

A Homotopy Theory of Algebraic Higher Categories

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To my beloved trivials.

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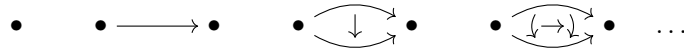
Introduction

This thesis is a reorganisation of three papers, jointly written with Soichiro Fujii and Yuki Maehara [FHM24; FHM25a; FHM25b], into a single coherent account. The most essential parts of these three papers are reflected in Chapters 2, 3, and 4, respectively, while the background material and technical preliminaries common to all three have been extracted and arranged into Chapter 1. Each chapter begins with an introduction based on the introduction of the corresponding paper, and so we refrain here from entering into the detailed discussion of each chapter separately.

In this Introduction, we first give a brief overview of Grothendieck’s *homotopy hypothesis* [Gro22], which is the common theme underlying the three papers, and we explain how the present work fits into this picture. We then describe the structure of the thesis, and finally list some of the notation that will be used throughout.

In his long, unpublished manuscript *Pursuing Stacks* [Gro22], Grothendieck suggested that homotopy types should be understood in terms of so-called ω -groupoids. The idea that homotopy types, usually modelled by Kan complexes or CW-complexes and their variants, might instead be described by algebraic objects such as ω -groupoids is now known as Grothendieck’s *homotopy hypothesis* [Mal10; Hen16].

The central idea itself is simple, and already visible at a very early stage in the study of homotopy and category theory. For instance, in an undergraduate course on algebraic topology, the notion of homotopy is often introduced as follows. A 1-homotopy is a path between points of a space, a 2-homotopy is a path between paths, a 3-homotopy is a path between 2-homotopies, and so on. In other words, the concept of homotopy is obtained by iterating and formalising a notion of ‘being connected by a path’ as a higher-dimensional structure. On the other hand, when one first encounters the notion of natural transformation in category theory, it is introduced as a ‘morphism between morphisms’, that is, a morphism between functors, and afterwards the idea of a 2-category is presented as a formalisation of this picture. In both explanations, it is natural to draw intuitive diagrams of the following ‘globular’ shapes.



If we forget, for the moment, the additional structure coming from homotopies or compositions and identities, and retain only these globular diagrams, we obtain what is called a globular set. Very roughly, Grothendieck’s homotopy hypothesis is the question of whether one can recover the notion of homotopy type completely by putting suitable algebraic structure on such globular sets, which can be thought of as a ‘bare shape’ of homotopy (cf. [Mal10; Ara10]).

In recent years, by enhancing homotopy types (Kan complexes) in various ways to encode a notion of ‘direction’ of homotopies, many different models of (∞, n) -categories have been constructed, and the equivalences between these models have been intensively studied [Lur09; CKM25; Ver08; BS20; CH24b]. On the algebraic side, however, there remain highly non-trivial issues: for example, it is now well understood that the most naive notion, that of a strict ω -groupoid, is not suitable as a model for general homotopy types [Sim98]. Motivated in part by this failure, Grothendieck himself and many other researchers have sought more appropriate algebraic notions of ω -groupoid or ω -category. A variety of models have been proposed, and the relationships between them have been clarified to a significant extent in recent years [Bat98; Lei04; Mal10; Che11; Ara10; Hen16].

Considering this background, it is reasonable to formulate the homotopy hypothesis in the following way.

Whether there exists a model of algebraic weak ω -categories on globular sets whose homotopy theory is (in the homotopical sense) equivalent to that of any given model of directed spaces, that is, of (∞, ∞) -categories.

The most essential step towards answering this question is to develop homotopy theory on the algebraic side. More concretely, one would like to equip the category of weak ω -categories with a (weak) model structure or a suitable variant, and to develop a homotopy theory that is rich enough to be comparable with the usual homotopy theory of spaces.

In this thesis we work with one particular candidate for such an algebraic model of ω -categories, namely the Batanin–Leinster style weak ω -categories introduced by Batanin [Bat98] and subsequently formulated by Leinster [Lei04], and we develop a fragment of homotopy theory in this setting. In Chapter 2 we introduce the notion of a *weakly invertible cell* as a way of measuring ‘internal equivalences’ within a single weak ω -category, and we study its basic properties in detail. In Chapter 3 we define and investigate a notion of equivalence between weak ω -categories, which plays a central role in the homotopy theory. Finally, in Chapter 4 we define a notion of fibration between weak ω -categories, extending the notions of isofibration between categories and equifibration between 2-categories, and we establish its fundamental properties. This notion of fibration is one of the basic building blocks for (weak) model structures on the category of weak ω -categories.

As indicated above, the problems addressed in this thesis arise from an attempt to extract directly the algebraic properties that the notion of homotopy should satisfy, without passing through ‘richer’ geometric objects such as topological spaces or simplicial sets. Consequently, phenomena that are usually regarded as almost ‘trivial’ in standard homotopy theory appear here in a highly non-trivial form. For instance, the main theorem of Chapter 2 shows that invertible cells, that is, equivalences internal to a weak ω -category, are closed under all the compositions defined in this setting. The main theorem of Chapter 3 establishes that weak equivalences between weak ω -categories satisfy the 2-out-of-3 property, a property that would normally be considered almost obvious for many familiar notions of homotopy equivalence and their natural extensions.

In this sense, the theory developed in this thesis could be described as taking a deliberately ‘naive’ approach to the phenomenon of homotopy. Precisely for this reason, however, homotopical phenomena present themselves here in their most elementary form. To deal with them adequately, the work in this thesis demands a close interaction between intricate combinatorial methods and categorical techniques for organising and structuring them. Indeed, all of the chapters deal with highly involved combinatorics concerning the manipulation of cells of globular sets. At the same time, cartesian monads, coinductive methods for capturing invertibility, and the theory of algebraic weak factorisation systems [Gar09b; BG16a; BG16b; GT06] play crucial roles in ensuring that these constructions can be carried out safely and smoothly. We collect this technical background in Chapter 1.

The three original papers [FHM24; FHM25a; FHM25b] lay the most basic foundations for a homotopy theory on Batanin–Leinster style weak ω -categories, and at the same time form part of an ongoing project towards resolving Grothendieck’s homotopy hypothesis. The present thesis is intended as an ‘interim report’, bringing together the techniques and results obtained so far into a single place. I hope that this report will serve as a modest guide for readers seeking to understand the nature of homotopy and of higher categorical structures.

Outline of this thesis. This thesis is organised as follows.

Chapter 1 fixes the basic setting and notation used in the rest of the thesis. We begin by recalling globular sets, strict ω -categories, and the Batanin–Leinster notion of weak ω -categories, and we describe the free weak ω -category monad on globular sets. We then review further categorical tools that underlie our constructions, including cartesian monads, algebraic weak factorisation systems, and *algebraic injectivity*, with particular emphasis on the small object argument and fibrant/cofibrant replacement. Finally, we introduce a simple coinductive framework which will later be used to define and analyse notions of invertibility in weak ω -categories.

Chapter 2 develops the notion of *weakly invertible cells* in a weak ω -category. These are intended to play the role of internal equivalences within a single object. The main result of the chapter shows that weakly invertible cells are closed under all the compositions defined.

Chapter 3 introduces a notion of ω -*weak equivalence* between weak ω -categories, modelled on the idea that a functor should be an equivalence if it is essentially surjective in all dimensions. After proving that ω -weak equivalences satisfy the 2-out-of-3 property, we extend our scope to *weak ω -functors*, and show that *weak ω -weak equivalences* also satisfy the 2-out-of-3 property. We also relate such weak ω -weak equivalences to more classical notions of equivalence in low-dimensional truncations of weak ω -categories.

Chapter 4 studies equivalence cells and fibrations between weak ω -categories. We first show that two natural definitions of equivalence cells in a weak ω -category—via a two-sided inverse, or via separate left and right inverses—agree. We then define ω -equifibrations and characterise them by a right lifting property against a set $\mathcal{J}$ of strict ω -functors, obtained from families of maps between marked weak ω -categories using the algebraic small object argument. By specifying a certain family $\mathcal{K}_{\mathcal{F}}$, we describe the resulting universal set $\mathcal{J}_{\mathcal{F}}$ explicitly, and show that this construction produces a weak version of Ozornova–Rovelli’s coherent walking ω -equivalence. Finally, we show that between strict ω -categories, ω -equifibrations coincide with the fibrations in the folk model structure.

Deferred proofs and additional remarks are collected in Chapter A. In Chapter B, we revisit the construction of the free strict ω -category monad on globular sets developed in [Lei04, Appendix F], from the viewpoint of *familial representability*, and provide more detailed expressions for some of the structures involved.

General notation. By a *category* we mean a locally small (possibly large) category. We write

Id_X for the identity morphism on X , Id when the object is clear from context.

We denote by **2** the walking arrow (the category with two objects and a single non-identity arrow $0 \rightarrow 1$). For any category $\mathbf{C}$, its *arrow category* is $\mathbf{C}^2$: its objects are morphisms $f: A \rightarrow B$ of $\mathbf{C}$, and a morphism $(u, v): f \rightarrow f'$ is a commutative square in $\mathbf{C}$, i.e. $v \circ f = f' \circ u$ with $u: A \rightarrow A'$, $v: B \rightarrow B'$. We write dom and cod for the domain and codomain, which form functors $\mathbf{C}^2 \rightarrow \mathbf{C}$ for each category $\mathbf{C}$.

Given categories $\mathbf{A}$ and $\mathbf{B}$, we write $[\mathbf{A}, \mathbf{B}]$ for the functor category whose objects are functors $\mathbf{A} \rightarrow \mathbf{B}$ and whose morphisms are natural transformations.

If $\mathbf{C}$ has an initial object $\emptyset$ and a terminal object 1 , then for each object $Y \in \text{Obj}(\mathbf{C})$ we write

$$!^Y: \emptyset \rightarrow Y \quad \text{and} \quad !_Y: Y \rightarrow 1$$

for the unique morphisms.

For a set S , we write S^* for the set of all finite lists (finite sequences) of elements of S ; the empty list is written `nil`.

We use the following size conventions.

Set is the category of (small) sets, **SET** is the (huge) category of classes,

Cat is the category of small categories, **CAT** is the (huge) category of possibly large categories,

Cat is the 2-category of small categories, *CAT* is the (huge) 2-category of possibly large categories.

Given a category $\mathbf{C}$, we write $\text{Obj}(\mathbf{C})$ for its (possibly large) class of objects. Given composable morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ in $\mathbf{C}$, we use any of the following equivalent notations for their composite:

- $g \circ f$ (the ‘classical’ order composition)
- gf (abbreviation of the above)
- $f \circledast g$ (the ‘diagrammatic’ (left-to-right) order composition)

We adopt the same convention for the composition of functors and natural transformations.

We write $\mathbb{N}$ for the set of natural numbers, taken to include 0: $\mathbb{N} = \{0, 1, 2, \dots\}$. Indices and generic natural numbers are denoted $i, j, k, l, m, n, r \in \mathbb{N}$. Unless stated otherwise, we use x, y, z, u, v, w, p, q for cells of appropriate dimensions, and f, g, h for morphisms (in particular, for strict or weak ω -functors when $\mathbf{C}$ is a category of ω -categories).

CHAPTER 1

Preliminary

In the first two sections of this chapter, we recall Leinster’s definition of weak ω -categories [Lei04], which is based on an earlier definition by Batanin [Bat98]. Along the way, we also introduce some notation. In short, we shall define a weak ω -category as an Eilenberg–Moore algebra of a suitable monad $\mathbf{L}$ on the category $\mathbf{GSet}$ of globular sets. The monad $\mathbf{L}$ for weak ω -categories is defined using the monad $\mathbf{T}$ on $\mathbf{GSet}$ for *strict* ω -categories: $\mathbf{L}$ is the *initial cartesian monad over $\mathbf{T}$ with contraction*.

The last two sections (Sections 1.3 and 1.4) of this chapter is devoted to some other preliminary notions and results that will play important roles throughout this thesis: that is, *algebraic injectivity* and *coinduction*.

1.1. Strict ω -categories as algebras over globular sets

Although there are several ways to define strict ω -categories, the one most directly related to the Batanin–Leinster definition of weak ω -categories is the approach that treats them as algebraic structures on globular sets. This rests on the idea that, among the ingredients appearing in the definitions of strict ω -categories, the only structure that weak ω -categories should (in a strict sense) share is the globular set structure — that is, the source and target data for cells. Put differently, a Batanin–Leinster-style weak ω -category is defined by weakening every part of the structure of a strict ω -category other than its underlying globular set structure.

This observation directly motivates the following advantage of this style of definition: suitably ‘weakening’ arguments that work for strict ω -categories allows one to convert them into statements and proofs for weak ω -categories. To realise this programme, we must capture the notion of ‘weakness’ with precision and weave it appropriately into the proofs in the strict setting. For this, all combinatorics concerning strict ω -categories need to be described in detail at the level of algebraic structure on globular sets.

In this section, we carry out a detailed analysis of the structure of strict ω -categories as algebras on globular sets — equivalently, the structure of the free strict ω -category monad $\mathbf{T}$ — and provide tools adequate to handle the various combinatorics that will arise in the later chapters.

Almost all of the contents of this section are taken from [FHM24, Section 2], although some results and descriptions are added or modified for clarity, and some results have also been extracted from the preliminary sections of [FHM25a; FHM25b]. For items (definitions, propositions, etc.) drawn from the latter two papers, we have attached explicit citations to the corresponding blocks to make this provenance clear.

1.1.1. Globular sets.

Definition 1.1.1.1. Let $\mathbf{G}$ be the category freely generated by the graph

$$0 \begin{array}{c} \xrightarrow{\sigma^0} \\ \xrightarrow{\tau^0} \end{array} 1 \begin{array}{c} \xrightarrow{\sigma^1} \\ \xrightarrow{\tau^1} \end{array} 2 \longrightarrow \cdots \longrightarrow n \begin{array}{c} \xrightarrow{\sigma^n} \\ \xrightarrow{\tau^n} \end{array} n+1 \longrightarrow \cdots$$

subject to the following equations

$$\sigma^n \circ \sigma^{n+1} = \sigma^n \circ \tau^{n+1}, \quad \tau^n \circ \sigma^{n+1} = \tau^n \circ \tau^{n+1}. \quad (1.1.1.1.1)$$

Explicitly, we have

$$\mathbf{G}([m], [n]) = \begin{cases} \{\sigma_n^m, \tau_n^m\} & \text{if } m < n; \\ \{\text{Id}_{[n]}\} & \text{if } m = n; \\ \emptyset & \text{if } m > n, \end{cases}$$

where $\sigma_n^m = \sigma^{n-1} \circ \sigma^{n-2} \circ \cdots \circ \sigma^m$ and $\tau_n^m = \tau^{n-1} \circ \tau^{n-2} \circ \cdots \circ \tau^m$. ┘

Definition 1.1.1.2. A *globular set* is a presheaf over $\mathbf{G}$. We write $\mathbf{GSet}$ for the presheaf category $[\mathbf{G}^{\text{op}}, \mathbf{Set}]$ of $\mathbf{G}$. Explicitly, a globular set X is a graded set $(X_n)_{n \in \mathbb{N}}$ equipped with functions

$$s_n^{m,X} (:= X_{\sigma_n^m}) : X_m \longrightarrow X_n, \quad t_n^{m,X} (:= X_{\tau_n^m}) : X_m \longrightarrow X_n$$

for each $n < m$ in $\mathbf{G}$, which are called the *source* and *target* maps, respectively. They are subject to the equations corresponding to (1.1.1.1.1). For convenience, we also write $s_n^{n,X}$ and $t_n^{n,X}$ for the identity map on X_n . We sometimes omit the superscripts, and write simply like s_n, t_n and s_n^X, t_n^X for $s_n^{n,X}$ and $t_n^{n,X}$.

Given a globular set X and a natural number $n \in \mathbb{N}$, elements of X_n are called n -cells of X . $\lrcorner$

Definition 1.1.1.3. Let X be a globular set, and let $n \in \mathbb{N}$ be a natural number. Two n -cells x and y of X are said to be *parallel* if $s_m^X(x) = s_m^X(y)$ and $t_m^X(x) = t_m^X(y)$ hold for each $m < n$. In other words, all 0-cells of X are parallel, and when $n \geq 1$, $x, y \in X_n$ are parallel if and only if $s_{n-1}^X(x) = s_{n-1}^X(y)$ and $t_{n-1}^X(x) = t_{n-1}^X(y)$ hold. For an n -cell u of X ($n \geq 1$), we write $u: x \rightarrow y$ to mean $s_{n-1}^X(u) = x$ and $t_{n-1}^X(u) = y$. $\lrcorner$

Definition 1.1.1.4. Let n be a natural number. We write G^n for the representable globular set $\mathbf{G}(-, [n])$; it has precisely one n -cell, two m -cells for $0 \leq m < n$, and no m -cells for $m > n$. We write $\sigma^n: G^n \rightarrow G^{n+1}$ and $\tau^n: G^n \rightarrow G^{n+1}$ for the morphisms in $\mathbf{GSet}$ corresponding to σ^n and τ^n in $\mathbf{G}$, respectively.

