding to certain quivers. These equations possess the affine Weyl group symmetries of type $A_1^{(1)}$ and $(A_1 + A_1')^{(1)}$, respectively. We show that these equations and symmetries can be realized as birational canonical transformations. A relationship between the quivers and the discrete KdV equation is also discussed.

§ 1. Introduction

It is known that some discrete soliton equations and q -Painlevé equations can be described as birational mappings generated by sequences of mutations in the cluster algebras corresponding to quivers appropriately chosen so that the quivers are invariant under the sequences of mutations. In [6, 11], q -analogues of the Painlevé I and II equations were obtained from cluster algebras corresponding to quivers with mutation-period one, i.e. the shape of the quiver is invariant under a single mutation; see also [4]. In [12], q -analogues of the Painlevé III and VI equations, together with the discrete KdV equation and some other soliton equations, were obtained from cluster algebras corresponding to more complicated quivers. In [1], all the q -Painlevé equations in Sakai's list [15] and their underlying Weyl group symmetries were derived from cluster algebras corresponding to the quivers arising from cluster integrable systems. In our previous work [8], we present a unified way to construct birational representations of Weyl groups, which are relevant to the q -Painlevé equations and their higher order generalizations, from cluster algebras by means of a graph-combinatorial point of view; the key ingredients of the argument is the *reflections associated with cycle-subgraphs* in the quiver

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*Department of Physics and Mathematics, Aoyama Gakuin University, 5-10-1 Fuchinobe, Chuo-ku, Sagami-hara-shi, Kanagawa 252-5258, Japan.

**Department of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601, Japan.

under consideration. In [14], a specific construction of a higher order generalization of the q -Painlevé VI equation was demonstrated by using such reflections in terms of a certain toroidal quiver.

However, even for the second order q -Painlevé equations in Sakai's list, a few relatively degenerated ones, i.e. the q -Painlevé equations of surface type $A_7^{(1)}$ and $A_6^{(1)}$, can not be dealt with in the general framework [8] since their underlying Weyl groups contain a reflection which seems not to be associated with any cycle-subgraphs. Recall here that, to each of the Painlevé equations and discrete Painlevé equations, we can associate a rational surface classified by Dynkin diagrams of affine root systems as the space of initial conditions [15].

The aim of this paper is to construct higher order generalizations of the q -Painlevé equations of surface type $A_7^{(1)}$ and $A_6^{(1)}$ together with their Weyl group symmetries based on the cluster algebras corresponding to certain quivers.

To illustrate our results, we prepare some basic notions of cluster algebras [2] minimum required. Let Q be a quiver (directed graph) with a vertex set $V = \{1, 2, \dots, N\}$ represented by the signed adjacency matrix $B = (b_{i,j})_{i,j=1}^N$, i.e. $b_{i,j}$ indicates the number of arrows from the vertex i to j if $b_{i,j} > 0$. Let $\mathbf{y} = (y_i)_{i \in V}$ be an N -tuple of variables. Define the transformation $\mu_k : (Q, \mathbf{y}) \mapsto (Q', \mathbf{y}')$ called the *mutation* at the vertex $k \in V$ as follows. A new quiver $Q' = \mu_k(Q)$ is obtained by the following procedure on the quiver Q .

1. For all subgraphs $i \rightarrow k \rightarrow j$, add $b_{i,k}b_{k,j}$ arrows from i to j ;
2. remove all 2-cycles;
3. reverse the orientation of all arrows which have edges at the vertex k .

In parallel, the new variables $\mathbf{y}' = (y'_i)_{i \in V}$ are defined by

$$y'_i = \mu_k(y_i) = \begin{cases} y_k^{-1} & (i = k) \\ y_i \frac{(1 + y_k)^{\max\{b_{i,k}, 0\}}}{(1 + y_k^{-1})^{\max\{b_{k,i}, 0\}}} & (i \neq k). \end{cases}$$

We suppose that a composition of mutations $w = \mu_{i_1} \circ \mu_{i_2} \circ \dots \circ \mu_{i_k}$ acts on a rational function $\varphi = \varphi(\mathbf{y})$ as $w.\varphi(\mathbf{y}) = \varphi(\mathbf{y}.w)$, i.e. w acts on the variables from the right. It holds that

$$(1.1) \quad \mu_i^2 = \text{id} \quad \text{and} \quad \mu_i \circ \mu_j = \mu_j \circ \mu_i \quad \text{if} \quad b_{i,j} = 0.$$

A permutation σ of the vertex set V is extended to the variables as $\sigma(y_i) = y_{\sigma(i)}$. An operation ι , called the *inversion*, which reverses the orientation of all arrows is extended as $\iota(y_i) = 1/y_i$. It is easy to see that

$$(1.2) \quad (i, j) \circ \mu_i = \mu_j \circ (i, j), \quad (i, j) \circ \mu_k = \mu_k \circ (i, j) \quad (\forall i, j, k; i \neq j \neq k \neq i)$$

and ι commutes with any mutation and permutation. We focus on the *stabilizer of the quiver*, i.e. the group consisting of compositions of mutations, permutations of the

vertices and the inversion that keeps the quiver invariant; cf. [1]. The birational action of the stabilizer of the quiver on the variables $\mathbf{y}$ provides a nontrivial group of birational transformations on the field $\mathbb{C}(\mathbf{y})$ of rational functions.