For any globular set X , the morphisms $G^n \rightarrow X$ in $\mathbf{GSet}$ correspond to the n -cells of X by the Yoneda lemma; for $x \in X_n$, we also denote the corresponding morphism by $x: G^n \rightarrow X$. $\lrcorner$

Definition 1.1.1.5 ([FHM25b, Section 2.1]). Let n be a natural number. We write ∂G^n for the *boundary* of G^n ; i.e., the largest proper globular subset of G^n . In other words, ∂G^n is obtained from G^n by removing its unique n -cell. We denote the associated inclusion by $\iota^n: \partial G^n \hookrightarrow G^n$. The factorisation of $\sigma^{n-1}: G^{n-1} \rightarrow G^n$ through ι^n is written as σ_∂^{n-1} :

$$\begin{array}{ccccc} & & \partial G^{n-1} & & \\ \sigma_\partial^{n-1} \nearrow & & \downarrow \iota^n & \nwarrow \tau_\partial^{n-1} & \\ G^{n-1} & \xrightarrow{\sigma^{n-1}} & G^n & \xleftarrow{\tau^{n-1}} & G^{n-1} \end{array} \quad (1.1.1.5.1)$$

τ_∂^{n-1} is defined similarly as the factorisation of τ^{n-1} through ι^n . $\lrcorner$

Remark 1.1.1.6 ([FHM25b, Section 2.1]). For a globular set X , we have $u: x \rightarrow y$ in X if and only if the diagram

$$\begin{array}{ccc} \partial G^n & \xrightarrow{\langle x, y \rangle} & X \\ \iota^n \downarrow & \nearrow u & \\ G^n & & \end{array}$$

commutes in $\mathbf{GSet}$. $\lrcorner$

Remark 1.1.1.7. These morphisms ι^n in Definition 1.1.1.5 may be depicted as follows:

$$\begin{array}{ccccccc} \left(\begin{array}{c} \\ \end{array} \right) & \left(\begin{array}{cc} \bullet & \bullet \end{array} \right) & \left(\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \end{array} \right) & \left(\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow & & \downarrow \end{array} \right) & \dots \\ \downarrow \iota^0 & \downarrow \iota^1 & \downarrow \iota^2 & \downarrow \iota^3 & & & \\ \left(\begin{array}{c} \bullet \end{array} \right) & \left(\begin{array}{ccc} \bullet & \longrightarrow & \bullet \end{array} \right) & \left(\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow & & \downarrow \end{array} \right) & \left(\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow & & \downarrow \end{array} \right) & & & \end{array}$$

Inductively, ∂G^0 is the empty (initial) globular set, and for $n \geq 1$, we have a pushout diagram

$$\begin{array}{ccc} \partial G^{n-1} & \xrightarrow{\iota^{n-1}} & G^{n-1} \\ \iota^{n-1} \downarrow & \lrcorner & \downarrow \sigma_\partial^{n-1} \\ G^{n-1} & \xrightarrow{\tau_\partial^{n-1}} & \partial G^n \end{array}$$

in $\mathbf{GSet}$. Therefore for any globular set X , the morphisms $\partial G^n \rightarrow X$ correspond to the parallel pairs of $(n-1)$ -cells if $n \geq 1$; for each pair (u, v) of parallel $(n-1)$ -cells of X , we denote the corresponding morphism by $\langle u, v \rangle: \partial G^n \rightarrow X$. $\lrcorner$

Definition 1.1.1.8. We write $\mathbf{GSet}_{**}$ for the under category $G^0 \sqcup G^0 / \mathbf{GSet}$, which is the category of 2-pointed globular sets. Given a 2-pointed globular set $(X, (x, y))$, we define the *hom globular set* $X(x, y)$ as the following

pullback in **GSet**:

$$\begin{array}{ccc} X(x, y) & \hookrightarrow & X_{-+1} \\ \downarrow & \lrcorner & \downarrow (s_0^X, t_0^X) \\ 1 & \xrightarrow{(x, y)} & \Delta(X_0) \times \Delta(X_0) \end{array}$$

Here, 1 is the terminal globular set and $\Delta: \mathbf{Set} \rightarrow \mathbf{GSet}$ is the constant presheaf functor. The globular set X_{-+1} is defined by $(X_{-+1})_n = X_{n+1}$ for each $n \in \mathbb{N}$. In more elementary terms, n -cells of $X(x, y)$ are $n+1$ -cells f of X satisfying $s_0^X(f) = x$ and $t_0^X(f) = y$. This extends to a functor $(X, (x, y)) \mapsto X(x, y): \mathbf{GSet}_{**} \rightarrow \mathbf{GSet}$ in an obvious way. $\lrcorner$

Remark 1.1.1.9. The hom-globular set functor has a left adjoint $X \mapsto (\underline{\Sigma} X, (\star, \star')): \mathbf{GSet} \rightarrow \mathbf{GSet}_{**}$, and we write $\underline{\Sigma}: \mathbf{GSet} \rightarrow \mathbf{GSet}$ for the composite of this left adjoint with the forgetful functor $\mathbf{GSet}_{**} \rightarrow \mathbf{GSet}$, which we call the *suspension* functor. Observe that $\underline{\Sigma}$ preserves connected colimits, since the forgetful functor $\mathbf{GSet}_{**} \rightarrow \mathbf{GSet}$ is a projection from an under category, and hence it creates connected colimits.

This sends a globular set A to the 2-pointed globular set $(\star, \star'): G^0 \sqcup G^0 \rightarrow \underline{\Sigma} A$ given by

$$(\underline{\Sigma} A)_n = \begin{cases} \{\star, \star'\} & n = 0, \\ A_{n-1} & n \geq 1 \end{cases}$$

where $s_0^{\underline{\Sigma} A}(a) = \star$ and $t_0^{\underline{\Sigma} A}(a) = \star'$ for all $a \in A'_1$. See also Remark B.3.18. $\lrcorner$

Notation 1.1.1.10. The terminal globular set 1 is the globular set such that 1_n is a singleton for each $n \in \mathbb{N}$. We write $*_n$ for the unique n -cell of 1. $\lrcorner$

1.1.2. The free strict ω -category monad $\mathbf{T}$. In this thesis, our treatment of strict ω -categories is as follows: we give an explicit description of the free strict ω -category monad $\mathbf{T}$ on **GSet**, and think of strict ω -categories as algebras over this monad. However, since it is useful to give a direct and concise definition of strict ω -categories, we present the following definitions of strict ω -categories and of the free strict ω -category monad.

Definition 1.1.2.1. A *strict ω -category* is a globular set X equipped with the following additional structure.

- A structure $(*_n^X, \text{id}_{n,m}^X)$ of a category on the graph $(s_n^X, t_n^X): X_m \rightrightarrows X_n$ for each $n < m$.
- * For each $n < m < l$, the 2-globular set $X_l \rightrightarrows X_m \rightrightarrows X_n$ is a 2-category with the structures of categories $(*_m^X, \text{id}_{m,l}^X)$, $(*_n^X, \text{id}_{n,l}^X)$, and $(*_n^X, \text{id}_{n,m}^X)$: i.e., for any cells x, y, a, b in $X_n \sqcup X_m \sqcup X_l$, the left- and the right-hand sides below are defined simultaneously, and these equations hold:

$$\begin{aligned} (x *_m^X y) *_n^X (a *_m^X b) &= (x *_n^X a) *_m^X (y *_n^X b) \\ (\text{id}_{m,l}^X x) *_n^X (\text{id}_{m,l}^X a) &= \text{id}_{m,l}^X (x *_n^X a) \\ \text{id}_{m,l}^X (\text{id}_{n,m}^X x) &= \text{id}_{n,l}^X x \end{aligned}$$

A *strict ω -functor* between strict ω -categories is a morphism of the underlying globular sets preserving all the additional structures above. Let $\mathbf{Str}\text{-}\omega\text{-Cat}_s$ be the category of small strict ω -categories and strict ω -functors. $\lrcorner$

Definition 1.1.2.2. The forgetful functor $\mathbf{Str}\text{-}\omega\text{-Cat}_s \rightarrow \mathbf{GSet}$ is monadic [Lei04, Theorem F.2.2], and the induced monad on **GSet** is denoted by $\mathbf{T} = (\mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}})$. $\lrcorner$

Reconstructing the proof of the theorem [Lei04, Theorem F.2.2] carefully, we obtain the following inductive description of the monad $\mathbf{T}$. See Chapter B for more details.

Fact 1.1.2.3 (Theorem B.4.1). Here, we summarise some known¹ facts about the monad $\mathbf{T}$, which in fact characterise it. Here we give these in an inductive form, which is discussed in detail in Section B.4.

- (i) $\mathbf{T}$ is a *familial monad* (see Definition B.3.13 for more details): i.e., it satisfies the following properties.
 - There exists a *familial representation* functor $\widehat{(-)}: \mathbf{G}/\mathbf{T}1 \rightarrow \mathbf{GSet}$ such that the sliced functor $\mathbf{T}_{/1}: \mathbf{GSet} \rightarrow \mathbf{GSet}/\mathbf{T}1$ is isomorphic to the nerve of $\widehat{(-)}$, through the canonical equivalence $\mathbf{GSet}/\mathbf{T}1 \simeq \mathbf{Psh}(\mathbf{G}/\mathbf{T}1)$. Here, $\mathbf{G}/\mathbf{T}1$ is the category of elements of the globular set $\mathbf{T}1$.

¹Indeed, the familial representation of $\mathbf{T}$ is discussed in all sort of literature like [Lei04], but here we give it in an inductive form, which seems not to have appeared explicitly in the literature. Moreover, as far as the author knows, the explicit description of the multiplication of $\mathbf{T}$ has not appeared in the literature.

- The natural transformations $\mu^T: \mathbf{T}\mathbf{T} \rightarrow \mathbf{T}$ and $\eta^T: \mathbf{Id}_{\mathbf{GSet}} \rightarrow \mathbf{T}$ are cartesian: i.e., their naturality squares are pullback diagrams (see Definition 1.2.2.1).
- (ii) The set of n -cells $(\mathbf{T}1)_n$ is 1^{*n} for each $n \in \mathbb{N}$, which is the n -fold iteration of $(-)^*$ applied to the terminal object in **Set**.
- (iii) The source and target maps of type $(\mathbf{T}1)_{n+1} \rightarrow (\mathbf{T}1)_n$ in **Set** coincide with each other, which we denote by $\partial_n^{\mathbf{T}1}$. This is inductively determined as follows.
 - $\partial_0^{\mathbf{T}1}: (\mathbf{T}1)_1 \rightarrow (\mathbf{T}1)_0$ is the unique one towards the terminal set.
 - $\partial_{n+1}^{\mathbf{T}1} = (\partial_n^{\mathbf{T}1})^*$ is the image of $\partial_n^{\mathbf{T}1}$ under the functor $(-)^*: \mathbf{Set} \rightarrow \mathbf{Set}$.
- (iv) The familial representation $(n+1, \vec{\pi})$ is the colimit of the diagram

$$\begin{array}{ccccccc}
 & \widehat{\Sigma(n, \pi_1)} & & \widehat{\Sigma(n, \pi_2)} & & \cdots & & \widehat{\Sigma(n, \pi_n)} \\
 & \swarrow \star & & \swarrow \star & & \swarrow \star & & \swarrow \star \\
 G^0 & & G^0 & & G^0 & & G^0 & & G^0
 \end{array}$$

(see also (B.3.25.1) for $c = n$), and $\widehat{(0, *)}$ is the representable globular set G^0 . Note that $(\mathbf{T}1)_{n+1}$ is the same as the set of lists $(\mathbf{T}1)_n^*$, and we regard the list $\vec{\pi} = (\pi_{i+1})_{i < n}$ of elements of $(\mathbf{T}1)_n$ as an element of $(\mathbf{T}1)_{n+1}$.

The leftmost $\star$ and the rightmost $\star'$ in the above diagram give the parallel morphisms

$$\widehat{\sigma_{n+1}^0}, \widehat{\tau_{n+1}^0}: \widehat{\partial_0^{\mathbf{T}1}(n+1, \vec{\pi})} = \widehat{(0, *)} \rightarrow \widehat{(n+1, \vec{\pi})},$$

and for each $m \leq n$, the parallel morphisms

$$\widehat{\sigma_{n+1}^{m+1}}, \widehat{\tau_{n+1}^{m+1}}: \widehat{\partial_{m+1}^{\mathbf{T}1}(n+1, \vec{\pi})} \rightarrow \widehat{(n+1, \vec{\pi})}$$

are those induced from the universal property of the above colimit by the morphisms

$$\underline{\Sigma}(\widehat{\sigma_n^m}), \underline{\Sigma}(\widehat{\tau_n^m}): \underline{\Sigma}(\widehat{\partial_m^{\mathbf{T}1}(n, \pi_{i+1})}) \rightarrow \underline{\Sigma}(\widehat{(n, \pi_{i+1})})$$

for each $i < n$, respectively.

- (v) $(\eta_1^T)_{n+1}(*_{n+1}) \in \mathbf{T}1_{n+1} = \mathbf{T}1_n^*$ is the list of length 1 whose unique element is $(\eta_1^T)_n(*_n) \in \mathbf{T}1_n$. $\lrcorner$

The remainder of this section is devoted to unpacking the above inductive description of the monad $\mathbf{T}$. To this end, we employ the following notions of *pasting schemes*, which gives a direct description of the globular set $\mathbf{T}1$.

Definition 1.1.2.4 (See [Web01, Section 2.1] and [Web04, Section 4]). A (globular) *pasting scheme* $\mathbf{k}$ is a pair $((k_i)_{i \leq r}, (\underline{k}_{i+1})_{i < r})$ of sequences of non-negative integers satisfying $k_i > \underline{k}_{i+1} < k_{i+1}$ for each $i < r$. We often denote a pasting scheme $\mathbf{k}$ as a table of non-negative integers

$$\mathbf{k} = \begin{bmatrix} k_0 & k_1 & \cdots & k_r \\ & \underline{k}_1 & \cdots & \underline{k}_r \end{bmatrix}.$$

We call r the *rank* of $\mathbf{k}$, which is denoted by $\text{rank}(\mathbf{k})$. The *level* of $\mathbf{k}$, defined by $\max\{k_i \mid i \leq r\}$, is denoted by $\text{level}(\mathbf{k})$. For $n \geq 0$, a *pasting scheme of dimension n* is a pasting scheme $\mathbf{k}$ as above which moreover satisfies $\text{level}(\mathbf{k}) \leq n$ for all $i \leq r$. We write $\mathbf{PS}_n$ for the set of all pasting schemes of dimension n , and write $\mathbf{PS} = \cup_{n \in \mathbb{N}} \mathbf{PS}_n$ for the set of all pasting schemes. Note that a single pasting scheme $\mathbf{k}$ might belong to several of the sets $\mathbf{PS}_n$ for different n 's, and the minimum such n is $\text{level}(\mathbf{k})$.

When a pasting scheme $\mathbf{k}$ is regarded as an element of $\mathbf{PS}_m$ for some $m \in \mathbb{N}$ with $m \geq \text{level}(\mathbf{k})$, we write $\mathbf{k}^{(m)}$ to indicate the (ambient) dimension m . $\lrcorner$

Definition 1.1.2.5. Define the *successor* pasting scheme of a pasting scheme $\mathbf{k}$ by $\mathbf{S}(\mathbf{k}) = ((k_i)_{i \leq r}, (\underline{k}_{i+1})_{i < r})$ by $\mathbf{S}\mathbf{k} = ((k_i + 1)_{i \leq r}, (\underline{k}_{i+1} + 1)_{i < r})$.

Given two pasting schemes $\mathbf{k}$ and $\mathbf{k}'$, we define the *concatenation*

$$\mathbf{k} \frown \mathbf{k}' := ((l_i)_{i \leq R}, (\underline{l}_{i+1})_{i < R})$$

as follows:

- If $\mathbf{k}' = [^0]$, then $\mathbf{l} = \mathbf{k}$ and $R = r$. If $\mathbf{k} = [^0]$, then $\mathbf{l} = \mathbf{k}'$ and $R = r'$.

- Otherwise, $R = r + r' + 1$ and $\mathbf{k} \frown \mathbf{k}'$ is defined by

$$l_i = \begin{cases} k_i & \text{if } i \leq r \\ k'_{i-r-1} & \text{if } i > r \end{cases}, \quad l_{i+1} = \begin{cases} \underline{k}_{i+1} & \text{if } i < r \\ 0 & \text{if } i = r \\ \underline{k}'_{i-r} & \text{if } i > r. \end{cases}$$

Using the table notation, it is expressed as

$$\mathbf{k} \frown \mathbf{k}' = \begin{bmatrix} k_0 & k_1 & \dots & k_r & k'_0 & k'_1 & \dots & k'_{r'} \\ \underline{k}_1 & \underline{k}_2 & \dots & \underline{k}_r & 0 & \underline{k}'_1 & \underline{k}'_2 & \dots & \underline{k}'_{r'} \end{bmatrix}.$$

Note that $- \frown -$ endows $\mathbf{PS}$ with a monoid structure with the unit $[^0]$. $\lrcorner$

Definition 1.1.2.6. Given a pasting scheme $\mathbf{k}$ of dimension n and of rank r , and a natural number m with $m \leq n$, an m -transversal component of $\mathbf{k}$ is a subsequence of $\mathbf{k}$ of the form

$$\begin{bmatrix} k_i & k_{i+1} & \dots & k_j \\ \underline{k}_{i+1} & \underline{k}_{i+2} & \dots & \underline{k}_j \end{bmatrix} \quad (1.1.2.6.1)$$

with $0 \leq i \leq j \leq r$ such that

- $k_l > m$ for all $i \leq l \leq j$,
- $\underline{k}_l \geq m$ for all $i + 1 \leq l \leq j$,
- either $i = 0$ or $\underline{k}_i < m$, and
- either $j = r$ or $\underline{k}_{j+1} < m$.