In this paper, we start from the specific quiver $Q = Q_N$ with a vertex set $V = \{1, 2, \dots, N\}$ represented by the signed adjacency matrix

$$\begin{aligned} B_N &= (b_{i,j})_{i,j=1}^N \\ &= E_{2,1} - E_{1,2} + 2 \sum_{i=2}^{N-2} (E_{i+1,i} - E_{i,i+1}) + E_{N,N-1} - E_{N-1,N} + \sum_{i=1}^{N-2} (E_{i,i+2} - E_{i+2,i}) \\ &\quad + E_{1,N-1} - E_{N-1,1} + E_{2,N} - E_{N,2} + E_{N-1,2} - E_{2,N-1} + E_{N,1} - E_{1,N}, \end{aligned}$$

where $N \geq 4$ and $E_{i,j}$ denotes a matrix element, i.e. a matrix in which the (i, j) entry is unity and all the others are zero. Some examples of Q_N are shown in (2.1) and (3.1). In the cases where N is even and odd, the affine Weyl groups of type $A_1^{(1)}$ and $(A_1 + A_1')^{(1)}$ respectively emerge as (subgroups of) the stabilizers of Q_N ; see Theorems 2.1 and 3.1. Via a certain normalization of the skew-symmetric integer matrix B_N by the use of unimodular matrices, we describe the Weyl group actions as the group of birational canonical transformations; see Theorems 2.2 and 3.2. The dimension of the phase space of the discrete dynamical system thus obtained turns out to be $2l$ for both $N = 2l + 2$ (even) and $N = 2l + 3$ (odd) cases. We regard the birational mapping arising from a lattice part of the affine Weyl group as the higher order generalization of the q -Painlevé equations. In fact the first two cases, i.e. $N = 4$ and $N = 5$, recover the q -Painlevé equations of surface type $A_7^{(1)}$ and $A_6^{(1)}$, respectively. In the last section, we discuss an interesting relationship between the quiver Q_N and an infinite quiver corresponding to the discrete KdV equation.

Remark 1.1. Two types of variables usually appear in the context of cluster algebras [2]; one is called a *cluster variable* and the other a *coefficient*. The variables $\mathbf{y} = (y_i)_{i \in V}$, mainly concerned in this paper, are the coefficients. The mutation μ_k at the vertex $k \in V$ acts birationally on the cluster variables $\mathbf{x} = (x_i)_{i \in V}$ as follows:

$$\mu_k(x_i) = \begin{cases} x_k^{-1} \left(\prod_{b_{j,k} > 0} x_j + \prod_{b_{j,k} < 0} x_j \right) & (i = k) \\ x_i & (i \neq k). \end{cases}$$

Remark 1.2. The higher order generalizations of the q -Painlevé equations of surface type $A_7^{(1)}$ and $A_6^{(1)}$ were firstly found by the second author, without concerning any Weyl group symmetries; see [13] in which the *coprimeness property*, an integrability criterion for nonlinear recurrence relations, was proven for these equations.

§ 2. The q -Painlevé equation of type $A_7^{(1)}$ and its higher order generalization

In this section we construct the q -Painlevé equation of surface type $A_7^{(1)}$ and its $2l$ -th order generalizations, each of which admits the symmetry of the affine Weyl group of type $A_1^{(1)}$, starting from the quiver Q_{2l+2} or the corresponding signed adjacency matrix B_{2l+2} . The first few examples of the quiver and the matrix are given by

$$(2.1) \quad Q_4 = \begin{array}{ccc} 2 & \longleftrightarrow & 3 \\ \downarrow & \nearrow & \downarrow \\ 1 & \longleftrightarrow & 4 \end{array}, \quad Q_6 = \begin{array}{ccccc} & 3 & \longleftrightarrow & 4 & \\ \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\ 2 & \longleftrightarrow & 5 & & \\ \searrow & \downarrow & \searrow & \downarrow & \searrow \\ & 1 & \longleftrightarrow & 6 & \end{array}, \quad Q_8 = \begin{array}{ccccccccc} & & & 4 & \longleftrightarrow & 5 & & & \\ \nearrow & & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & \nearrow \\ 3 & \longleftrightarrow & 6 & & 7 & & 8 & & \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ 2 & \longleftrightarrow & 1 & \longleftrightarrow & 8 & & 7 & & 6 \end{array}$$

and

$$B_4 = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -2 & 1 & -1 & 1 \\ -1 & 2 & 0 & -2 & 1 & 0 \\ 0 & -1 & 2 & 0 & -2 & 1 \\ -1 & 1 & -1 & 2 & 0 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \end{pmatrix},$$

$$B_8 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & -2 & 1 & 0 & 0 & -1 & 1 \\ -1 & 2 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & -2 & 1 \\ -1 & 1 & 0 & 0 & -1 & 2 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix},$$

respectively.

Theorem 2.1. *Let t_0, t_1 and π be the compositions of mutations, permutations of vertices and the inversion ι defined by*

$$\begin{aligned} t_0 &= (2l+2, 2l+1, \dots, 2, 1) \circ \mu_1, \\ t_1 &= (1, 2, \dots, 2l+1, 2l+2) \circ \mu_{2l+2}, \\ \pi &= (1, 2l+2) \circ (2, 2l+1) \circ \dots \circ (l+1, l+2) \circ \iota, \end{aligned}$$

where $(i_1, i_2, \dots, i_k)$ denotes a cyclic permutation of vertices. Then these elements preserve the quiver Q_{2l+2} invariant and thus generate a group. Also these satisfy the fundamental relations

$$(2.2) \quad t_0 t_1 = t_1 t_0 = \pi^2 = \text{id}, \quad t_0 = \pi t_1 \pi.$$

Proof. The invariance of the quiver follows directly from the definition of mutations. The fundamental relations are verified by using the relations (1.1) and (1.2). $\square$

The fundamental relations (2.2) tell us that the group $G = \langle t_0, t_1, \pi \rangle$ is isomorphic to the affine Weyl group of type $A_1^{(1)}$. For instance, if we put $r_0 = t_1 \pi$ and $r_1 = \pi t_1$, then it holds that $r_0^2 = r_1^2 = \pi^2 = \text{id}$ and $r_0 = \pi r_1 \pi$. The elements r_i ($i = 0, 1$) are simple reflections and π is the Dynkin automorphism of $A_1^{(1)}$.