(The first clause is not completely redundant because of the case $i = j$.) We also use the phrase m -transversal component $0 \leq i \leq j \leq r$ of $\mathbf{k}$ to refer to the subsequence (1.1.2.6.1). $\lrcorner$

We say that $\mathbf{k}$ is m -transversal if $\mathbf{k}$ itself is an m -transversal component. Note that any pasting scheme $\mathbf{k}$ is 0-transversal unless it is of dimension 0. $\lrcorner$

Definition 1.1.2.7. Define a function $\mathbf{br}: \mathbf{PS}^* \rightarrow \mathbf{PS}$ by

$$\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n] = (\mathbf{S}\mathbf{k}_1) \frown (\mathbf{S}\mathbf{k}_2) \frown \dots \frown (\mathbf{S}\mathbf{k}_n), \quad \mathbf{br}[\mathbf{nil}] = [^0]. \quad \lrcorner$$

This function $\mathbf{br}$ unpacks the inductive characterisation of $\mathbf{T1}$ in Fact 1.1.2.3.(ii) as follows.

Proposition 1.1.2.8. For each natural number $m \in \mathbb{N}$, the function $\mathbf{br}$ gives a bijection $\mathbf{PS}_{m-1}^* \cong \mathbf{PS}_m$, where we set $\mathbf{PS}_{-1} = \emptyset$. $\lrcorner$

Proof. Clearly we have $\mathbf{PS}_0 = \{\mathbf{br}[\mathbf{nil}]\}$, since the rank of a pasting scheme must be 0 when its level is 0. We prove that $\mathbf{br}$ restricts to a bijection $\mathbf{PS}_m^* \cong \mathbf{PS}_{m+1}$ for each $m \in \mathbb{N}$. Since the level of $\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n]$ is equal to $\max\{\text{level}(\mathbf{k}_{i+1}) + 1 \mid i < n\}$ for each list $\vec{\mathbf{k}} = (\mathbf{k}_{i+1})_{i < n}$ of pasting schemes, $\mathbf{br}$ restricts to a function $\mathbf{PS}_m^* \rightarrow \mathbf{PS}_{m+1}$.

To show this is a bijection, suppose we are given a pasting scheme $\mathbf{k}$ of dimension $m + 1$, and we show that there exists a unique list $(\mathbf{k}_{i+1})_{i < n}$ of pasting schemes of dimension m satisfying $\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n] = \mathbf{k}$. Since we have $\mathbf{PS}_0 = \{\mathbf{br}[\mathbf{nil}]\}$, we may assume $\text{level}(\mathbf{k}) > 0$. Let us define $\mathbf{k}'_{i+1}$ as the $i + 1$ -th 1-transversal component of $\mathbf{k}$, which is uniquely determined by the definition of 1-transversal components. We write n for the number of 1-transversal components of $\mathbf{k}$, which is larger than 0 since $\text{level}(\mathbf{k}) > 0$. Then for each $\mathbf{k}'_{i+1}$, there exists a unique pasting scheme $\mathbf{k}_{i+1}$ satisfying $\mathbf{k}'_{i+1} = \mathbf{S}\mathbf{k}_{i+1}$, and we have $\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n] = \mathbf{k}$ by the definition of $\mathbf{br}$. $\square$

We thus have the bijection $\mathbf{PS}_m \cong (\mathbf{T1})_m$ for each $m \in \mathbb{N}$ by identifying $\mathbf{PS}_m$ with $\mathbf{PS}_{m-1}^*$ via $\mathbf{br}$ and using the description of $(\mathbf{T1})_m$ appeared as Fact 1.1.2.3.(ii). In the rest of this thesis, we identify $\mathbf{PS}_m$ with $(\mathbf{T1})_m$ via this bijection.

Notation 1.1.2.9. An n -cell $\mathbf{k} = \mathbf{k}^{(n)} \in (\mathbf{T1})_n$ is *degenerate* if $\text{level}(\mathbf{k}) < n$. $\lrcorner$

Moreover, by the above proposition, we have the following characterisation of $\mathbf{PS}$, which underlies several inductive definitions of $\mathbf{T1}$ in the literature (see e.g., the definition of Batanin trees in [Dea+24]).

Corollary 1.1.2.10. $(\mathbf{PS}, \mathbf{br})$ is the initial algebra of the endofunctor $(-)^*: \mathbf{Set} \rightarrow \mathbf{Set}$. $\lrcorner$

Proof. Given a $(-)^*$ -algebra (X, x) , a homomorphism $\bar{x}: (\mathbf{PS}, \mathbf{br}) \rightarrow (X, x)$ is a function $\bar{x}: \mathbf{PS} \rightarrow X$ satisfying

$$\bar{x}(\mathbf{br}[\mathbf{nil}]) = x(\mathbf{nil}) \text{ and } \bar{x}(\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n]) = x(\bar{x}(\mathbf{k}_1), \dots, \bar{x}(\mathbf{k}_n)).$$

Since we have $\mathbf{PS} \cong \bigcup_{m \in \mathbb{N}} \mathbf{PS}_{m-1}^*$ by Proposition 1.1.2.8, these equations uniquely determine such $\bar{x}$, which shows that $(\mathbf{PS}, \mathbf{br})$ is the initial algebra of $(-)^*$. $\square$

The source (or target) map of $\mathbf{T1}$, which is denoted by $\partial_n^{\mathbf{T1}}$ in Fact 1.1.2.3.(iii), is described in terms of pasting schemes as follows.

Proposition 1.1.2.11. For each pasting scheme $\mathbf{k} \in \mathbf{PS}_n$, the pasting scheme $\partial_m^{\mathbf{T1}} \mathbf{k}^{(n)}$ is obtained from $\mathbf{k}$ by replacing each of its m -transversal components with the sequence

$$\begin{bmatrix} m \end{bmatrix}.$$

In particular, $\partial_m^{\mathbf{T1}} \mathbf{k}^{(n)}$ does not depend on the (ambient) dimension n , and hence we just write $\partial_m^{\mathbf{T1}} \mathbf{k}$ for $\partial_m^{\mathbf{T1}} \mathbf{k}^{(n)}$. $\lrcorner$

Proof. Let us write $\partial_m^{\mathbf{T1}}$ for the operation on pasting schemes defined in the statement. Considering the inductive characterisation of $\partial_m^{\mathbf{T1}}$ in Fact 1.1.2.3.(iii), it suffices to show that

$$\partial_{m+1}^{\mathbf{T1}}(\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n]) = \mathbf{br}[\partial_m^{\mathbf{T1}} \mathbf{k}_1, \dots, \partial_m^{\mathbf{T1}} \mathbf{k}_n] \quad (1.1.2.11.1)$$

for each list $(\mathbf{k}_1, \dots, \mathbf{k}_n)$ of pasting schemes. This is straightforward when we observe that $m+1$ -transversal components of $\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n]$ are exactly those coming from m -transversal components of each $\mathbf{k}_i$ via the construction of $\mathbf{br}$. $\square$

Example 1.1.2.12. We have

$$\partial_4^{\mathbf{T1}} \left(\begin{bmatrix} 3 & 6 & 5 & 7 & 2 & 6 \\ & 2 & 3 & 4 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 3 & 4 & 4 & 2 & 4 \\ & 2 & 3 & 0 & 1 \end{bmatrix}.$$

Proposition 1.1.2.13 (See [Lei04, Section 8.1], [Web01, Section 2.1], and [Web04, Section 4]). Let $\mathbf{k} = ((k_i)_{i \leq r}, (\underline{k}_{i+1})_{i < r})$ be a pasting scheme, seen as an element of $\mathbf{T1}_m = \mathbf{PS}_m$ for some $m \geq \text{level}(\mathbf{k})$. Then the familial representation $\widehat{\mathbf{k}}^{(m)}$ is isomorphic to the colimit of the diagram

$$\begin{array}{ccccccc} G^{k_0} & & G^{k_1} & & \dots & & G^{k_r} \\ & \nwarrow \tau_{k_0}^{k_1} & & \nearrow \sigma_{k_1}^{k_1} & & \nwarrow \tau_{k_2}^{k_2} & \nearrow \sigma_{k_r}^{k_r} \\ & & G^{k_1} & & G^{k_2} & & G^{k_r} \end{array} \quad (1.1.2.13.1)$$

In particular, it does not depend on the (ambient) dimension m , and hence we just write $\widehat{\mathbf{k}}$ for $\widehat{\mathbf{k}}^{(m)}$, and call it the *realisation* of the pasting scheme $\mathbf{k}$. $\lrcorner$

Proof. By Fact 1.1.2.3.(iv), it suffices to show that, for each list $(\mathbf{k}_1, \dots, \mathbf{k}_n)$ of pasting schemes, the realisation $\widehat{\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n]}$ is isomorphic to the colimit of the diagram

$$\begin{array}{ccccccc} \underline{\Sigma} \widehat{\mathbf{k}}_1 & & \underline{\Sigma} \widehat{\mathbf{k}}_2 & & \dots & & \underline{\Sigma} \widehat{\mathbf{k}}_n \\ & \nearrow \star & \nwarrow \star' & & \nearrow \star & \nwarrow \star' & \nearrow \star \\ G^0 & & G^0 & & G^0 & & G^0 \end{array} \quad (1.1.2.13.2)$$

Observe first that the suspension $\underline{\Sigma}$ sends a representable G^n to G^{n+1} , and morphisms $\tau_n^k, \sigma_n^k: G^k \rightarrow G^n$ to $\tau_{n+1}^{k+1}, \sigma_{n+1}^{k+1}: G^{k+1} \rightarrow G^{n+1}$ respectively, for each $n \in \mathbb{N}$ and $k \leq n$. Moreover, through this identification, the points $\star, \star'$ in $\underline{\Sigma} G^n$ correspond to $\sigma_{n+1}^0, \tau_{n+1}^0: G^0 \rightarrow G^{n+1}$ respectively. Therefore, since $\underline{\Sigma}$ preserves connected colimits, we have an isomorphism $\underline{\Sigma} \widehat{\mathbf{k}} \cong \widehat{\mathbf{S} \mathbf{k}}$ for each pasting scheme $\mathbf{k} = ((k_i)_{i \leq r}, (\underline{k}_{i+1})_{i < r})$, and the point $\star$ in $\underline{\Sigma} \widehat{\mathbf{k}}$ corresponds to the source 0-cell of G^{k_1+1} and $\star'$ to the target 0-cell of G^{k_r+1} in $\widehat{\mathbf{S} \mathbf{k}}$. On the other hand, $\widehat{\mathbf{k}}_1 \frown \widehat{\mathbf{k}}_2$ is isomorphic to the pushout of the cospan

$$\widehat{\mathbf{k}}_1 \xleftarrow{\overline{\tau^0}} G^0 \xrightarrow{\overline{\sigma^0}} \widehat{\mathbf{k}}_2,$$

where $\overline{\tau^0}$ is the target 0-cell of the last component of $\mathbf{k}_1$ and $\overline{\sigma^0}$ is the source 0-cell of the first component of $\mathbf{k}_2$. These observations conclude that $\widehat{\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_n]}$ is isomorphic to the colimit of the diagram above. $\square$

Accordingly, using the morphism part of Fact 1.1.2.3.(iv), we obtain the following characterisation of the action of $(-)$ on morphisms in $\mathbf{G}/\mathbf{T1}$.

Proposition 1.1.2.14. For each $m < n$ and each n -dimensional pasting scheme $\mathbf{k}$, the parallel morphisms $\widehat{\sigma}_m^m, \widehat{\tau}_m^m : \widehat{\partial}_m^{\mathbf{T}1} \mathbf{k} \rightarrow \widehat{\mathbf{k}}$ are characterised as follows:

- If $\mathbf{k}$ is m -transversal, then we have $\widehat{\partial}_m^{\mathbf{T}1} \mathbf{k} = \begin{bmatrix} m \end{bmatrix}$, and $\widehat{\sigma}_m^m, \widehat{\tau}_m^m : G^m \rightarrow \widehat{\mathbf{k}}$ are the source and target maps of G^{k_0} and G^{k_r} respectively.
- Let $\mathbf{l}_1, \dots, \mathbf{l}_N$ be the m -transversal components of $\mathbf{k}$. Then by replacing each m -transversal component appearing in (1.1.2.13.1) with their realisations, $\mathbf{k}$ is expressed as the colimit

$$\begin{array}{ccccccc} X_0 & & X_1 & & \dots & & X_{r'} \\ \swarrow \tau_{n_1}^m & & \searrow \sigma_{n_1}^m & & \swarrow \tau_{n_2}^m & & \searrow \sigma_{n_{r'}}^m \\ & G^{n_1} & & G^{n_2} & & & G^{n_{r'}} \end{array} \quad (1.1.2.14.1)$$

where X_i is either G^{k_j} for some j with $k_j \leq m$ or $\widehat{\mathbf{l}}_{j'+1}$ for some $j' < N$, and r' is the rank of $\widehat{\partial}_m^{\mathbf{T}1} \mathbf{k}$. Morphisms defined in the above item for each $\mathbf{l}_{j'+1}$ and the identity on G^{k_j} for each $k_j \leq m$ define a parallel pair of morphisms of diagrams from the diagram (1.1.2.13.1) for $\widehat{\partial}_m^{\mathbf{T}1} \mathbf{k}$ to the above diagram, which result in the parallel morphisms $\widehat{\sigma}_n^m, \widehat{\tau}_n^m : \widehat{\partial}_m^{\mathbf{T}1} \mathbf{k} \rightarrow \widehat{\mathbf{k}}$.

In particular, the morphisms $\widehat{\sigma}_n^m, \widehat{\tau}_n^m$ are independent of the (ambient) dimension n of $\mathbf{k}$, so we just write them as $\widehat{\sigma}_m^m, \widehat{\tau}_m^m : \widehat{\partial}_m^{\mathbf{T}1} \mathbf{k} \rightarrow \widehat{\mathbf{k}}$. $\lrcorner$

These results explain the finitariness of the underlying endofunctor $\mathbf{T}$ on $\mathbf{GSet}$.

Proposition 1.1.2.15 ([Lei04, Theorem F.2.2]). $\mathbf{T}$ preserves filtered colimits and coproducts. $\lrcorner$

Proof. By Proposition 1.1.2.13, the familial representation $(-): \mathbf{G}/\mathbf{T}1 \rightarrow \mathbf{GSet}$ has its image contained in the full subcategory of finitely presentable and connected globular sets, since each realisation of a pasting scheme is a finite connected colimit of representables. Now the result follows from Fact 1.1.2.3.(i). $\square$

Example 1.1.2.16. The following are some examples of pasting schemes and their realisations:

$$\begin{array}{c} \begin{bmatrix} 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 2 & 2 \\ & 0 & 0 & 1 \end{bmatrix} \\ \\ \bullet \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \bullet \begin{array}{c} \curvearrowright \\ \downarrow \\ \curvearrowleft \end{array} \bullet \longrightarrow \bullet \begin{array}{c} \curvearrowright \\ \downarrow \\ \curvearrowleft \end{array} \bullet \end{array}$$

Here, all n -cells appearing in the realisation of a pasting scheme are distinct. $\lrcorner$

Using the inductive description (1.1.2.13.2) in the proof of Proposition 1.1.2.13, we also obtain the following property of the realisation of pasting schemes, which plays a key role in the next subsection. We defer its proof to Section A.1.

Proposition 1.1.2.17. For any pasting schemes $\mathbf{k}$ and $\mathbf{k}'$, if there is an isomorphism $\widehat{\mathbf{k}} \cong \widehat{\mathbf{k}'}$ in $\mathbf{GSet}$, then we have $\mathbf{k} = \mathbf{k}'$ and the isomorphism is the identity morphism. $\lrcorner$

Definition 1.1.2.18. Let X be a globular set and $\mathbf{k}$ be a pasting scheme of rank r . A *pasting diagram of shape $\mathbf{k}$ in X* is a sequence $\mathbf{u} = ((u_i)_{i \leq r}, (\underline{u}_{i+1})_{i < r})$ of cells in X satisfying $u_i \in X_{k_i}$ for each $i \leq r$, $\underline{u}_{i+1} \in X_{\underline{k}_{i+1}}$ for each $i < r$, and

$$\mathbf{t}_{\underline{k}_{i+1}}^X(u_i) = \underline{u}_{i+1} = \mathbf{s}_{\underline{k}_{i+1}}^X(u_{i+1})$$

for each $i < r$. The level and the rank of $\mathbf{u}$ are defined as those of $\mathbf{k}$. We often denote a pasting diagram $\mathbf{u}$ as a table of cells in X

$$\mathbf{u} = \begin{bmatrix} u_0 & u_1 & \dots & u_r \\ \underline{u}_1 & \underline{u}_2 & \dots & \underline{u}_r \end{bmatrix}.$$

Remark 1.1.2.19. For each globular set X , the set of all pasting diagrams of shape $\mathbf{k}$ in X is in bijection with the hom-set $\mathbf{GSet}(\widehat{\mathbf{k}}, X)$. Through this bijection, we identify an element of $\mathbf{TX}_m$ with a pasting diagram together with an (ambient) dimension m for its shape. In the same way as in Definition 1.1.2.18, when a pasting diagram $\mathbf{u}$ is regarded as an element of $(\mathbf{TX})_m$, we write $\mathbf{u}^{(m)}$ to indicate the (ambient) dimension m . $\lrcorner$

The action of $\mathbf{T}$ on morphisms of globular sets is straightforward: if $f: X \rightarrow Y$ is a morphism of globular sets, then $\mathbf{T}f: \mathbf{T}X \rightarrow \mathbf{T}Y$ is a morphism of globular sets which maps each pasting diagram

$$\mathbf{u} = \begin{bmatrix} u_0 & u_1 & \dots & u_r \\ \underline{u}_1 & \underline{u}_2 & \dots & \underline{u}_r \end{bmatrix}$$

in X to the pasting diagram

$$(\mathbf{T}f)(\mathbf{u}) = \begin{bmatrix} fu_0 & fu_1 & \dots & fu_r \\ f\underline{u}_1 & f\underline{u}_2 & \dots & f\underline{u}_r \end{bmatrix}$$

in Y . In particular, for the unique morphism $!: X \rightarrow 1$ to the terminal globular set 1 , the induced morphism $\mathbf{T}!: \mathbf{T}X \rightarrow \mathbf{T}1$ maps each pasting diagram $\mathbf{u}$ in X to its shape $\mathbf{k}$. $\lrcorner$

The following is a direct consequence of Proposition 1.1.2.14.