The birational action of the group G on the field $\mathbb{C}(\mathbf{y})$ of rational functions is easily obtained. When $l = 1$, it is described by

$$\begin{aligned} t_0 : y_1 &\mapsto y_2(1 + y_1), & y_2 &\mapsto y_3 \frac{y_1^2}{(1 + y_1)^2}, & y_3 &\mapsto y_4(1 + y_1), & y_4 &\mapsto \frac{1}{y_1}, \\ t_1 : y_1 &\mapsto \frac{1}{y_4}, & y_2 &\mapsto y_1 \frac{y_4}{1 + y_4}, & y_3 &\mapsto y_2(1 + y_4)^2, & y_4 &\mapsto y_3 \frac{y_4}{1 + y_4} \end{aligned}$$

and

$$\pi : y_i \mapsto \frac{1}{y_{5-i}} \quad (1 \leq i \leq 4).$$

When $l \geq 2$, the action is described by

$$\begin{aligned} t_0 : y_1 &\mapsto y_2(1 + y_1), & y_2 &\mapsto y_3 \frac{y_1}{1 + y_1}, & y_i &\mapsto y_{i+1} \quad (3 \leq i \leq 2l - 1), \\ y_{2l} &\mapsto y_{2l+1} \frac{y_1}{1 + y_1}, & y_{2l+1} &\mapsto y_{2l+2}(1 + y_1), & y_{2l+2} &\mapsto \frac{1}{y_1}, \\ t_1 : y_1 &\mapsto \frac{1}{y_{2l+2}}, & y_2 &\mapsto y_1 \frac{y_{2l+2}}{1 + y_{2l+2}}, & y_3 &\mapsto y_2(1 + y_{2l+2}), \\ y_i &\mapsto y_{i-1} \quad (4 \leq i \leq 2l), & y_{2l+1} &\mapsto y_{2l}(1 + y_{2l+2}), & y_{2l+2} &\mapsto y_{2l+1} \frac{y_{2l+2}}{1 + y_{2l+2}} \end{aligned}$$

and

$$\pi : y_i \mapsto \frac{1}{y_{2l+3-i}} \quad (1 \leq i \leq 2l + 2).$$

Prepare the column vectors in $\mathbb{Z}^{2l+2}$

$$\mathbf{v}_0 = \sum_{k=2}^{2l+2} (k-1) \mathbf{e}_k, \quad \mathbf{v}_1 = \sum_{k=1}^{2l+1} (2l+2-k) \mathbf{e}_k,$$

where $\mathbf{e}_k = {}^T(0, \dots, 0, \overset{k}{1}, 0, \dots, 0)$ is the k -th standard basis vector. Note that these two vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ span $\text{Ker } B_{2l+2}$. We introduce the variables b_0 and b_1 defined by

$$b_0 = \mathbf{y}^{\mathbf{v}_0} = \prod_{k=2}^{2l+2} y_k^{k-1}, \quad b_1 = \mathbf{y}^{\mathbf{v}_1} = \prod_{k=1}^{2l+1} y_k^{2l+2-k},$$

where we use the multi-index notation $\mathbf{y}^{\mathbf{v}} = \prod_{k=1}^{2l+2} y_k^{v_k}$ for $\mathbf{v} = (v_k)_{k=1}^{2l+2} \in \mathbb{Z}^{2l+2}$. It is easy to see that b_0 and b_1 simply behave under the action of G as

$$(2.3) \quad \begin{aligned} t_0 : b_0 &\mapsto q^{-1}b_0, & b_1 &\mapsto qb_1, \\ t_1 : b_0 &\mapsto qb_0, & b_1 &\mapsto q^{-1}b_1, \\ \pi : b_0 &\mapsto b_1^{-1}, & b_1 &\mapsto b_0^{-1}, \end{aligned}$$

where $q = \prod_{k=1}^{2l+2} y_k$, and therefore $b_0b_1 = q^{2l+1}$, $t_0(q) = t_1(q) = q$ and $\pi(q) = q^{-1}$.

Let us introduce the matrix $U = (\mathbf{u}_1, \dots, \mathbf{u}_{2l}, \mathbf{v}_0, \mathbf{v}_1) \in \text{Mat}_{2l+2}(\mathbb{Z})$, where the column vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{2l} \in \mathbb{Z}^{2l+2}$ are given by

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{e}_2, & \mathbf{u}_2 &= \mathbf{e}_1, \\ \mathbf{u}_{2i-1} &= \mathbf{e}_{2i} - \mathbf{u}_{2i-2} \quad (2 \leq i \leq l), \\ \mathbf{u}_{2i} &= i\mathbf{e}_1 + \sum_{k=1}^{i-1} (i-k)(\mathbf{e}_{2k} + \mathbf{e}_{2k+1}) \quad (2 \leq i \leq l). \end{aligned}$$