Proposition 1.1.2.20. Let $m < n$ be natural numbers and $\mathbf{u} = \mathbf{u}^{[n]}$ be a pasting diagram as in Definition 1.1.2.18 of shape $\mathbf{k}$, seen as an element of $(\mathbf{T}X)_n$. Then $s_m^{\mathbf{T}X}(\mathbf{u})$ is obtained from $\mathbf{u}$ by, for each of the m -transversal components $0 \leq i \leq j \leq r$ of $\mathbf{k}$, replacing the corresponding subsequence

$$\begin{bmatrix} u_i & u_{i+1} & \dots & u_j \\ \underline{u}_{i+1} & \underline{u}_{i+2} & \dots & \underline{u}_j \end{bmatrix} \quad (1.1.2.20.1)$$

of $\mathbf{u}$ with the sequence

$$\left[s_m^X(u_i) \right].$$

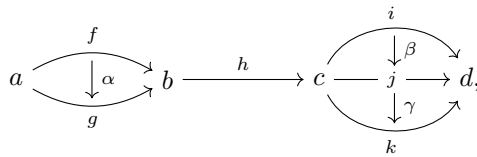
Similarly, $t_m^{\mathbf{T}X}(\mathbf{u}) \in (\mathbf{T}X)_m$ is obtained from $\mathbf{u}$ by replacing each instance of (1.1.2.20.1) with

$$\left[t_m^X(u_j) \right].$$

Example 1.1.2.21. Given a 2-dimensional pasting diagram

$$\mathbf{u} = \begin{bmatrix} \alpha & h & \beta & \gamma \\ b & c & j \end{bmatrix}$$

in X , which may be depicted as



we have

$$s_1^{\mathbf{T}X}(\mathbf{u}) = \begin{bmatrix} s_1^X(\alpha) & h & s_1^X(\beta) \\ b & c & j \end{bmatrix} = \begin{bmatrix} f & h & i \\ b & c & \end{bmatrix},$$

which may be depicted as

$$a \xrightarrow{f} b \xrightarrow{h} c \xrightarrow{i} d,$$

and

$$t_1^{\mathbf{T}X}(\mathbf{u}) = \begin{bmatrix} t_1^X(\alpha) & h & t_1^X(\gamma) \\ b & c & j \end{bmatrix} = \begin{bmatrix} g & h & k \\ b & c & \end{bmatrix},$$

which may be depicted as

$$a \xrightarrow{g} b \xrightarrow{h} c \xrightarrow{k} d,$$

as expected. $\lrcorner$

1.1.3. Unit and multiplication of the monad $\mathbf{T}$. In this subsection, we give a way to deal with the unit and multiplication of the monad $\mathbf{T}$ on $\mathbf{GSet}$. Before getting into the details, we first observe that cartesian transformations between familially represented functors are characterised in the following way.

Lemma 1.1.3.1 ([GH18, Proposition 3.8], [Sha25, Proposition 1.7]. See also Remark B.3.11.). Suppose we are given two endofunctors $F, F': \mathbf{GSet} \rightarrow \mathbf{GSet}$ and two functors $\widehat{(-)}: \mathbf{G}/F1 \rightarrow \mathbf{GSet}$ and $\widehat{(-)}': \mathbf{G}/F'1 \rightarrow \mathbf{GSet}$ such that the sliced functors $F_{/1}: \mathbf{GSet} \rightarrow \mathbf{GSet}/F1$ and $F'_{/1}: \mathbf{GSet} \rightarrow \mathbf{GSet}/F'1$ are nerve functors of $\widehat{(-)}$ and $\widehat{(-)}'$ through the canonical equivalences $\mathbf{GSet}/F1 \simeq \mathbf{Psh}(\mathbf{G}/F1)$ and $\mathbf{GSet}/F'1 \simeq \mathbf{Psh}(\mathbf{G}/F'1)$ respectively.

Then a natural transformation $\alpha: F \Rightarrow F'$ is cartesian if and only if there exists a natural isomorphism $\varphi: \widehat{\alpha_1(-)}' \cong \widehat{(-)}: \mathbf{G}/F1 \rightarrow \mathbf{GSet}$ such that the following diagram commutes in $\mathbf{Set}$ for each object $X \in \mathbf{GSet}$:

$$\begin{array}{ccc} (FX)_n & \xrightarrow{\sim} & \coprod_{\pi \in (F1)_n} \mathbf{GSet}(\widehat{\pi}, X) \\ (\alpha_X)_n \downarrow & & \downarrow (\pi, x) \mapsto (\alpha_1(\pi), \varphi_{\pi} x) \\ (F'X)_n & \xrightarrow{\sim} & \coprod_{\rho \in (F'1)_n} \mathbf{GSet}(\widehat{\rho}', X) \end{array}$$

where the horizontal maps are the isomorphisms given by the familial representations. $\lrcorner$

For the unit, we obtain the following explicit description using Fact 1.1.2.3.(v) and cartesianness of η (Fact 1.1.2.3.(i)).

Proposition 1.1.3.2. The unit $\eta_X^{\mathbf{T}}: X \rightarrow \mathbf{T}X$ maps each n -cell x of X to the n -cell

$$\begin{bmatrix} x \end{bmatrix}$$

of $\mathbf{T}X$. In particular, $\eta_1^{\mathbf{T}}: 1 \rightarrow \mathbf{T}1$ maps the unique n -cell of 1 to the n -cell

$$\begin{bmatrix} n \end{bmatrix}$$

of $\mathbf{T}1$. We have $\begin{bmatrix} n \end{bmatrix}: \begin{bmatrix} n-1 \end{bmatrix} \rightarrow \begin{bmatrix} n-1 \end{bmatrix}$ for each $n \geq 1$. $\lrcorner$

Proof. Rewriting Fact 1.1.2.3.(v) via $\mathbf{br}$, we have $\eta_1^{\mathbf{T}}(*_0) = \begin{bmatrix} 0 \end{bmatrix}$ and $\eta_1^{\mathbf{T}}(*_{n+1}) = \mathbf{br}[\eta_1^{\mathbf{T}}(*_n)] = \mathbf{S}(\eta_1^{\mathbf{T}}(*_n))$, which concludes the latter part ($X = 1$) of the statement. For the general case, because of the cartesianness of $\eta^{\mathbf{T}}$, there is an isomorphism $\widehat{\eta_1^{\mathbf{T}}(*_n)} \cong G^n$ and $\eta_X^{\mathbf{T}}(x)$ is obtained by precomposing it with the morphism $x: G^n \rightarrow X$ corresponding to an n -cell x . By virtue of Proposition 1.1.2.17, the isomorphism $\widehat{\eta_1^{\mathbf{T}}(*_n)} (= G^n) \cong G^n$ is the identity morphism, which concludes the former part. $\square$

For the multiplication $\mu^{\mathbf{T}}$, we propose a general procedure to compute it by introducing the notion of *2-pasting schemes*, which are just pasting diagrams in $\mathbf{T}1$.

Definition 1.1.3.3. A *2-pasting scheme* $\mathfrak{n} = ((\mathbf{n}_i)_{i \leq r}, (\mathbf{n}_{i+1})_{i < r})$ is a sequence of pasting schemes equipped with its *shape* $\mathbf{k} \in \mathbf{PS}$ such that for each $i \leq r$, $\mathbf{n}_i$ is of dimension k_i , for each $i < r$, $\mathbf{n}_{i+1}$ is of dimension k_{i+1} , and $\partial_{k_{i+1}}^{\mathbf{T}1} \mathbf{n}_i = \mathbf{n}_{i+1} = \partial_{k_{i+1}}^{\mathbf{T}1} \mathbf{n}_{i+1}$ for each $i < r$. To indicate the shape of a 2-pasting scheme, we often write it as $\mathfrak{n}^{(\mathbf{k})}$. One can readily observe that a 2-pasting scheme is nothing but a pasting diagram in $\mathbf{T}1$.

We write $\mathbf{P}^2\mathbf{S}$ for the set of all 2-pasting schemes. We write $\mathbf{shp}: \mathbf{P}^2\mathbf{S} \rightarrow \mathbf{PS}$ for the function assigning to each 2-pasting scheme its shape, and write $\mathbf{P}^2\mathbf{S}_{\mathbf{k}}$ for the fibre of $\mathbf{shp}$ over a pasting scheme $\mathbf{k}$. $\lrcorner$

Remark 1.1.3.4. For each globular set X , an n -cell $\mathfrak{u}$ in $\mathbf{TT}X$ of dimension n is a pasting diagram $\mathfrak{u}$ in $\mathbf{T}X$ with shape $\mathbf{k}$: i.e., it is specified as a table

$$\mathfrak{u} = \begin{bmatrix} \mathbf{u}_0^{(k_0)} & \mathbf{u}_1^{(k_1)} & \dots & \mathbf{u}_r^{(k_r)} \\ & \mathbf{u}_1^{(k_1)} & & \\ & \mathbf{u}_2^{(k_2)} & \dots & \mathbf{u}_r^{(k_r)} \end{bmatrix}^{(n)}$$

of pasting diagrams in X with their ambient dimensions indicated.

Taking shapes of each component, we obtain a 2-pasting scheme $\mathfrak{n}$ of shape $\mathbf{k}$, which is the image of $(\mathbf{k}, \mathfrak{u})$ under the morphism $\mathbf{TT}!_X: \mathbf{TT}X \rightarrow \mathbf{TT}1$. We call $\mathfrak{n}$ the *underlying 2-pasting scheme* of the n -cell $\mathfrak{u}$ in $\mathbf{TT}X$. $\lrcorner$

This remark immediately leads to the following realisation result for 2-pasting schemes, which indeed gives the familial representation of the composite endofunctor $\mathbf{TT}$.

Proposition 1.1.3.5. Let us define the *realisation* $\widehat{\mathfrak{n}}$ of a 2-pasting scheme $\mathfrak{n} = ((\mathbf{n}_i^{(k_i)})_{i \leq r}, (\mathbf{n}_{i+1}^{(\underline{k}_{i+1})})_{i < r})$ by the colimit

$$\begin{array}{ccccccc} \widehat{\mathbf{n}}_0 & & \widehat{\mathbf{n}}_1 & & \dots & & \widehat{\mathbf{n}}_r \\ & \nwarrow & \nearrow & \nwarrow & \nearrow & \nearrow & \nearrow \\ \widehat{\mathbf{n}}_{-1} & & \widehat{\mathbf{n}}_2 & & & & \widehat{\mathbf{n}}_r \end{array}$$

Then we have a (natural) bijection from $\mathbf{GSet}(\widehat{n}, X)$ to the fibre of the function $(\mathbf{TT}!_X)_n: \mathbf{TT}X \rightarrow \mathbf{TT}1$.

Now by virtue of cartesianness of the multiplication $\mu^\top$ (Fact 1.1.2.3.(i)) and Proposition 1.1.2.17, we immediately obtain the following characterisation of $\mu_1^\top$:

Proposition 1.1.3.6. For any 2-pasting scheme $\mathfrak{n}$, there is a unique isomorphism $\widehat{\mathfrak{n}} \cong \widehat{\mu_1^{\mathsf{T}}(\mathfrak{n})}$ in **GSet**. Moreover, if we have a pasting scheme $\mathfrak{l}$ and an isomorphism $\widehat{\mathfrak{n}} \cong \widehat{\mathfrak{l}}$ in **GSet**, then we have $\mathfrak{l} = \mu_1^{\mathsf{T}}(\mathfrak{n})$ and the isomorphism coincides with the one above. $\square$

Remark 1.1.3.7. Proposition 1.1.3.6 indeed gives a general procedure to compute the multiplication. To see this, let $\mathfrak{u}$ be an n -cell of $\mathbf{TTX}$ (as in Remark 1.1.3.4), whose underlying 2-pasting scheme is $\mathfrak{n}$. The above proposition shows that there is a unique pasting scheme $\mathfrak{l}$ with an isomorphism $\widehat{\mathfrak{l}} \cong \widehat{\mathfrak{n}}$, and by Lemma 1.1.3.1, the multiplication $\mu_X^{\mathbf{T}}(\mathfrak{u})$ is obtained as the composite

$$\hat{\mathbb{I}} \cong \hat{\mathbb{S}} \xrightarrow{e} X,$$

where the latter morphism $\widehat{n} \rightarrow X$ is that corresponding to $\mathfrak{u} \in \mathbf{TT}X$. Therefore, to compute $\mu_X^{\mathbf{T}}(\mathfrak{u})$, it suffices to find at least one pasting scheme $\mathbf{l}$ with $\widehat{\mathbf{l}} \cong \widehat{n}$. $\square$

In particular, since the realisation $\widehat{\mathfrak{n}}$ of a 2-pasting scheme $\mathfrak{n}$ does not depend on the ambient dimension and its shape, we conclude:

Corollary 1.1.3.8. For any 2-pasting scheme $\mathfrak{n} = \left(\left(\mathbf{n}_i^{(k_i)} \right)_{i \leq n}, \left(\mathbf{n}_{i+1}^{(k_{i+1})} \right)_{i \leq n} \right)$ seen as a cell of **TT1**, the multiplication $\mu_1^{\mathbf{T}(\mathfrak{n}^{(k)})}$ is independent of the ambient dimension of $\mathfrak{n}$ and those of the pasting schemes appearing in the table $\mathfrak{n}$. Consequently, given a pasting diagram $\mathfrak{u}$ in **TX** seen as an n -cell of **TTX**, the multiplication $\mu_X^{\mathbf{T}(\mathfrak{u}^{(n)})}$ is independent of the ambient dimension n . $\square$

Example 1.1.3.9. For a pasting scheme $\mathbf{k}$ of dimension m and any natural number $n \geq m$, we have

$$\mu_1^{\mathbf{T}} \left(\left(\begin{bmatrix} \mathbf{k} \end{bmatrix}^{(m)} \right)^{(n)} \right) = \mathbf{k}^{(n)}.$$

This follows from Corollary 1.1.3.8 and the unit law of the monad T .

Example 1.1.3.10. Suppose we are given natural numbers $k < m_1 \leq n$ and $k < m_2 \leq n$, and a 2-pasting scheme

$$\mathfrak{n} = \begin{bmatrix} \mathbf{m}^{(m_1)} & & \mathbf{n}^{(m_2)} \\ & \mathbf{l}^{(k)} & \end{bmatrix}.$$

Then the multiplication $\mu_1^T(\mathfrak{n})$ is the pasting scheme obtained as follows. Let $\mathbf{m}_1, \dots, \mathbf{m}_N$ be the components of the form $\begin{bmatrix} k \end{bmatrix}$ and the k -transversal components of $\mathbf{m}$, ordered from left to right. In the same way, let $\mathbf{n}_1, \dots, \mathbf{n}_{N'}$ be those of $\mathbf{n}$. Because $\mathfrak{n}$ forms a 2-pasting scheme, we have $N = N'$ and $\mathfrak{t}_k^{T1} \mathbf{m}_{i+1} = \begin{bmatrix} k \end{bmatrix} = s_k^{T1} \mathbf{n}_{i+1}$ for each $i < N$. The pasting scheme $\mu_1^T(\mathfrak{n})$ is obtained from $\mathbf{m}$ by replacing each $\mathbf{m}_{i+1}$ with the ‘concatenation of $\mathbf{m}_{i+1}$ and $\mathbf{n}_{i+1}$ at k ’:

$$\begin{array}{ll} \mathbf{m}_{i+1} & \text{(if } \mathbf{n}_{i+1} = \begin{bmatrix} k \\ \end{bmatrix} \text{)} \\ \mathbf{n}_{i+1} & \text{(if } \mathbf{m}_{i+1} = \begin{bmatrix} k \\ \end{bmatrix} \text{)} \\ \left[\begin{array}{cccccc} m_{i+1,0} & \dots & m_{i+1,r_{i+1}} & n_{i+1,0} & \dots & n_{i+1,r'_{i+1}} \\ \underline{m}_{i+1,1} & \dots & \underline{m}_{i+1,r_{i+1}} & k & \underline{n}_{i+1,1} & \dots & \underline{n}_{i+1,r'_{i+1}} \end{array} \right] & \text{(otherwise)} \end{array}$$

where r_{i+1} and r'_{i+1} are the ranks of $\mathbf{m}_{i+1}$ and $\mathbf{n}_{i+1}$ respectively.

This can be shown as follows. First observe that by Proposition 1.1.2.14 the realisation $\widehat{\mathfrak{n}}$ is obtained as the colimit of the following diagram:

$$\begin{array}{ccccccccccccccc}
 X_0 & \leftarrow \circ & G^{l_1} & \longrightarrow & X_1 & \leftarrow \circ & G^{l_2} & \longrightarrow & X_2 & \leftarrow \circ & \cdots & \longrightarrow & X_{r-1} & \leftarrow \circ & G^{l_r} & \longrightarrow & X_r \\
 \uparrow & & \parallel & & \uparrow & & \parallel & & \uparrow & & \parallel & & \uparrow & & \parallel & & \uparrow \\
 G^{l_0} & \leftarrow \circ & G^{l_1} & \longrightarrow & G^{l_1} & \leftarrow \circ & G^{l_2} & \longrightarrow & G^{l_2} & \leftarrow \circ & \cdots & \longrightarrow & G^{l_{r-1}} & \leftarrow \circ & G^{l_r} & \longrightarrow & G^{l_r} \\
 \downarrow & & \parallel & & \downarrow & & \parallel & & \downarrow & & \parallel & & \downarrow & & \parallel & & \downarrow \\
 Y_0 & \longleftarrow & G^{l_1} & \longrightarrow & Y_1 & \longleftarrow & G^{l_2} & \longrightarrow & Y_2 & \longleftarrow & \cdots & \longrightarrow & Y_{r-1} & \longleftarrow & G^{l_r} & \longrightarrow & Y_r
 \end{array}$$

Here, the top and bottom rows are the diagrams for $\widehat{\mathfrak{m}}$ and $\widehat{\mathfrak{n}}$ considered in Proposition 1.1.2.14 respectively. The middle row is the diagram for $\widehat{\mathfrak{l}}$ considered in Proposition 1.1.2.13, and the vertical arrow from mid to top comprises the morphism of diagrams inducing $\widehat{\tau_{m_1}^k}$ in Proposition 1.1.2.14, while that from mid to bottom comprises the one inducing $\widehat{\sigma_{m_2}^k}$. One can remove from this diagram all the morphisms marked by $\circ$ without changing the colimit, and unpacking the colimits defining each X_i and Y_i ($i \leq r$), we obtain the diagram for $\widehat{\mu_1^T(\mathfrak{n})}$ as desired. $\lrcorner$

1.2. Weak ω -categories

1.2.1. The idea of weak ω -categories. In this subsection, we shall give a heuristic discussion that motivates the definition of the monad $\mathbf{L}$ (on $\mathbf{GSet}$) for weak ω -categories; an actual definition is carried out in the next subsection. First note that, whatever we define $\mathbf{L}$ to be, there should be a monad morphism $\mathbf{ar}^{\mathbf{L}}: \mathbf{L} \rightarrow \mathbf{T}$ since we would want strict ω -categories to be a special case of weak ω -categories. (Here the notation $\mathbf{ar}^{\mathbf{L}}$ stands for *arity*, and this terminology will be justified below where we consider its component $\mathbf{ar}_1^{\mathbf{L}}$ at the terminal globular set 1.)

Suppose that a globular set X contains a sequence of 1-cells:

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d.$$

In contrast to the strict case where we would have a unique composite hgf in $\mathbf{TX}$, the free weak ω -category $\mathbf{LX}$ should contain *distinct* 1-cells $(hg)f$ and $h(gf)$ (and also other composites with identities inserted in various positions). Motivated by such examples, we claim that $(\mathbf{LX})_n$ should consist of the *pasting instructions* $(\phi, \mathbf{u})$ of dimension n in X , by which we mean a pasting diagram $\mathbf{u} \in (\mathbf{TX})_n$ of dimension n in X together with an additional piece of data ϕ that encodes *how* to compose the pasting diagram $\mathbf{u}$.

Since it should be the *shape* of the pasting diagram $\mathbf{u}$ (rather than the labels u_i and $\underline{u}_i$) that determines what ϕ can be, we may simply take ϕ to be an element of $(\mathbf{L1})_n$ with appropriate arity; that is, we want the globular set $\mathbf{LX}$ to be the pullback

$$\begin{array}{ccc}
 \mathbf{LX} & \xrightarrow{\mathbf{L}!} & \mathbf{L1} \\
 \mathbf{ar}_X^{\mathbf{L}} \downarrow & \lrcorner & \downarrow \mathbf{ar}_1^{\mathbf{L}} \\
 \mathbf{TX} & \xrightarrow{\mathbf{T}!} & \mathbf{T1}
 \end{array} \tag{1.2.1.0.1}$$

where $\mathbf{ar}^{\mathbf{L}}: \mathbf{L} \rightarrow \mathbf{T}$ is the monad morphism mentioned earlier. We also call elements ϕ of $(\mathbf{L1})_n$ *pasting instructions*.

Thus, a major part of the definition of the monad $\mathbf{L}$ is reduced to describing what $\mathbf{L1}$ is. Recall that, in the previous subsection, we saw a description of strict ω -categories as those globular sets equipped with a *unique* composite for each pasting diagram. The structure of $\mathbf{L1}$ should somehow encode a weak version of this, so we want each pasting diagram to have *some* composite, which should moreover be unique up to suitably invertible higher cell. To make this precise, we draw insight from the *pasting theorem* in the 2-dimensional case.

The pasting theorem for *strict* 2-categories [Pow90] tells us that a pasting diagram such as

$$\begin{array}{ccccc}
 & f & & i & \\
 a & \xrightarrow{\quad} & b & \xrightarrow{h} & c & \xrightarrow{\quad} & d \\
 & g & & j & \\
 & \alpha & & \beta & \\
 & \downarrow & & \downarrow & \\
 & \gamma & & k &
 \end{array}$$

(or much more complicated ones) admits a unique composite 2-cell $ihf \rightarrow khg$. In the *weak* case of bicategories [Ver11, Appendix A], however, we must first specify how to interpret ‘ ihf ’ and ‘ khg ’; once we have fixed the

bracketing on these 1-cells, we obtain a unique composite 2-cell. When defining $\mathbf{L}$, we can follow the same pattern at least for the existence part: *given a pasting scheme of dimension n together with pasting instructions on its $(n-1)$ -dimensional source and target, it must be possible to extend them to a pasting instruction on the whole pasting scheme.*

More precisely, we ask that $\text{ar}_1^{\mathbf{L}}: \mathbf{L}1 \rightarrow \mathbf{T}1$ have the right lifting property with respect to the boundary inclusion $\iota^n: \partial G^n \rightarrow G^n$ for each $n \geq 1$ (cf. [Gar09a]). This means that given any u and v making the outer square in the following diagram commutative, there exists a (not necessarily unique) morphism w making both triangles in the diagram commutative:

$$\begin{array}{ccc} \partial G^n & \xrightarrow{u} & \mathbf{L}1 \\ \iota^n \downarrow & \nearrow w & \downarrow v \\ G^n & \xrightarrow{\text{ar}_1^{\mathbf{L}}} & \mathbf{T}1 \end{array}$$

Note that, thanks to the pullback condition (1.2.1.0.1), the component $\text{ar}_X^{\mathbf{L}}$ at any globular set X inherits the right lifting property from $\text{ar}_1^{\mathbf{L}}$; given a commutative square as below left, we can first find a lift of the outer square using the right lifting property of $\text{ar}_1^{\mathbf{L}}$, and then use the universal property of the pullback to obtain the desired lift:

$$\begin{array}{ccccc} \partial G^n & \xrightarrow{u} & \mathbf{L}X & \xrightarrow{\mathbf{L}!} & \mathbf{L}1 \\ \iota^n \downarrow & \nearrow & \downarrow \text{ar}_X^{\mathbf{L}} & \nearrow & \downarrow \text{ar}_1^{\mathbf{L}} \\ G^n & \xrightarrow{v} & \mathbf{T}X & \xrightarrow{\mathbf{T}!} & \mathbf{T}1 \end{array}$$

Thus we have taken care of the existence part of the pasting theorem, and in fact formalising these ideas (combined with a suitable universal property) essentially leads to the precise definition given in the next subsection. The uniqueness part turns out to be a theorem (see Proposition 2.3.7 and Remark 2.3.9) rather than part of the definition.

1.2.2. The definition of weak ω -category. Before turning to the definition, we recall the notion of *cartesianness*, which has already appeared in Fact 1.1.2.3 and will play an important role in defining weak ω -categories.

Definition 1.2.2.1. A natural transformation $\alpha: F \Rightarrow G: \mathbf{A} \rightarrow \mathbf{B}$ is said to be *cartesian* if for each morphism $f: a \rightarrow a'$ in $\mathbf{A}$, the naturality square

$$\begin{array}{ccc} Fa & \xrightarrow{\alpha_a} & Ga \\ Ff \downarrow & & \downarrow Gf \\ Fa' & \xrightarrow{\alpha_{a'}} & Ga' \end{array}$$

is a pullback square in $\mathbf{B}$.

A monad (S, η^S, μ^S) on a category with pullbacks is *cartesian* if its functor part S preserves all pullbacks and its unit η^S and multiplication μ^S are cartesian natural transformations. If S and S' are monads on a category with pullbacks, then an (icon) monad morphism $\alpha: S \rightarrow S'$ is *cartesian* if it is cartesian as a natural transformation. $\lrcorner$

The following are immediate from the pasting lemma of pullback squares.

Lemma 1.2.2.2. Let $\mathbf{A}$ and $\mathbf{B}$ be categories with pullbacks.

- For any functor $G: \mathbf{A} \rightarrow \mathbf{B}$ that preserves pullbacks and a cartesian natural transformation $\alpha: F \Rightarrow G$, the domain F also preserves pullbacks.
- Suppose we are given a commutative square

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & G \\ & \searrow \gamma & \downarrow \beta \\ & & H \end{array}$$

in $[\mathbf{A}, \mathbf{B}]$ such that β is cartesian. Then α is cartesian if and only if γ is cartesian. $\lrcorner$

Lemma 1.2.2.3. Given a monad morphism $\alpha: S \rightarrow S'$ between monads S and S' on a category with pullbacks, if S' is a cartesian monad and α is cartesian, then S is also a cartesian monad. $\lrcorner$

As is mentioned in Fact 1.1.2.3.(i), the monad $\mathbf{T}$ for strict ω -categories is cartesian (see also [Lei04, Theorem F.2.2]).

Definition 1.2.2.4. A *cartesian monad over $\mathbf{T}$* is a monad $P = (P, \eta^P, \mu^P)$ on $\mathbf{GSet}$ equipped with a cartesian monad morphism $\text{ar}^P: P \rightarrow \mathbf{T}$, which is necessarily cartesian as a monad by Lemma 1.2.2.3. $\lrcorner$

Definition 1.2.2.5. Let $P = (P, \text{ar}^P)$ be a cartesian monad over $\mathbf{T}$. A *contraction* on P is a function $\bar{\kappa}^P$ which assigns for each $n \geq 1$, $u: \partial G^n \rightarrow P1$, and $v: G^n \rightarrow \mathbf{T}1$ such that $\text{ar}_1^P \circ u = v \circ \iota^n$, a morphism $\bar{\kappa}^P(u, v): G^n \rightarrow P1$ such that $u = \bar{\kappa}^P(u, v) \circ \iota^n$ and $v = \text{ar}_1^P \circ \bar{\kappa}^P(u, v)$:

$$\begin{array}{ccc} \partial G^n & \xrightarrow{u} & P1 \\ \downarrow \iota^n & \nearrow \bar{\kappa}^P(u, v) & \downarrow \text{ar}_1^P \\ G^n & \xrightarrow{v} & \mathbf{T}1 \end{array}$$

Let Q be another cartesian monad over $\mathbf{T}$ with a contraction $\bar{\kappa}^Q$, and let $\alpha: P \rightarrow Q$ be a monad morphism over $\mathbf{T}$, which is necessarily cartesian by Lemma 1.2.2.2. Then we say α *preserves contractions* if the following diagram commutes (i.e., $\alpha_1 \circ \bar{\kappa}^P(u, v) = \bar{\kappa}^Q(\alpha_1 \circ u, v)$ holds) for each $n \geq 1$, $u: \partial G^n \rightarrow P1$, and $v: G^n \rightarrow \mathbf{T}1$ such that $\text{ar}_1^P \circ u = v \circ \iota^n$:

$$\begin{array}{ccccc} \partial G^n & \xrightarrow{u} & P1 & \xrightarrow{\alpha_1} & Q1 \\ \downarrow \iota^n & \nearrow \bar{\kappa}^P(u, v) & \downarrow \text{ar}_X^P & \nearrow \bar{\kappa}^Q(\alpha_1 \circ u, v) & \downarrow \text{ar}_1^Q \\ G^n & \xrightarrow{v} & \mathbf{T}1 & \xlongequal{\quad} & \mathbf{T}1 \end{array}$$

Example 1.2.2.6. Let us slightly unpack the definition of a contraction. Let $P = (P, \text{ar}^P, \bar{\kappa}^P)$ be a cartesian monad over $\mathbf{T}$ with a contraction. Suppose we are given $n \geq 1$, a parallel pair of $(n-1)$ -cells ϕ, ϕ' in $P1$ (inducing the morphism $\langle \phi, \phi' \rangle: \partial G^n \rightarrow P1$), and a pasting scheme $\mathbf{k}$ of dimension n such that $\text{ar}_1^P(\phi) = \partial_{n-1}^{\mathbf{T}1}(\mathbf{k}) = \text{ar}_1^P(\phi')$; then we have an n -cell $\bar{\kappa}^P(\langle \phi, \phi' \rangle, \mathbf{k})$ in $P1$ such that $\bar{\kappa}^P(\langle \phi, \phi' \rangle, \mathbf{k}): \phi \rightarrow \phi'$ and $\text{ar}_1^P(\bar{\kappa}^P(\langle \phi, \phi' \rangle, \mathbf{k})) = \mathbf{k}$.

$$\begin{array}{ccc} \partial G^n & \xrightarrow{\langle \phi, \phi' \rangle} & P1 \\ \downarrow \iota^n & \nearrow \bar{\kappa}^P(\langle \phi, \phi' \rangle, \mathbf{k}) & \downarrow \text{ar}_1^P \\ G^n & \xrightarrow{\mathbf{k}} & \mathbf{T}1 \end{array}$$

The following is (an alternative formulation of) [Lei04, Corollary G.1.2].² See also Example 1.3.22 for a sketch of a proof of this proposition.

Proposition 1.2.2.7. The category of cartesian monads over $\mathbf{T}$ with contractions and monad morphisms preserving contractions has an initial object, which we write as $\mathbf{L} = (\mathbf{L}, \eta^{\mathbf{L}}, \mu^{\mathbf{L}}, \text{ar}^{\mathbf{L}}, \bar{\kappa}^{\mathbf{L}})$. $\lrcorner$

Definition 1.2.2.8 ([Lei04]). A *weak ω -category* is an Eilenberg–Moore algebra $(X, \xi: \mathbf{L}X \rightarrow X)$ for the monad $\mathbf{L} = (\mathbf{L}, \eta^{\mathbf{L}}, \mu^{\mathbf{L}})$.

We define the category $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ of weak ω -categories and *strict ω -functors* between them as the Eilenberg–Moore category of the monad $\mathbf{L}$. $\lrcorner$

Proposition 1.2.2.9 ([FHM25b, 2.2.1]). Observe that, in terms of Eilenberg–Moore categories, the monad morphism $\text{ar}^{\mathbf{L}}: \mathbf{L} \rightarrow \mathbf{T}$ is described as a commutative triangle

$$\begin{array}{ccc} \mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s & \xrightarrow{\text{inclusion}} & \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \\ & \searrow U_{\mathbf{T}} & \swarrow U_{\mathbf{L}} \\ & \mathbf{GSet} & \end{array} \quad (1.2.2.9.1)$$

of right adjoint functors, where the top horizontal functor sends a strict ω -category (X, ξ) to the weak ω -category $(X, \xi \circ \text{ar}_X^{\mathbf{L}})$. This inclusion functor is full and faithful. $\lrcorner$

²In [Lei04], cartesian monads over $\mathbf{T}$ are (equivalently) described as $\mathbf{T}$ -operads (or *globular operads*), and its Corollary G.1.2 states the existence of an initial $\mathbf{T}$ -operad with contraction. See Section A.2.1 for more details.

Proof. Since the forgetful functors U_T and U_L are both faithful, the inclusion functor is also faithful. To see that it is full, it suffices to check that each component of ar^L is an epimorphism in **GSet**. To this end, it suffices to check that the component $\text{ar}_1^L: L1 \rightarrow T1$ at the terminal globular set 1 is an epimorphism since epimorphisms are pullback stable in a presheaf category like **GSet**.

This³ follows because ar_1^L has the right lifting property with respect to the morphisms $\iota^n: \partial G^n \rightarrow G^n$ for every $n > 0$ by definition, and with respect to $\iota^0: \emptyset \rightarrow G^0$ as well, since $(\eta_1^T)_0$ is a bijection and $\text{ar}_1^L \circ \eta_1^L = \eta_1^T$. $\square$

Through this fully faithful right adjoint, we often regard **Str- ω -Cat_s** as a reflective full subcategory of **Wk- ω -Cat_s**, and identify a strict ω -category (X, ξ) with the weak ω -category $(X, \xi \circ \text{ar}_X)$.