Then the matrix U is unimodular, i.e. $U \in GL_{2l+2}(\mathbb{Z})$, and the skew-symmetric integer matrix B_{2l+2} is transformed into the following normal form [10]

$${}^TUB_{2l+2}U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus l} \oplus O_2.$$

This fact leads to a choice of canonical variables; cf. [3, 8]. Let us introduce the variables f_i, g_i ($1 \leq i \leq l$) defined by

$$f_i = \mathbf{y}^{\mathbf{u}_{2i-1}}, \quad g_i = \mathbf{y}^{\mathbf{u}_{2i}} \quad (1 \leq i \leq l).$$

Theorem 2.2. *Let $\mathbb{K}(\mathbf{f}, \mathbf{g}) = \mathbb{K}(f_1, \dots, f_l, g_1, \dots, g_l)$ be the field of rational functions of f_i, g_i ($1 \leq i \leq l$), where the coefficient field is $\mathbb{K} = \mathbb{C}(b_0, b_1)$. Then the action of $G = \langle t_0, t_1, \pi \rangle$ on $\mathbb{K}(\mathbf{f}, \mathbf{g})$ is birational. The 2-form $\omega = \sum_{i=1}^l \frac{df_i \wedge dg_i}{f_i g_i}$ is invariant under the action of t_0 and t_1 and $\pi(\omega) = -\omega$.*

Proof. By definition the variables f_i, g_i ($1 \leq i \leq l$) are monomials in y_k ($1 \leq k \leq 2l + 2$). Conversely the variables y_k ($1 \leq k \leq 2l + 2$) are also monomials in f_i, g_i ($1 \leq i \leq l$) since the matrix U is unimodular. Then the action on $\mathbb{C}(\mathbf{y})$ yields the birational action of G on $\mathbb{K}(\mathbf{f}, \mathbf{g})$. The invariance of the 2-form ω has been shown in a general framework [8]. $\square$

The second statement of this theorem means that $\log f_i, \log g_i$ ($1 \leq i \leq l$) are Darboux coordinates for the symplectic 2-form ω and that the birational action of G on $\mathbb{K}(\mathbf{f}, \mathbf{g})$ gives rise to canonical transformations.

Let us present the birational action on $\mathbb{K}(\mathbf{f}, \mathbf{g})$. When $l = 1$, it is described by

$$\begin{aligned} t_0 : f_1 &\mapsto \frac{b_1}{f_1^2 g_1 (1 + g_1)^2}, & g_1 &\mapsto f_1 (1 + g_1), \\ t_1 : f_1 &\mapsto \frac{q f_1 g_1^3}{b_1 + q f_1 g_1^2}, & g_1 &\mapsto \frac{b_1}{q f_1 g_1^2}, \\ \pi : f_1 &\mapsto \frac{f_1^2 g_1^3}{b_1}, & g_1 &\mapsto \frac{b_1}{q f_1 g_1^2}. \end{aligned}$$

When $l \geq 2$, the action is described by

$$\begin{aligned} t_0 : f_i &\mapsto \frac{g_1 g_{i+1}}{(1 + g_1) g_i} \prod_{k=1}^i \frac{1}{f_k g_k} \quad (1 \leq i \leq l - 1), & f_l &\mapsto \frac{b_1 g_1^2}{(1 + g_1)^2 g_l} \prod_{k=1}^l \frac{1}{f_k^2 g_k^2}, \\ g_i &\mapsto \frac{1 + g_1}{g_1} \prod_{k=1}^i f_k g_k \quad (1 \leq i \leq l), \\ t_1 : f_1 &\mapsto \frac{q g_1 g_l \prod_{k=1}^l f_k g_k}{b_1 + q g_l \prod_{k=1}^l f_k g_k}, & f_i &\mapsto \frac{q g_i g_l}{b_1 g_{i-1}} \prod_{k=i}^l f_k g_k \quad (2 \leq i \leq l), \\ g_i &\mapsto \frac{b_1}{q g_l} \prod_{k=i}^l \frac{1}{f_k g_k} \quad (1 \leq i \leq l), \\ \pi : f_i &\mapsto \frac{q^{i-1} f_{l-i+1} g_{l-i+1}^2}{b_1} \prod_{k=1}^l f_k g_k, & g_i &\mapsto \frac{b_1}{q^i g_{l-i+1}} \prod_{k=1}^l \frac{1}{f_k g_k} \quad (1 \leq i \leq l). \end{aligned}$$

When $l = 1$, the transformation $T = t_0$ is exactly the time evolution of the q -Painlevé equation of surface type $A_7^{(1)}$. In fact, one obtains the second order q -difference equation

$$\varphi[n + 2] \varphi[n] = q^n b_1 \frac{1 + \varphi[n + 1]}{\varphi[n + 1]^2}$$

for $\varphi[n] = T^n(g_1)$, which is known as a q -analogue of the Painlevé I equation; see e.g. [7].

§ 3. The q -Painlevé equation of type $A_6^{(1)}$ and its higher order generalization

In a similar way to Section 2, one can obtain the q -Painlevé equation of surface type $A_6^{(1)}$ and its $2l$ -th order generalizations starting from the quiver Q_{2l+3} . The first few examples of the quiver are given by

$$(3.1) \quad Q_5 = \begin{array}{c} \text{Diagram of } Q_5 \end{array}, \quad Q_7 = \begin{array}{c} \text{Diagram of } Q_7 \end{array}, \quad Q_9 = \begin{array}{c} \text{Diagram of } Q_9 \end{array}.$$

Note that the quiver Q_{2l+3} contains the $(l+2)$ -cycle $1 \rightarrow 3 \rightarrow \cdots \rightarrow 2l+3 \rightarrow 1$ and the $(l+1)$ -cycle $2 \rightarrow 4 \rightarrow \cdots \rightarrow 2l+2 \rightarrow 2$ as subgraphs. As is shown in [8] the stabilizer of Q_{2l+3} thus contains reflection transformations associated with these cycle-subgraphs.

Theorem 3.1. *Let s_0, s_1, t_0, t_1 and π be the compositions of mutations, permutations of vertices and the inversion ι defined by*

$$\begin{aligned} s_0 &= \mu_{1,3,\dots,2l+1} \circ (2l+1, 2l+3) \circ \mu_{2l+1,\dots,3,1}, \\ s_1 &= \mu_{2,4,\dots,2l} \circ (2l, 2l+2) \circ \mu_{2l,\dots,4,2}, \\ t_0 &= (2l+3, 2l+2, \dots, 2, 1) \circ \mu_1, \\ t_1 &= (1, 2, \dots, 2l+2, 2l+3) \circ \mu_{2l+3}, \\ \pi &= (1, 2l+3) \circ (2, 2l+2) \circ \cdots \circ (l+1, l+3) \circ \iota, \end{aligned}$$

where $\mu_{i_1, i_2, \dots, i_k} = \mu_{i_1} \circ \mu_{i_2} \circ \cdots \circ \mu_{i_k}$. Then these elements preserve the quiver Q_{2l+3} invariant and thus generate a group. Also these satisfy the fundamental relations

$$(3.2) \quad \begin{aligned} s_0^2 &= s_1^2 = t_0 t_1 = t_1 t_0 = \pi^2 = \text{id}, \\ \pi s_i &= s_i \pi \quad (i = 0, 1), \quad t_0 = \pi t_1 \pi, \quad s_0 = t_0 s_1 t_1 = t_1 s_1 t_0. \end{aligned}$$

Proof. The invariance of the quiver follows directly from the definition of mutations. As is shown in [8], reflection transformations associated with cycle-subgraphs are invariant under cyclic permutations of indices, namely one can alternatively express s_0 as

$$s_0 = \mu_{2l+3,1,3,\dots,2l-1} \circ (2l-1, 2l+1) \circ \mu_{2l-1,\dots,3,1,2l+3}.$$

Then it is easy to verify the relation $t_0 s_1 t_1 = s_0$ by using (1.2). The relation $s_0 = t_1 s_1 t_0$ can be shown similarly. The other fundamental relations are obvious. $\square$

The fundamental relations (3.2) tell us that the group $G = \langle s_0, s_1, t_0, t_1, \pi \rangle$ is isomorphic to the affine Weyl group of type $(A_1 + A'_1)^{(1)}$. For instance, if we put $r_0 = \pi, r_1 = \pi t_1^2$ and $\rho = \pi t_1$, then it holds that

$$r_0^2 = r_1^2 = \rho^2 = \text{id}, \quad s_0 = \rho s_1 \rho, \quad r_0 = \rho r_1 \rho, \quad s_i r_j = r_j s_i \quad (i, j = 0, 1).$$

Both $\langle s_0, s_1 \rangle$ and $\langle r_0, r_1 \rangle$ are isomorphic to the affine Weyl group of type $A_1^{(1)}$. They mutually commute and ρ plays the role of a Dynkin automorphism; cf. [9, 15].

For the purpose of describing the transformations s_0 and s_1 on the field $\mathbb{C}(\mathbf{y})$ of rational functions explicitly, we introduce two families of polynomials F_i ($i \in \mathbb{Z}/(l+2)\mathbb{Z}$) and G_j ($j \in \mathbb{Z}/(l+1)\mathbb{Z}$). The polynomials F_i are defined by

$$F_i = \sum_{m=1}^{l+2} \prod_{k=1}^m y_{2i+2k-3},$$

where the subscripts of the variables $\mathbf{y}$ are regarded as the elements of $\mathbb{Z}/(2l+4)\mathbb{Z}$. Similarly the polynomials G_j are defined by

$$G_j = \sum_{m=1}^{l+1} \prod_{k=1}^m y_{2j+2k-2},$$

where the subscripts of the variables $\mathbf{y}$ are regarded as the elements of $\mathbb{Z}/(2l+2)\mathbb{Z}$. Then the transformations s_0 and s_1 are described by

$$\begin{aligned} s_0 : y_1 &\mapsto \frac{F_{l+2}}{y_{2l+3}F_2}, \quad y_{2i-1} \mapsto \frac{F_{i-1}}{y_{2i-3}F_{i+1}} \quad (2 \leq i \leq l+2), \\ y_2 &\mapsto y_2 \frac{y_1^2 y_{2l+3} F_2^2}{F_{l+2} F_1}, \quad y_{2i} \mapsto y_{2i} \left(\frac{y_{2i-1} F_{i+1}}{F_i} \right)^2 \quad (2 \leq i \leq l), \\ y_{2l+2} &\mapsto y_{2l+2} \frac{y_{2l+1}^2 y_{2l+3} F_1 F_{l+2}}{F_{l+1}^2}, \\ s_1 : y_2 &\mapsto \frac{G_{l+1}}{y_{2l+2} G_2}, \quad y_{2j} \mapsto \frac{G_{j-1}}{y_{2j-2} G_{j+1}} \quad (2 \leq j \leq l+1), \\ y_1 &\mapsto y_1 \frac{y_{2l+2} G_1}{G_{l+1}}, \quad y_{2j-1} \mapsto y_{2j-1} \left(\frac{y_{2j-2} G_j}{G_{j-1}} \right)^2 \quad (2 \leq j \leq l+1), \\ y_{2l+3} &\mapsto y_{2l+3} \frac{y_{2l+2} G_1}{G_{l+1}} \end{aligned}$$