Note that, although there is such an inclusion identifying strict ω -categories with weak ω -categories, this inclusion does not, in general, preserve the free ‘walking objects’; rather, these are preserved by the associated free functors.

Notation 1.2.2.10. Let us write F_T and F_L for the free functors for the Eilenberg–Moore categories of the monads **T** and **L**, respectively. The left adjoint of the inclusion functor in (1.2.2.9.1) is referred to as the *strict ω -categorical reflection*. $\lrcorner$

Notation 1.2.2.11 (cf. [FHM25b, Section 2.2]). For each $n \geq 0$, we write

- C^n for the free strict ω -category $F_T G^n$ on G^n ,
- ∂C^n for the free strict ω -category $F_T \partial G^n$ on the boundary of G^n ,
- $\mathcal{C}^n$ for the free weak ω -category $F_L G^n$ on G^n , and
- $\partial \mathcal{C}^n$ for the free weak ω -category $F_L \partial G^n$ on the boundary of G^n .

(The letter C stands for ‘cell’.) Abusing the notation, the images of (1.1.1.5.1) under F_T and F_L are written as

$$\begin{array}{ccc} & \partial C^{n-1} & \\ \sigma_\partial^{n-1} \nearrow & \downarrow \iota^n & \nwarrow \tau_\partial^{n-1} \\ C^{n-1} & \xrightarrow{\sigma^{n-1}} C^n \xleftarrow{\tau^{n-1}} & C^{n-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} & \partial \mathcal{C}^{n-1} & \\ \sigma_\partial^{n-1} \nearrow & \downarrow \iota^n & \nwarrow \tau_\partial^{n-1} \\ \mathcal{C}^{n-1} & \xrightarrow{\sigma^{n-1}} \mathcal{C}^n \xleftarrow{\tau^{n-1}} & \mathcal{C}^{n-1} \end{array} \quad (1.2.2.11.1)$$

Since the free functors preserve colimits, those form the same pushout squares as in Remark 1.1.1.7. $\lrcorner$

1.2.3. The free weak ω -category monad. In this subsection, we shall describe the structure of the monad **L** in some detail. From now on, we employ the following abbreviations and notations.

Notation 1.2.3.1. We write ar_X^L as ar_X , ar_1^L as ar , and $\bar{\kappa}^L$ as $\bar{\kappa}$ for brevity.

- By a *pasting instruction*, we mean an element ϕ of $(L1)_n$ for some $n \in \mathbb{N}$. n is called the *dimension* of the pasting instruction ϕ , and $\text{ar}(\phi) \in (T1)_n$ is called the *arity* of ϕ .
- For each $n \geq 0$, we denote the image of the unique n -cell of 1 under the unit $\eta_1^L: 1 \rightarrow L1$ of **L** by $\tilde{e}_n \in (L1)_n$, whose arity is $[n]$ (see Proposition 1.1.3.2). $\lrcorner$

Remark 1.2.3.2 (cf. [FHM24, Section 2.5], [FHM25b, proof of Proposition B.0.7]). Given a globular set X , the globular set LX is obtained as the pullback

$$\begin{array}{ccc} LX & \xrightarrow{L!} & L1 \\ \text{ar}_X \downarrow & \lrcorner & \downarrow \text{ar}_1 \\ TX & \xrightarrow{T!} & T1 \end{array} \quad (1.2.3.2.1)$$

Moreover, they fit into the following commutative cubes

$$\begin{array}{ccccccc} LLX & \xrightarrow{\mu_X^L} & LX & \xleftarrow{\eta_X^L} & X & \searrow & 1 \\ \text{ar}_{LX} \downarrow & \lrcorner & \downarrow L! & \downarrow \mu_1^L & \downarrow L! & \downarrow \eta_1^L & \downarrow \\ TLX & \xrightarrow{\mu_X^L} & L1 & \xleftarrow{\eta_1^L} & 1 & \searrow & 1 \\ \text{Tar}_X \downarrow & \lrcorner & \downarrow \text{ar}_{L1} & \downarrow \mu_1^L & \downarrow \text{ar}_1 & \downarrow \eta_1^L & \downarrow \\ TTX & \xrightarrow{\mu_X^T} & TX & \xleftarrow{\eta_X^T} & X & \searrow & 1 \\ TT! \downarrow & \lrcorner & \downarrow T! & \downarrow \mu_1^T & \downarrow T! & \downarrow \eta_1^T & \downarrow \\ TT1 & \xrightarrow{\mu_1^T} & T1 & \xleftarrow{\eta_1^T} & 1 & \searrow & 1 \end{array}$$

³ ar_1^L is in fact a split epimorphism, with a section given in Definition 1.2.3.6.

such that all the squares on the left, the central sliced square, and the vertical squares on the right are pullback squares. Thus we have the following concrete description of $\mathbf{L}$:⁴

- (i) An n -cell of $\mathbf{L}X$ is a pair $(\phi, \mathbf{u})$ consisting of a pasting instruction ϕ and a pasting diagram $\mathbf{u}$ in X whose shape is the arity of ϕ .
- (ii) We have

$$s_m^{LX}(\phi, \mathbf{u}) = (s_m^{L1}(\phi), s_m^{TX}(\mathbf{u})) \quad \text{and} \quad t_m^{LX}(\phi, \mathbf{u}) = (t_m^{L1}(\phi), t_m^{TX}(\mathbf{u})).$$

- (iii) For each morphism $f: X \rightarrow Y$ in $\mathbf{GSet}$, the morphism $Lf: \mathbf{L}X \rightarrow \mathbf{L}Y$ maps an n -cell $(\phi, \mathbf{u})$ of $\mathbf{L}X$ to the n -cell $(\phi, Tf(\mathbf{u}))$ of $\mathbf{L}Y$.
- (iv) By Proposition 1.1.3.2 and the pullback (1.2.3.2.1) defining $\mathbf{L}X$, the unit $\eta_X^L: X \rightarrow \mathbf{L}X$ maps each n -cell x of X to the n -cell

$$\left(\tilde{e}_n, \begin{bmatrix} x \end{bmatrix} \right)$$

of $\mathbf{L}X$.

- (v) An n -cell of $\mathbf{L}\mathbf{L}X$ is a pair $(\phi, \tilde{\mathbf{u}})$ of a pasting instruction ϕ and a pasting diagram $\tilde{\mathbf{u}}$ in $\mathbf{L}X$ of shape $\text{ar}(\phi)$. Using the above description of $\mathbf{L}X$, we may write $\tilde{\mathbf{u}}$ as

$$\tilde{\mathbf{u}} = \begin{bmatrix} (\chi_0, \mathbf{u}_0) & \cdots & (\chi_r, \mathbf{u}_r) \\ (\underline{\chi}_1, \mathbf{u}_1) & \cdots & (\underline{\chi}_r, \mathbf{u}_r) \end{bmatrix}.$$

Thus $\tilde{\mathbf{u}}$ can be decomposed into

$$\chi = \begin{bmatrix} \chi_0 & \cdots & \chi_r \\ \underline{\chi}_1 & \cdots & \underline{\chi}_r \end{bmatrix} \in (\mathbf{TL}1)_n \quad \text{and} \quad \bar{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_0 & \cdots & \mathbf{u}_r \\ \mathbf{u}_1 & \cdots & \mathbf{u}_r \end{bmatrix} \in (\mathbf{TT}X)_n.$$

Now we have $\mu_X^L(\phi, \tilde{\mathbf{u}}) = (\mu_1^L(\phi, \chi), \mu_X^T(\bar{\mathbf{u}}))$. ┘

This description (in particular, the third item) immediately yields the following lemma, which explains the terminology ‘strict ω -functor’.

Lemma 1.2.3.3. Suppose we are given a strict ω -functor $f: X \rightarrow Y$ between weak ω -categories, and a pasting instruction ϕ of dimension $n \geq 0$ with arity $\mathbf{k}$. Then f preserves the ‘paste according to ϕ ’: for any pasting diagram $\mathbf{u}$ in X of shape $\mathbf{k}$ and of dimension $n \geq 0$, we have

$$f(\xi^X(\phi, \mathbf{u})) = \xi^Y(\phi, (Tf)(\mathbf{u})). \quad \text{┘}$$

Proof. The left hand side is equal to $\xi^Y((Lf)(\phi, \mathbf{u}))$ since f is a morphism of $\mathbf{L}$ -algebras, and it is equal to the right hand side because of the definition of $\mathbf{L}$ (see Remark 1.2.3.2.(iii)). □

On the other hand, considering the pullback description (1.2.3.2.1) functorially, we can observe the following fact (mentioned in, e.g., [Gar10, Section 5]).

Proposition 1.2.3.4 ([FHM25a, Proposition 2.2.5]). The functor $\mathbf{L}: \mathbf{GSet} \rightarrow \mathbf{GSet}$ is finitary; that is, it preserves filtered colimits. ┘

Proof. Up to natural isomorphism, we can factorise $\mathbf{L}$ as

$$\mathbf{GSet} \rightarrow \mathbf{GSet}/\mathbf{T}1 \xrightarrow{\text{ar}_1^*} \mathbf{GSet}/\mathbf{L}1 \rightarrow \mathbf{GSet}$$

where the first factor sends $X \in \mathbf{GSet}$ to $\mathbf{T}! : \mathbf{T}X \rightarrow \mathbf{T}1$ (which preserves filtered colimits because $\mathbf{T}$ does by [Lei04, Theorem F.2.2] or [Ara+25, Proposition 14.2.8]), the second factor is pullback along ar_1 (which preserves filtered colimits because it is a left adjoint; note that $\mathbf{GSet}$, being a presheaf category, is locally cartesian closed), and the last factor is the canonical projection. □

Corollary 1.2.3.5 ([FHM25b, Proposition 2.2.2]). All categories in the triangle (1.2.2.9.1) are locally finitely presentable, and all functors in it are finitary right adjoints. ┘

Proof. The category $\mathbf{GSet}$ is a presheaf category and hence is locally finitely presentable. The remaining claims follow from the facts that the monads $\mathbf{T}$ and $\mathbf{L}$ are finitary; the former is [Lei04, Theorem F.2.2] or Proposition 1.1.2.15 and the latter is Proposition 1.2.3.4. □

Definition 1.2.3.6. For any $n \geq 0$ and a pasting scheme $\mathbf{k}$ of dimension n , we define a pasting instruction $\text{spi}_n(\mathbf{k})$ of dimension n with arity $\mathbf{k}$ inductively as follows.

⁴In fact, the description applies to an arbitrary cartesian monad P over $\mathbf{T}$ in place of $\mathbf{L}$.

- If $\mathbf{k} = \begin{bmatrix} n \end{bmatrix}$, then $\text{spi}_n(\begin{bmatrix} n \end{bmatrix}) = \tilde{e}_n$.
- Otherwise, suppose that $\mathbf{k}$ is of the form $\mathbf{k}' \rightarrow \mathbf{k}'$ in $\mathbf{T1}$, where $\mathbf{k}'$ is a pasting scheme of dimension $n-1$ (note that necessarily $n \geq 1$). We set $\text{spi}_n(\mathbf{k}) = \bar{\kappa}(\langle \text{spi}_{n-1}(\mathbf{k}'), \text{spi}_{n-1}(\mathbf{k}') \rangle, \mathbf{k}) \in (\mathbf{L1})_n$.

The pasting instruction $\text{spi}_n(\mathbf{k})$ is called the *standard pasting instruction* of shape $\mathbf{k}$. We often abbreviate the subscript n when the dimension is clear from the context. The graded function $\text{spi}: \mathbf{T1} \rightarrow \mathbf{L1}$ is a globular map and a section of ar , and it commutes with η_1^T and η_1^L . $\lrcorner$

Notation 1.2.3.7. Let $X = (X, \xi)$ be a weak ω -category. For any $n \geq 0$ and any pasting scheme $\mathbf{k}$ of dimension n , we have the *standard pasting operation of arity $\mathbf{k}$* in X :

$$\mathbf{u} \mapsto \xi(\text{spi}_n(\mathbf{k}), \mathbf{u}): \mathbf{GSet}(\widehat{\mathbf{k}}, X) \rightarrow X_n \quad \lrcorner$$

We introduce a notation (extending the corresponding notation for strict ω -categories) for certain standard pasting operations we shall use in the sequel.

Definition 1.2.3.8 ([FHM25a, Definition 2.3.5]). Let $X = (X, \xi)$ be a weak ω -category.

- (i) Given natural numbers $k \leq n$ and a k -cell x in X , we define the n -cell

$$\text{id}_{k,n}^X(x) = \xi \left(\text{spi}_n \left(\begin{bmatrix} n-1 \end{bmatrix}^{(n)} \right), \begin{bmatrix} x \end{bmatrix}^{(n)} \right)$$

of X . We also write $\text{id}_n^X(x)$ for $\text{id}_{k,n}^X(x)$ when the dimension k of x is clear from the context.

- (ii) Given natural numbers k, m_1, m_2, n with $k < m_i \leq n$ for each $i = 1, 2$, an m_1 -cell u and an m_2 -cell v in X such that $t_k^X(u) = s_k^X(v)$, we define the n -cell

$$u *_{k,n}^X v = \xi \left(\text{spi}_n \left(\begin{bmatrix} m_1 & m_2 \\ & k \end{bmatrix}^{(n)} \right), \begin{bmatrix} u & v \\ & x \end{bmatrix}^{(n)} \right)$$

of X , where $x = t_k^X(u) = s_k^X(v)$. When $n = \max\{m_1, m_2\}$, we also write $u *_{k,n}^X v$ for $u *_{k,n}^X v$.⁵ $\lrcorner$

Lemma 1.2.3.9. For any weak ω -category $X = (X, \xi)$ and any n -cell x in X , we have $\text{id}_{n,n}^X(x) = x$. $\lrcorner$

Proof. Using the description Remark 1.2.3.2.(iv) of $\mathbf{LX}$, we have

$$\begin{aligned} \text{id}_{n,n}^X(x) &= \xi \left(\text{spi}_n \left(\begin{bmatrix} n \end{bmatrix} \right), \begin{bmatrix} x \end{bmatrix} \right) \\ &= \xi \left(\tilde{e}_n, \begin{bmatrix} x \end{bmatrix} \right) \\ &= \xi \circ \eta_X^L(x) \\ &= x. \end{aligned} \quad \square$$

These operations satisfy the following source and target formulas.

Proposition 1.2.3.10 ([FHM25a, Proposition 2.3.6]). Let $X = (X, \xi)$ be a weak ω -category.

- (i) Let m, n be natural numbers with $m < n$ and x an m -cell in X . Let ℓ be another natural number.

- (a) If $\ell < m$, then we have

$$s_\ell(\text{id}_n(x)) = s_\ell(x) \quad \text{and} \quad t_\ell(\text{id}_n(x)) = t_\ell(x).$$

- (b) We have

$$s_m(\text{id}_n(x)) = x = t_m(\text{id}_n(x)).$$

- (c) If $m < \ell < n$, then we have

$$s_\ell(\text{id}_n(x)) = \text{id}_\ell(x) = t_\ell(\text{id}_n(x)).$$

- (ii) Let k, m_1, m_2, n be natural numbers with $k < m_i \leq n$ for each $i = 1, 2$, u an m_1 -cell and v an m_2 -cell in X such that $t_k(u) = s_k(v)$ holds. Let ℓ be another natural number.

- (a) If $\ell < k$, then we have

$$s_\ell(u *_{k,n} v) = s_\ell(u) = s_\ell(v) \quad \text{and} \quad t_\ell(u *_{k,n} v) = t_\ell(u) = t_\ell(v).$$

⁵Note that, in the abbreviated notation, the displayed subscripts serve different purposes: in $*_{k,n}^X$, the number k indicates the lowest dimension present in the input pasting diagram, whereas in id_n^X the number n indicates the dimension of the output of the operation.

(b) We have

$$s_k(u *_{k,n} v) = s_k u \quad \text{and} \quad t_k(u *_{k,n} v) = t_k v.$$

(c) If $k < \ell < \min\{m_1, m_2\}$, then we have

$$s_\ell(u *_{k,n} v) = s_\ell(u) *_{k,\ell} s_\ell(v) \quad \text{and} \quad t_\ell(u *_{k,n} v) = t_\ell(u) *_{k,\ell} t_\ell(v).$$

(d) If $m_1 \leq \ell < m_2$, then we have

$$s_\ell(u *_{k,n} v) = u *_{k,\ell} s_\ell(v) \quad \text{and} \quad t_\ell(u *_{k,n} v) = u *_{k,\ell} t_\ell(v).$$

(e) If $m_2 \leq \ell < m_1$, then we have

$$s_\ell(u *_{k,n} v) = s_\ell(u) *_{k,\ell} v \quad \text{and} \quad t_\ell(u *_{k,n} v) = t_\ell(u) *_{k,\ell} v.$$

(f) If $\max\{m_1, m_2\} \leq \ell < n$, then we have

$$s_\ell(u *_{k,n} v) = u *_{k,\ell} v = t_\ell(u *_{k,n} v).$$

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Proof. These are all straightforward consequences of the description of the source and target operations of LX given in Proposition 1.1.2.20 and the fact that the standard pasting instructions form a globular subset of $L1$. For example, the first equation in Proposition 1.2.3.10.(ii).(d) can be proved as follows, writing the k -cell $t_k(u) = s_k(v)$ as x :

$$\begin{aligned} s_\ell^X(u *_{k,n}^X v) &= s_\ell^X \left(\xi \left(\text{spi} \left(\begin{bmatrix} m_1 & & m_2 \\ & k & \end{bmatrix}^{(n)} \right), \begin{bmatrix} u & & v \\ & x & \end{bmatrix}^{(n)} \right) \right) \\ &= \xi \left(s_\ell^{L1} \left(\text{spi} \left(\begin{bmatrix} m_1 & & m_2 \\ & k & \end{bmatrix}^{(n)} \right) \right), s_\ell^{TX} \left(\begin{bmatrix} u & & v \\ & x & \end{bmatrix}^{(n)} \right) \right) \\ &= \xi \left(\text{spi} \left(\begin{bmatrix} m_1 & & \ell \\ & k & \end{bmatrix}^{(\ell)} \right), \begin{bmatrix} u & & s_\ell^X(v) \\ & x & \end{bmatrix}^{(\ell)} \right) \\ &= u *_{k,\ell}^X s_\ell^X(v). \end{aligned}$$

□

In later sections, whenever we use the notation $\text{id}_n^X(x)$ for an m -cell x of X , we have $m = n - 1$.