in terms of the family of polynomials F_i and G_j . The transformations t_0, t_1 and π are

described by

$$\begin{aligned}
t_0 : y_1 &\mapsto y_2(1 + y_1), & y_2 &\mapsto y_3 \frac{y_1}{1 + y_1}, & y_i &\mapsto y_{i+1} \quad (3 \leq i \leq 2l), \\
y_{2l+1} &\mapsto y_{2l+2} \frac{y_1}{1 + y_1}, & y_{2l+2} &\mapsto y_{2l+3}(1 + y_1), & y_{2l+3} &\mapsto \frac{1}{y_1}, \\
t_1 : y_1 &\mapsto \frac{1}{y_{2l+3}}, & y_2 &\mapsto y_1 \frac{y_{2l+3}}{1 + y_{2l+3}}, & y_3 &\mapsto y_2(1 + y_{2l+3}), \\
y_i &\mapsto y_{i-1} \quad (4 \leq i \leq 2l + 1), & y_{2l+2} &\mapsto y_{2l+1}(1 + y_{2l+3}), & y_{2l+3} &\mapsto y_{2l+2} \frac{y_{2l+3}}{1 + y_{2l+3}}, \\
\pi : y_i &\mapsto \frac{1}{y_{2l+4-i}} \quad (1 \leq i \leq 2l + 3).
\end{aligned}$$

Prepare the column vectors in $\mathbb{Z}^{2l+3}$

$$\begin{aligned}
\mathbf{u}_0 &= \sum_{k=1}^{l+2} \mathbf{e}_{2k-1}, & \mathbf{u}_1 &= \sum_{k=1}^{l+1} \mathbf{e}_{2k}, \\
\mathbf{v}_0 &= (l+1)\mathbf{e}_1 + \sum_{k=1}^l (l-k+1)(\mathbf{e}_{2k} + \mathbf{e}_{2k+1}), \\
\mathbf{v}_1 &= (l+1)\mathbf{e}_{2l+3} + \sum_{k=1}^l k(\mathbf{e}_{2k+1} + \mathbf{e}_{2k+2}).
\end{aligned}$$

These are elements of $\text{Ker } B_{2l+3}$ and one can choose $\mathbf{u}_0, \mathbf{u}_1$ and $\mathbf{v}_0$ as the basis vectors of $\text{Ker } B_{2l+3}$. We introduce the variables a_0, a_1, b_0 and b_1 defined by

$$\begin{aligned}
a_0 &= \mathbf{y}^{\mathbf{u}_0} = \prod_{k=1}^{l+2} y_{2k-1}, & a_1 &= \mathbf{y}^{\mathbf{u}_1} = \prod_{k=1}^{l+1} y_{2k}, \\
b_0 &= \mathbf{y}^{\mathbf{v}_0} = y_1^{l+1} \prod_{k=1}^l (y_{2k} y_{2k+1})^{l-k+1}, & b_1 &= \mathbf{y}^{\mathbf{v}_1} = y_{2l+3}^{l+1} \prod_{k=1}^l (y_{2k+1} y_{2k+2})^k.
\end{aligned}$$

These variables simply behave under the action of G as

$$\begin{aligned}
s_0 : a_0 &\mapsto a_0^{-1}, & a_1 &\mapsto a_1 a_0^2, & b_0 &\mapsto b_0 a_0^{-1}, & b_1 &\mapsto b_1 a_0^{-1}, \\
s_1 : a_0 &\mapsto a_0 a_1^2, & a_1 &\mapsto a_1^{-1}, & b_0 &\mapsto a_1 b_0, & b_1 &\mapsto a_1 b_1, \\
t_0 : a_0 &\mapsto a_1, & a_1 &\mapsto a_0, & b_0 &\mapsto a_1 b_0, & b_1 &\mapsto b_1 a_0^{-1}, \\
t_1 : a_0 &\mapsto a_1, & a_1 &\mapsto a_0, & b_0 &\mapsto b_0 a_0^{-1}, & b_1 &\mapsto a_1 b_1, \\
\pi : a_0 &\mapsto a_0^{-1}, & a_1 &\mapsto a_1^{-1}, & b_0 &\mapsto b_1^{-1}, & b_1 &\mapsto b_0^{-1}.
\end{aligned}$$

Let us introduce the matrix $U = (\mathbf{w}_1, \dots, \mathbf{w}_{2l}, \mathbf{u}_0, \mathbf{u}_1, \mathbf{v}_0) \in \text{Mat}_{2l+3}(\mathbb{Z})$, where the

column vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{2l} \in \mathbb{Z}^{2l+3}$ are given by

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{e}_2, & \mathbf{w}_2 &= \mathbf{e}_1, \\ \mathbf{w}_{2j-1} &= \mathbf{e}_{2j} - \mathbf{w}_{2j-2} \quad (2 \leq j \leq l), \\ \mathbf{w}_{2j} &= j\mathbf{e}_1 + \sum_{k=1}^{j-1} (j-k)(\mathbf{e}_{2k} + \mathbf{e}_{2k+1}) \quad (2 \leq j \leq l). \end{aligned}$$

Then the matrix U is unimodular, i.e. $U \in GL_{2l+3}(\mathbb{Z})$, and the skew-symmetric integer matrix B_{2l+3} is transformed into the following normal form

$${}^tUB_{2l+3}U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus l} \oplus O_3.$$

Introduce the variables f_i, g_i ($1 \leq i \leq l$) defined by

$$f_i = \mathbf{y}^{\mathbf{w}_{2i-1}}, \quad g_i = \mathbf{y}^{\mathbf{w}_{2i}} \quad (1 \leq i \leq l).$$

Then we have the following.

Theorem 3.2. *Let $\mathbb{K}(\mathbf{f}, \mathbf{g}) = \mathbb{K}(f_1, \dots, f_l, g_1, \dots, g_l)$ be the field of rational functions of f_i, g_i ($1 \leq i \leq l$), where the coefficient field is $\mathbb{K} = \mathbb{C}(a_0, a_1, b_0, b_1)$. Then the action of $G = \langle s_0, s_1, t_0, t_1, \pi \rangle$ on $\mathbb{K}(\mathbf{f}, \mathbf{g})$ is birational. The 2-form $\omega = \sum_{i=1}^l \frac{df_i \wedge dg_i}{f_i g_i}$ is invariant under the action of s_0, s_1, t_0 and t_1 and $\pi(\omega) = -\omega$.*

When $l = 1$, the birational action on $\mathbb{K}(f_1, g_1)$ is described by

$$\begin{aligned} s_0 : f_1 &\mapsto \frac{f_1(b_0 + a_0 f_1 g_1 + a_0 f_1 g_1^2)^2}{(b_0 + f_1 g_1 + f_1 g_1^2)(b_0 + a_0 f_1 g_1 + f_1 g_1^2)}, & g_1 &\mapsto \frac{g_1(b_0 + f_1 g_1 + f_1 g_1^2)}{b_0 + a_0 f_1 g_1 + a_0 f_1 g_1^2}, \\ s_1 : f_1 &\mapsto \frac{f_1}{a_1}, & g_1 &\mapsto \frac{g_1(a_1 + f_1)}{1 + f_1}, \\ t_0 : f_1 &\mapsto \frac{b_0}{f_1 g_1 (1 + g_1)}, & g_1 &\mapsto f_1 (1 + g_1), \\ t_1 : f_1 &\mapsto \frac{a_0 f_1 g_1^2}{b_0 + a_0 f_1 g_1}, & g_1 &\mapsto \frac{b_0}{a_0 f_1 g_1}, \\ \pi : f_1 &\mapsto \frac{f_1}{a_1}, & g_1 &\mapsto \frac{b_0}{a_0 f_1 g_1}. \end{aligned}$$

It is not difficult to describe the birational action on $\mathbb{K}(\mathbf{f}, \mathbf{g})$ for $l \geq 2$, but the explicit formula is slightly complicated. Then we show only the transformation t_0 , which is

given as

$$t_0 : f_i \mapsto \frac{g_1 g_{i+1}}{(1+g_1)g_i} \prod_{k=1}^i \frac{1}{f_k g_k} \quad (1 \leq i \leq l-1), \quad f_l \mapsto \frac{b_0 g_1}{(1+g_1)g_l} \prod_{k=1}^l \frac{1}{f_k g_k},$$

$$g_i \mapsto \frac{1+g_1}{g_1} \prod_{k=1}^i f_k g_k \quad (1 \leq i \leq l).$$

Note that the transformation t_0^2 acts on the variables a_i, b_i ($i = 0, 1$) as

$$t_0^2 : a_0 \mapsto a_0, \quad a_1 \mapsto a_1, \quad b_0 \mapsto q b_0, \quad b_1 \mapsto q^{-1} b_1,$$

where $q = \prod_{k=1}^{2l+3} y_k$. Also we see that

$$a_0 a_1 = q, \quad b_0 b_1 = q^l a_0$$

and

$$s_i(q) = t_i(q) = q \quad (i = 0, 1), \quad \pi(q) = q^{-1}.$$

Hence the transformation $T = t_0^2$ is regarded as a time evolution and the variable b_0 plays the role of independent variable.

When $l = 1$, the transformation T gives the time evolution of the q -Painlevé equation of surface type $A_6^{(1)}$. In fact, one obtains the system of q -difference equations

$$\varphi[n+1] = a_1 \frac{\psi[n]}{1 + \varphi[n](1 + \psi[n])}, \quad \psi[n+1] = q^n b_0 \frac{1 + \varphi[n](1 + \psi[n])}{\varphi[n]\psi[n](1 + \psi[n])}$$

for $\varphi[n] = T^n(f_1)$ and $\psi[n] = T^n(g_1)$, which is known as a q -analogue of the Painlevé II equation; see e.g. [7].

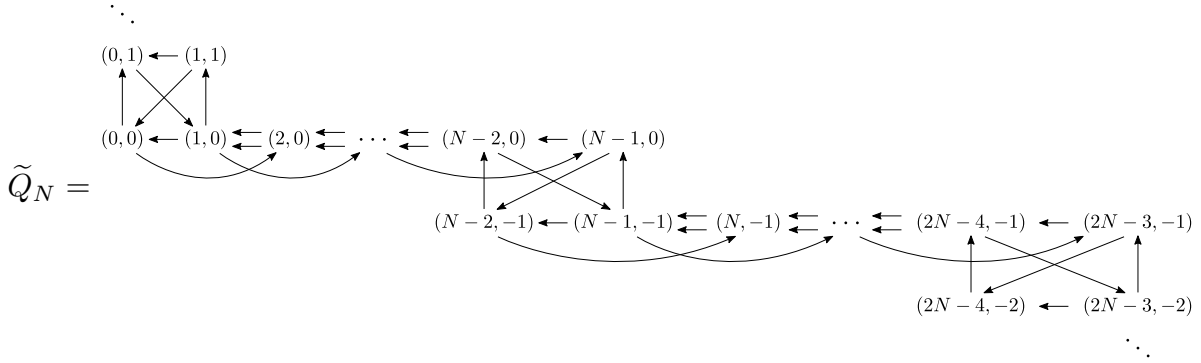
§ 4. Relation with the discrete KdV equation