Remark 1.2.3.11 (cf. [FHM25a, Remark 2.3.8]). Under the identification remarked immediately after Proposition 1.2.2.9, strict ω -categories are precisely those weak ω -categories X for which, for every $n \geq 0$ and every n -dimensional pasting diagram $\mathbf{u}$ in X , the ‘paste according to ϕ ’ map

$$\mathbf{u} \mapsto \xi(\phi, \mathbf{u}) : \mathbf{GSet}(\text{ar}(\mathbf{u}), X) \longrightarrow X_n$$

is independent of the choice of pasting instruction $\phi \in (L1)_n$. This way, we identify these operations appearing in Definition 1.1.2.1 for strict ω -categories with the standard pasting operations in Definition 1.2.3.8 (although $*_{k,n}^X$ in Definition 1.1.2.1 is defined only when the input cells have the same dimension n).

In particular, for a strict ω -category X , $\text{id}_{n,m}^X$ and $*_{k,n}^X$ defined in Definition 1.2.3.8 satisfy all the axioms listed in Definition 1.1.2.1. 」

Remark 1.2.3.12. For the case of the free strict ω -category TX on a globular set X , we have

$$\begin{aligned} \begin{bmatrix} u \\ \end{bmatrix} *_{k,n}^{TX} \begin{bmatrix} v \\ \end{bmatrix} &= \mu_X^T \left(\begin{bmatrix} \begin{bmatrix} u \\ \end{bmatrix} & \begin{bmatrix} v \\ \end{bmatrix} \\ & x \end{bmatrix} \right) \\ &= \mu_X^T \left(T(\eta_X^T) \left(\begin{bmatrix} u & & v \\ & x & \end{bmatrix}^{(n)} \right) \right) \\ &= \begin{bmatrix} u & & v \\ & x & \end{bmatrix}^{(n)} \end{aligned}$$

in the situation of Definition 1.2.3.8.(ii).⁶

More generally, suppose that we are given natural numbers $k < m_1 \leq n$ and $k < m_2 \leq n$, and pasting diagrams $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{x}$ in X of dimensions m_1 , m_2 , and k respectively. Then by Example 1.1.3.10 and Remark 1.1.3.7, the composition

$$\mathbf{u}^{(m_1)} *_{k,n}^{TX} \mathbf{v}^{(m_2)} = \mu_X^T \left(\begin{bmatrix} \mathbf{u} & & \mathbf{v} \\ & \mathbf{x} & \end{bmatrix} \right)$$

⁶One can also show this equation using Proposition 1.1.3.6 and Remark 1.1.3.7, just by checking that the realisation of the underlying 2-pasting scheme concerned on the first line is precisely the realisation of the shape of the right-most pasting diagram.

is obtained as follows. Let $\mathbf{u}_1, \dots, \mathbf{u}_N$ be the components of the form $[^{u_{i+1}}]$ with $u_{i+1} \in X_k$ and the k -transversal components of $\mathbf{u}$, ordered from left to right. In the same way, let $\mathbf{v}_1, \dots, \mathbf{v}_{N'}$ be those of $\mathbf{n}$. Because $\mathfrak{n}$ forms a 2-pasting scheme, we have $N = N'$ and $\mathfrak{t}_k^{\mathbf{T}1} \mathbf{u}_{i+1} = \mathfrak{s}_k^{\mathbf{T}1} \mathbf{v}_{i+1}$ for each $i < N$. We write $[\bar{x}_{i+1}]$ for this common k -cell $\mathfrak{t}_k^{\mathbf{T}1} \mathbf{u}_{i+1} = \mathfrak{s}_k^{\mathbf{T}1} \mathbf{v}_{i+1}$ appearing as a segment of $\mathbf{x}$. The pasting scheme $\mathbf{u}^{(m_1)} *_{k,n}^{\mathbf{T}X} \mathbf{v}^{(m_2)}$ is obtained from $\mathbf{u}$ by replacing each $\mathbf{u}_{i+1}$ with $\mathbf{u}_{i+1} *_{k,n}^{\mathbf{T}X} \mathbf{v}_{i+1}$, which is explicitly given as the ‘concatenation of $\mathbf{u}_{i+1}$ and $\mathbf{v}_{i+1}$ at $\bar{x}_{i+1}$ ’:

$$\begin{array}{ll} \mathbf{m}_{i+1} & (\text{if } \mathbf{u}_{i+1} = [\bar{x}_{i+1}]) \\ \mathbf{n}_{i+1} & (\text{if } \mathbf{m}_{i+1} = [\bar{x}_{i+1}]) \\ \left[\begin{array}{ccccccccc} u_{i+1,0} & \dots & u_{i+1,r_{i+1}} & v_{i+1,0} & \dots & v_{i+1,r'_{i+1}} \\ & u_{i+1,1} & \dots & u_{i+1,r_{i+1}} & \bar{x}_{i+1} & v_{i+1,1} & \dots & v_{i+1,r'_{i+1}} \end{array} \right] & (\text{otherwise}) \end{array}$$

where r_{i+1} and r'_{i+1} are the ranks of $\mathbf{u}_{i+1}$ and $\mathbf{v}_{i+1}$ respectively.

Similarly, we have $\text{id}_n^{\mathbf{T}X}(\mathbf{u}^{(m)}) = \mathbf{u}^{(n)}$ by using Example 1.1.3.9. $\square$

The pasting instructions for the identity and binary composition operations are themselves given by the identity and binary composition in the free weak ω -category $\mathbf{L1} = (\mathbf{L1}, \mu_1^{\mathbf{L}})$ in the following sense.

Proposition 1.2.3.13. For any pasting instruction $\phi \in (\mathbf{L1})_n$ of arity

$$\mathbf{k} = \begin{bmatrix} k_0 & k_1 & \dots & k_r \\ \underline{k}_1 & \underline{k}_2 & \dots & \underline{k}_r \end{bmatrix}$$

we have $\phi = \mu_1^{\mathbf{L}}(\phi, \tilde{e}_{\mathbf{k}})$ where

$$\tilde{e}_{\mathbf{k}} = \begin{bmatrix} \tilde{e}_{k_0} & \tilde{e}_{k_1} & \dots & \tilde{e}_{k_r} \\ & \tilde{e}_{\underline{k}_1} & \dots & \tilde{e}_{\underline{k}_r} \end{bmatrix}$$

Consequently, given a weak ω -category $X = (X, \xi)$,

- (i) we have $\text{id}_n^X(x) = \xi(\text{id}_n^{\mathbf{L1}}(\tilde{e}_m), [^x]^{(n)})$ for all natural numbers $m < n$ and m -cells x in X , and
- (ii) we have

$$u *_{k,n}^X v = \xi \left(\tilde{e}_{m_1} *_{k,n}^{\mathbf{L1}} \tilde{e}_{m_2}, \begin{bmatrix} u & v \\ x & \end{bmatrix}^{(n)} \right)$$

for all natural numbers k, m_1, m_2, n with $k < m_i \leq n$ for each $i = 1, 2$, and all m_1 -cells u and m_2 -cells v in X such that $\mathfrak{t}_k^X(u) = \mathfrak{s}_k^X(v)$ holds. $\square$

Proof. Let us first make the following observations.

- (1) If we denote the unique n -cell $*_n$ of the terminal globular set $\mathbf{1}$ simply by ‘ n ’, then we may regard each pasting scheme $\mathbf{k}$ as a pasting diagram $\mathbf{u}$ in the globular set $\mathbf{1}$, and this is unambiguous in the sense that both $\mathbf{k}$ and $\mathbf{u}$ specify the same n -cell in $\mathbf{T1}$.
- (2) When we apply the description of the cells in $\mathbf{LX}$ as pairs $(\phi, \mathbf{u})$ given in Remark 1.2.3.2.(i) to the case $X = \mathbf{1}$, we end up identifying each pasting instruction ϕ of arity $\mathbf{k}$ with the pair $(\phi, \mathbf{k})$.

It follows that, for any pasting instruction ϕ of arity $\mathbf{k}$ and of dimension n as in the statement, we have

$$\begin{aligned} \phi &= (\mu_1^{\mathbf{L}} \circ \mathbf{L}\eta_1^{\mathbf{L}})(\phi) & (\mu^{\mathbf{L}} \circ \mathbf{L}\eta^{\mathbf{L}} = \mathbf{1}_{\mathbf{L}}) \\ &= (\mu_1^{\mathbf{L}} \circ \mathbf{L}\eta_1^{\mathbf{L}})(\phi, \mathbf{k}) & (\text{ii}) \\ &= \mu_1^{\mathbf{L}}(\phi, (\mathbf{T}\eta_1^{\mathbf{L}})(\mathbf{k})). & (\text{action of } \mathbf{L}f \text{ in the case } f = \eta_1^{\mathbf{L}}) \end{aligned}$$

Using (1) and Remark 1.1.2.19 and Notation 1.2.3.1, we can compute $(\mathbf{T}\eta_1^{\mathbf{L}})(\mathbf{k})$ as

$$(\mathbf{T}\eta_1^{\mathbf{L}}) \left(\begin{bmatrix} k_0 & k_1 & \dots & k_r \\ \underline{k}_1 & \underline{k}_2 & \dots & \underline{k}_r \end{bmatrix} \right) = \begin{bmatrix} \eta_1^{\mathbf{L}}(k_0) & \eta_1^{\mathbf{L}}(k_1) & \dots & \eta_1^{\mathbf{L}}(k_r) \\ \eta_1^{\mathbf{L}}(\underline{k}_1) & \eta_1^{\mathbf{L}}(\underline{k}_2) & \dots & \eta_1^{\mathbf{L}}(\underline{k}_r) \end{bmatrix} = \tilde{e}_{\mathbf{k}}$$

This proves the first assertion. Moreover, the case $\phi = \text{spi}_n([^m]^{(n)})$ yields

$$\text{spi}_n \left(\begin{bmatrix} m \end{bmatrix} \right) = \mu_1^{\mathbf{L}} \left(\text{spi}_n \left(\begin{bmatrix} m \end{bmatrix} \right), \begin{bmatrix} \tilde{e}_m \end{bmatrix} \right) = \text{id}_n^{\mathbf{L1}}(\tilde{e}_m)$$

where the second equality is by the definition Definition 1.2.3.8.(i) of $\text{id}_n^{\mathbf{L1}}(\tilde{e}_m)$. This completes the proof of the assertion 1.2.3.13.(i), and similarly 1.2.3.13.(ii) follows from the case $\phi = \text{spi}_n \left(\begin{bmatrix} m_1 & m_2 \\ k \end{bmatrix}^{(n)} \right)$. $\square$

Remark 1.2.3.14 ([FHM25a, Remark 2.3.8]). Let X be a weak ω -category. Using identities and compositions in the free strict ω -category $\mathbf{TX}$ observed in Remark 1.2.3.12, Proposition 1.2.3.13.(i) is expressed as $\text{id}_n^X(x) = \xi(\text{id}_n^{L1}(\tilde{e}_m), \text{id}_n^{TX}([x]))$. Here is an alternative proof of this:

$$\begin{aligned} \xi\left(\text{id}_n^{L1}(\tilde{e}_m), \text{id}_n^{TX}\left([x]\right)\right) &= \xi\left(\text{id}_n^{LX}\left(\tilde{e}_m, [x]\right)\right) && (\mathbf{LX} \text{ is a pullback of } \mathbf{L1} \text{ and } \mathbf{TX} \text{ in } \mathbf{Wk-}\omega\text{-Cat}_s) \\ &= \text{id}_n^X\left(\xi\left(\tilde{e}_m, [x]\right)\right) && (\xi: (\mathbf{LX}, \mu_X^L) \rightarrow (X, \xi) \text{ is a strict } \omega\text{-functor}) \\ &= \text{id}_n^X(x) && (\xi \circ \eta_X^L = \text{id}_X) \end{aligned}$$

Similarly, in the situation of Proposition 1.2.3.13.(ii), Remark 1.2.3.12 allows us to express the main claim there as $u *_{k,n}^X v = \xi(\tilde{e}_{m_1} *_{k,n}^{L1} \tilde{e}_{m_2}, [u] *_{k,n}^{TX} [v])$. This can be proved similarly:

$$\begin{aligned} \xi\left(\tilde{e}_{m_1} *_{k,n}^{L1} \tilde{e}_{m_2}, [u] *_{k,n}^{TX} [v]\right) &= \xi\left(\left(\tilde{e}_{m_1}, [u]\right) *_{k,n}^{LX} \left(\tilde{e}_{m_2}, [v]\right)\right) \\ &= \xi\left(\tilde{e}_{m_1}, [u]\right) *_{k,n}^X \xi\left(\tilde{e}_{m_2}, [v]\right) \\ &= u *_{k,n}^X v. \end{aligned}$$

1.2.4. The hom weak ω -category. Given a weak ω -category X and two 0-cells a, b in X , we have the *hom weak ω -category* $X(a, b)$ defined in [Lei04, Section 9.3] and [CFP18], whose underlying globular set is that given in Definition 1.1.1.8. The latter is captured by the forgetful functor

$$U_s: \mathbf{Wk-}\omega\text{-Cat}_s \rightarrow (\mathbf{Wk-}\omega\text{-Cat}_s)\text{-Gph}$$

mapping each weak ω -category X to the $(\mathbf{Wk-}\omega\text{-Cat}_s)$ -enriched graph $[Wol74] U_s X$ consisting of the same objects as X together with the hom weak ω -categories of X . This subsection is devoted to recalling the definition of this functor U_s .

To this end, first observe that the successor $\mathbf{S}: \mathbf{PS} \rightarrow \mathbf{PS}$ (Definition 1.1.2.5) restricts to a function $(\mathbf{S}_1^T)_n: (\mathbf{T1})_n \rightarrow (\mathbf{T1})_{n+1}$ for each $n \geq 0$, which assembles into a globular map $\mathbf{S}_1^T: \mathbf{T1} \rightarrow \mathbf{T1}_{-+1}$. Moreover, given a globular set X and two 0-cells a, b in X , a pasting diagram in $X(a, b)$ of dimension n is identified with a pasting diagram in X of dimension $n+1$ whose source and target 0-cells are a and b , respectively. This identification means there is an inclusion $(\mathbf{T}(X(a, b)))_n \hookrightarrow ((\mathbf{TX})(a, b))_n$ that sends a pasting diagram $\hat{\mathbf{k}} \xrightarrow{\mathbf{u}} X(a, b)$ to another pasting diagram $\widehat{\mathbf{S}(\mathbf{k})} \rightarrow X$ of shape $\mathbf{Sk}$, which we denote by $\mathbf{S}_{X,a,b}^T(\mathbf{u})$.

In [CFP18, Section 3], it is shown that there is a canonical lift

$$\begin{array}{ccc} \mathbf{L1} & \xrightarrow{\mathbf{S}_1^L} & (\mathbf{L1})_{-+1} \\ \text{ar} \downarrow & & \downarrow \text{ar}_{-+1} \\ \mathbf{T1} & \xrightarrow{\mathbf{S}_1^T} & (\mathbf{T1})_{-+1} \end{array}$$

of $\mathbf{S}_1^T: \mathbf{T1} \rightarrow (\mathbf{T1})_{-+1}$, compatible with the monad structure μ_1^L and η_1^L of $\mathbf{L}$ at 1 in a suitable sense. Using these data, we can define the hom weak ω -category as follows.