We shall discuss a relationship between our quiver Q_N and one for the discrete KdV equation. The bilinear form of the discrete KdV equation [5] is given by

$$(4.1) \quad x_{i+2,j+1} x_{i,j} = x_{i+1,j+1} x_{i+1,j} + x_{i,j+1} x_{i+2,j},$$

where $(i, j) \in \mathbb{Z}^2$ are regarded as independent variables. In [12], a family of quivers with infinite number of vertices corresponding to the discrete KdV equation was proposed. Each quiver of the family represents the type of boundary conditions.

Among them, we consider the following quiver:



which consists of the vertices

$$V = \{(i + (N - 2)k, -k) \mid k \in \mathbb{Z}; 0 \leq i \leq N - 1\} \subset \mathbb{Z}^2$$

and the arrows of multiplicity one

$$\begin{aligned} (i + 1 + (N - 2)k, -k) &\rightarrow (i + (N - 2)k, -k) \quad (k \in \mathbb{Z}; i = 0, N - 2), \\ (i + (N - 2)k, -k) &\rightarrow (i + 2 + (N - 2)k, -k) \quad (k \in \mathbb{Z}; 0 \leq i \leq N - 3), \\ (i + (N - 2)k, -k) &\rightarrow (i + (N - 2)k, 1 - k) \quad (k \in \mathbb{Z}; i = 0, 1), \\ (1 + (N - 2)k, 1 - k) &\rightarrow ((N - 2)k, -k) \quad (k \in \mathbb{Z}), \\ ((N - 2)k, 1 - k) &\rightarrow (1 + (N - 2)k, -k) \quad (k \in \mathbb{Z}) \end{aligned}$$

and the arrows of multiplicity two

$$(i + (N - 2)k, -k) \rightleftarrows (i + 1 + (N - 2)k, -k) \quad (k \in \mathbb{Z}; 1 \leq i \leq N - 3),$$

where $N \geq 4$. Let us consider the composition

$$T = \sigma \circ \prod_{k \in \mathbb{Z}} \mu_{((N-2)k, -k)}$$

of an infinite sequence of commuting mutations $\{\mu_{((N-2)k, -k)} \mid k \in \mathbb{Z}\}$ and a permutation of vertices

$$\begin{aligned} \sigma : (i + (N - 2)k, -k) &\mapsto (i - 1 + (N - 2)k, -k) \quad (k \in \mathbb{Z}; 1 \leq i \leq N - 1), \\ ((N - 2)k, -k) &\mapsto (1 + (N - 2)k, 1 - k) \quad (k \in \mathbb{Z}). \end{aligned}$$

Then the quiver $\tilde{Q}_N$ is invariant under T . This yields the birational mapping of the cluster variables $x_{i,j} := x_{(i,j)}$ of the form (see Remark 1.1)

$$T(x_{i,j}) = \frac{x_{i+1,j+1}x_{i+1,j} + x_{i,j+1}x_{i+2,j}}{x_{i,j}}.$$

If we extend the discrete domain V of the variable $x_{i,j}$ to the whole plane $\mathbb{Z}^2$ by setting

$$(4.2) \quad x_{i+2,j+1} = T(x_{i,j}),$$

then the iteration of this birational mapping and its inverse determines $x_{i,j}$ for any $(i, j) \in \mathbb{Z}^2$ from the boundary values $\{x_{i,j} \mid (i, j) \in V\}$. Notice that (4.2) is exactly the bilinear form of the discrete KdV equation (4.1).

Since the shape of $\tilde{Q}_N$ is periodic with the period $(N-2, -1)$, one can fold up this quiver by identifying the infinite number of vertices $\{(i + (N-2)k, -k) \mid k \in \mathbb{Z}\}$ with the vertex $(i, 0)$ for each $i \in \{0, 1, \dots, N-1\}$. The resulting quiver is exactly the quiver Q_N corresponding to the higher order generalizations of the q -Painlevé equations of surface type $A_7^{(1)}$ and $A_6^{(1)}$. Note that this fact was already pointed out by the second author [11] for the first two cases, i.e. $N = 4$ and $N = 5$. One can impose the $(N-2, -1)$ -periodicity on the cluster variables as $x_{i+N-2,j-1} = x_{i,j}$, which is compatible with the above reduction of the quiver $\tilde{Q}_N$ to Q_N . The recurrence relation

$$\tau_{i+N}\tau_i = \tau_{i+N-1}\tau_{i+1} + \tau_{i+N-2}\tau_{i+2}$$

thus obtained for $\tau_i := x_{i,0}$ ($i \in \mathbb{Z}$) is regarded as the bilinear form of the higher order generalization of the q -Painlevé equation of surface type $A_7^{(1)}$ (resp. $A_6^{(1)}$) with special values of parameters $b_0 = b_1 = 1$ (resp. $a_0 = a_1 = b_0 = b_1 = 1$) and thereby $q = 1$ if N is even (resp. odd); cf. Section 2 (resp. 3). In the case where $N = 4$ (resp. 5), the above recurrence relation is known as the Somos-4 (resp. Somos-5) recurrence; the general case also appeared in the investigation [6].

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