Fact 1.2.4.1 ([CFP18, Section 3]). For each weak ω -category $X = (X, \xi)$ and each pair of 0-cells a, b in X , the globular map $\xi_{a,b}: \mathbf{L}(X(a, b)) \rightarrow X(a, b)$ defined by

$$\xi_{a,b}(\phi, \mathbf{u}) = \xi(\mathbf{S}_1^L(\phi), \mathbf{S}_{X,a,b}^T(\mathbf{u}))$$

makes $X(a, b) = (X(a, b), \xi_{a,b})$ a weak ω -category. ┐

Proposition 1.2.4.2. $\mathbf{S}_1^L$ preserves standard pasting instructions. ┐

Proof. Among the compatibility conditions of $\mathbf{S}_1^L$ with the structure of $\mathbf{L}$ and $\mathbf{S}_1^T$ are the properties that we have $\mathbf{S}_1^L(\tilde{e}_n) = \tilde{e}_{n+1}$ for all $n \geq 0$, and that we have

$$\mathbf{S}_1^L(\bar{\kappa}(\langle \phi, \phi' \rangle, \mathbf{k})) = \bar{\kappa}(\langle \mathbf{S}_1^L(\phi), \mathbf{S}_1^L(\phi') \rangle, \mathbf{S}_1^T(\mathbf{k}))$$

for all $n \geq 1$, all parallel $\phi, \phi' \in (\mathbf{L1})_{n-1}$, and all $\mathbf{k} \in (\mathbf{T1})_n$ with $\text{ar}(\phi) = \text{ar}(\phi') = \partial_{n-1}^{\mathbf{T1}}(\mathbf{k})$. The claim follows from a straightforward induction on n . □

It follows that the standard pasting operation of arity $\mathbf{k} \in (\mathbf{T1})_n$ on a hom weak ω -category $X(x, y)$ is given by (a restriction of) the standard pasting operation of arity $\mathbf{S}_1^T(\mathbf{k}) \in (\mathbf{T1})_{n+1}$ on X . In particular, we shall use the following.

Corollary 1.2.4.3. Let X be a weak ω -category and $x, y \in X_0$.

- (i) Given a natural number $n \geq 1$ and an $(n-1)$ -cell z of $X(x, y)$, we have

$$\mathrm{id}_n^{X(x, y)}(z) = \mathrm{id}_{n+1}^X(z).$$

- (ii) Given natural numbers k, m_1, m_2, n with $k < m_1 \leq n$ and $k < m_2 \leq n$, an m_1 -cell u of $X(x, y)$, and an m_2 -cell v of $X(x, y)$ such that $t_k^{X(x, y)}(u) = s_k^{X(x, y)}(v)$, we have

$$u *_{k, n}^{X(x, y)} v = u *_{k+1, n+1}^X v. \quad \lrcorner$$

1.3. Algebraic injectivity

The algebraic weak factorisation system (AWFS) introduced in [GT06] is an ‘algebraic’ refinement of the notion of weak factorisation system. A weak factorisation system is defined by imposing a property expressed using an existential quantifier on a pair of classes of morphisms, whereas an AWFS recovers, as algebraic structure, the data that are forgotten when one passes to such an existential formulation. In this thesis we use AWFS, and in particular the theory of the small object argument for AWFS [Gar09b; BG16a], in order to introduce several of our central notions.

For example, this point of view already appears in the definition of $\mathbf{L}$ (see Example 1.3.22, whose underlying idea goes back to [Gar09a]). We also adopt, as our definition of weak ω -functor between weak ω -categories, the one introduced in [Gar10], which is based on the notion of cofibrant replacement comonad associated with an AWFS. Furthermore, in Chapter 4 we shall make use of techniques from AWFS in order to give a presentation of the coherent walking equivalence. However, these definitions depend only on a small portion of the data of an AWFS, and in this subsection we therefore refrain from introducing the full structure of an AWFS. Instead, we introduce a smaller structure, which we call *algebraic injectivity*.⁷ This is an ‘algebraic’ refinement, in the same sense as above, of the notions of injective morphism and injective object. Most of the definitions and results in this section are taken from [FHM25a, Section 4].

For each category $\mathbf{C}$, we write $\mathbf{SET} \downarrow \mathbf{C}^2$ for the comma category:

$$\begin{array}{ccc} \mathbf{SET} \downarrow \mathbf{C}^2 & \longrightarrow & \mathbf{CAT}/\mathbf{C}^2 \\ \downarrow & \lrcorner & \downarrow \text{domain} \\ \mathbf{SET} & \hookrightarrow & \mathbf{CAT} \end{array}$$

whose objects are called *indexed families of morphisms in $\mathbf{C}$* . Suppose we are given a category $\mathbf{C}$, and let $\mathcal{K} \in \mathbf{SET} \downarrow \mathbf{C}^2$ be an indexed family of morphisms in $\mathbf{C}$. Abusing the notation, we write $\mathcal{K} = (\mathcal{K}, | - |)$ where $\mathcal{K}$ is a class and $| - |: \mathcal{K} \rightarrow \mathbf{C}^2$ is a functor from the discrete category.

Remark 1.3.1. In almost all cases of interest to us, $\mathcal{K}$ is just a class of morphisms. Given such a class $\mathcal{K}$ in $\mathbf{C}$, we may regard it as an indexed family of morphisms by taking $| - |: \mathcal{K} \rightarrow \mathbf{C}^2$ to be the inclusion functor. However, it is sometimes convenient to keep the functor $| - |$ explicit; moreover, this slightly more general notion fits better with the general theory of the algebraic small object argument developed in [Gar09b; BG16a]. $\lrcorner$

We say $\mathcal{K}$ is *small* if the indexing class $\mathcal{K}$ is a (small) set.

Definition 1.3.2. We say that a morphism (or an object) in $\mathbf{C}^2$ is $\mathcal{K}$ -*injective* if it has the right lifting property with respect to morphisms in the image of the functor $| - |: \mathcal{K} \rightarrow \mathbf{C}^2$. If $\mathbf{C}$ has a terminal object $1 \in \mathbf{C}$, then an object A is $\mathcal{K}$ -injective if and only if the unique morphism $!_A: A \rightarrow 1$ is a $\mathcal{K}$ -injective morphism. We write $\mathbf{K}\text{-}\mathbf{Inj}$ for the full subcategory of $\mathbf{C}$ spanned by $\mathcal{K}$ -injective objects.

We say a morphism $f: C \rightarrow D$ in $\mathbf{C}$ is $\mathcal{K}$ -*cofibration* if it has the left lifting property with respect to all $\mathcal{K}$ -injective morphisms, and write $\mathbf{K}\text{-}\mathbf{cof}$ for the class of all $\mathcal{K}$ -cofibrations in $\mathbf{C}$. $\lrcorner$

⁷In [FHM25a], this notion is called *algebraic trivial fibration*. This is because in that paper, the main focus is to capture an algebraic refinement of the notion of trivial fibration Example 1.3.3. However, since the same notion can be used to refine several notions which are no longer considered trivial fibrations, we change the name to *algebraically injective morphism* in this thesis, so that it also suits the terminology in [FHM25b, Section 5].

Example 1.3.3 ([FHM25a, Definition 4.1.8]). Let $f: X \rightarrow Y$ be a morphism in **GSet**. We say that f is a *trivial fibration* if it is $\mathcal{I}_o$ -injective, where we set $\mathcal{I}_o = (\mathcal{I}_o, | - |)$ as

$$\mathcal{I}_o = \{\partial G^n \xrightarrow{\iota^n} G^n \mid n \in \mathbb{N}\}$$

and $| - |$ is the inclusion.

A strict ω -functor $f: X \rightarrow Y$ between weak ω -categories is a *trivial fibration* if it is a trivial fibration as a morphism in **GSet**. Using the universality of the left adjoint of the adjunction (see Notation 1.2.2.11):

$$F_L: \mathbf{GSet} \xleftarrow{\quad} \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s: U_L, \quad (1.3.3.1)$$

we can also see a trivial fibration between weak ω -categories as a $\mathcal{I}$ -injective morphism in **Wk- ω -Cat_s**, where we set $\mathcal{I} = (\mathcal{I}, | - |)$ as

$$\mathcal{I} = \{\partial C^n \xrightarrow{\iota^n} C^n \mid n \in \mathbb{N}\}$$

and $| - |$ is the inclusion. $\lrcorner$

Example 1.3.4 (Definition 2.3.2). Removing the 0-dimensional boundary inclusion from Example 1.3.3 and setting

$$\mathcal{I}_o^\times = \{\partial G^n \xrightarrow{\iota^n} G^n \mid n \geq 1\},$$

we define a *contractible morphism* in **GSet** as an $\mathcal{I}_o^\times$ -injective morphism, and a *contractible strict ω -functor* between weak ω -categories as an $\mathcal{I}^\times$ -injective morphism in **Wk- ω -Cat_s** where we set $\mathcal{I}^\times = \{\partial C^n \xrightarrow{\iota^n} C^n \mid n \geq 1\}$. The latter is defined and used in Section 2.3. $\lrcorner$

Remark 1.3.5. Let $f: C \rightarrow D$ be a morphism in **C**, and consider the following composite of morphisms in $[\mathcal{K}, \mathbf{Set}]$:

$$\mathbf{C}(\mathrm{cod}(| - |), C) \cong \mathbf{C}^2(\mathrm{Id}_{\mathrm{cod}(| - |)}, f) \rightarrow \mathbf{C}^2(| - |, f) \quad (1.3.5.1)$$

Here, the first isomorphism is the adjunction isomorphism for $\mathrm{Id} \dashv \mathrm{dom}$, and the second morphism is precomposition of units of the adjunction $\mathrm{cod} \dashv \mathrm{Id}$. For each $j \in \mathcal{K}$, this composite sends $u: \mathrm{cod} |j| \rightarrow C$ to the pair $(|j| \circ u, u \circ f)$. Then, f is a $\mathcal{K}$ -injective morphism if and only if the above morphism in $[\mathcal{K}, \mathbf{Set}]$ is a surjection.

Similarly, if we are given an object A in **C**, we have the following morphism in $[\mathcal{K}, \mathbf{Set}]$:

$$\mathbf{C}(\mathrm{cod} | - |, A) \rightarrow \mathbf{C}(\mathrm{dom} | - |, A) \quad (1.3.5.2)$$

which sends $u: \mathrm{cod} |j| \rightarrow A$ to $|j| \circ u: \mathrm{dom} |j| \rightarrow A$ for each $j \in \mathcal{K}$. Then, A is a $\mathcal{K}$ -injective object if and only if the above morphism in $[\mathcal{K}, \mathbf{Set}]$ is a surjection. $\lrcorner$

Definition 1.3.6 (cf. [Gar09b, Proposition 3.8], [BG16a, Proposition 15]). Suppose we are given a morphism $f: C \rightarrow D$ in **C**. A (*right*) $\mathcal{K}$ -lifting structure κ on f is a section in $[\mathcal{K}, \mathbf{Set}]$ of the morphism (1.3.5.1). More explicitly, a $\mathcal{K}$ -lifting structure κ on a morphism f is a function which assigns to each commutative square with $j \in \mathcal{K}$ as in the solid part below, a morphism $\kappa(j; x, y)$ making both triangles commutative:

$$\begin{array}{ccc} \cdot & \xrightarrow{x} & C \\ |j| \downarrow & \kappa(j; x, y) \nearrow & \downarrow f \\ \cdot & \xrightarrow{y} & D. \end{array}$$

In other words, $\kappa(j; x, y)$ is the solution for the lifting problem given by the solid part of the above diagram.

Given two morphisms f, g in **C** and $\mathcal{K}$ -lifting structure κ^f, κ^g on them, we say a morphism $(u, v): f \rightarrow g$ in **C²** preserves the $\mathcal{K}$ -lifting structure if we have

$$\kappa^g(j; x \circ u, y \circ v) = \kappa^f(j; x, y) \circ u$$

for all $j \in \mathcal{K}$, $(x, y): |j| \rightarrow f$, and $(u, v): f \rightarrow g$:

$$\begin{array}{ccccc} \cdot & \xrightarrow{x} & \cdot & \xrightarrow{u} & \cdot \\ |j| \downarrow & \kappa^f(j; x, y) \nearrow & \downarrow f & \kappa^g(j; x \circ u, y \circ v) \nearrow & \downarrow g \\ \cdot & \xrightarrow{y} & \cdot & \xrightarrow{v} & \cdot \end{array}$$

An *algebraically $\mathcal{K}$ -injective morphism* is a morphism f equipped with a $\mathcal{K}$ -lifting structure κ . We write **K-AIMor** for the category of algebraically $\mathcal{K}$ -injective morphisms in **C** and morphisms preserving $\mathcal{K}$ -lifting structures. We often write $\mathbf{f}, \mathbf{g}, \mathbf{h}: C \rightarrow D$ for algebraically $\mathcal{K}$ -injective morphisms whose underlying morphisms

are $f, g, h: C \rightarrow D$ respectively. The category $\mathcal{K}\text{-}\mathbf{AIMor}$ is precisely the same as $\mathcal{J}^\flat$ defined in [BG16a, Proposition 15] where $\mathcal{J} = \mathcal{K}$.

For each $X \in \mathbf{C}$, we write $\mathcal{K}\text{-}\mathbf{AIMor}_X$ for the fibre over X of the composite

$$\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}^2 \xrightarrow{\text{cod}} \mathbf{C} \quad (1.3.6.1)$$

where the first functor is the forgetful and the second functor is the codomain functor. In other words, $\mathcal{K}\text{-}\mathbf{AIMor}_X$ is the pullback of $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ in $\mathbf{CAT}$ along the functor $\mathbf{1} \rightarrow \mathbf{C}$ picking out X . $\lrcorner$

Definition 1.3.7. Suppose we are given an object A in $\mathbf{C}$. A $\mathcal{K}$ -lifting structure κ on A is a section in $[\mathcal{K}, \mathbf{Set}]$ of the morphism (1.3.5.2). Equivalently, a $\mathcal{K}$ -lifting structure κ on an object A is a function which assigns to each morphism $x: \text{dom } |j| \rightarrow A$ with $j \in \mathcal{K}$, its lift $\kappa(j; x): \text{cod } |j| \rightarrow A$ satisfying $|j| \circ \kappa(j; x) = x$.

An *algebraically $\mathcal{K}$ -injective object* is an object C equipped with a $\mathcal{K}$ -lifting structure κ . We write $\mathcal{K}\text{-}\mathbf{AInj}$ for the category of algebraically $\mathcal{K}$ -injective objects in $\mathbf{C}$ and morphisms preserving $\mathcal{K}$ -lifting structures: i.e. those morphisms $u: C \rightarrow D$ in $\mathbf{C}$ satisfying

$$\kappa^D(j; x \circ u) = \kappa^C(j; x) \circ u,$$

where κ^C and κ^D are the $\mathcal{K}$ -lifting structures on C and D , respectively. $\lrcorner$

Remark 1.3.8. Consider the composite

$$|-|': \mathcal{K} \xrightarrow{|-|} \mathbf{C}^2 \rightarrow (\mathbf{C}^2)^2$$

where the second functor is the one induced by the join $\mathbf{2} \times \mathbf{2} \rightarrow \mathbf{2}$ of the lattice $\mathbf{2}$. Let $\mathcal{K}'$ be the pair $(\mathcal{K}, |-|') \in \mathbf{SET} \downarrow (\mathbf{C}^2)^2$. Then the category $\mathcal{K}\text{-}\mathbf{AIMor}$ is isomorphic to the category $\mathcal{K}'\text{-}\mathbf{AInj}$.

For the converse, suppose the category $\mathbf{C}$ has a terminal object $1 \in \mathbf{C}$. Then the category $\mathcal{K}\text{-}\mathbf{AInj}$ is isomorphic to the fibre $\mathcal{K}\text{-}\mathbf{AIMor}_1$ at 1 of the composite cod in (1.3.6.1). $\lrcorner$

Remark 1.3.9 (cf. [BG16a, 4.5]. See also the proof of [BG16a, Proposition 16]). The forgetful functor $\mathcal{K}\text{-}\mathbf{AInj} \rightarrow \mathbf{C}$ fits into the following pullback square:

$$\begin{array}{ccc} \mathcal{K}\text{-}\mathbf{AInj} & \longrightarrow & \mathbf{SpEpi}([\mathcal{K}, \mathbf{Set}]) \\ \downarrow & \lrcorner & \downarrow \text{forget} \\ \mathbf{C} & \longrightarrow & [\mathcal{K}, \mathbf{Set}]^2 \end{array} \quad (1.3.9.1)$$

Here the bottom horizontal functor is the nerve of the uncurrying $\mathcal{K} \times \mathbf{2} \rightarrow \mathbf{C}$ of $|-|$. The category $\mathbf{SpEpi}(\mathbf{D})$ ⁸ for a category $\mathbf{D}$ is the category whose objects are morphisms in $\mathbf{D}$, each equipped with a chosen section, and whose morphisms are morphisms in $\mathbf{D}^2$ that preserve the chosen sections. $\lrcorner$

Remark 1.3.10. If $\mathbf{C}$ is locally presentable and $\mathcal{K}$ is small, we can apply the algebraic small object argument (see [Gar09b, Theorem 4.4] and [BG16a, Proposition 16]) to $\mathcal{K}$, and obtain the *algebraic weak factorisation system* (L, R) on $\mathbf{C}$ *cofibrantly generated* by $\mathcal{K}$ (in the sense of [BG16a, 3.2]), which is a pair of a comonad L and a monad R on $\mathbf{C}^2$. More precisely, the forgetful functor $\mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}^2$ is strictly monadic and the induced monad on $\mathbf{C}^2$ is isomorphic to R . An algebraically $\mathcal{K}$ -injective object is precisely an *algebraically fibrant object* in the sense of [BG16a, 2.3] with respect to this AWFS.

We now give a direct proof for the strict monadicity of the forgetful functor $\mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}^2$ without relying on the algebraic small object argument, which is slightly different from the proof of [BG16a, Proposition 16]. First observe that the right vertical forgetful functor in (1.3.9.1) is strictly monadic, whose monad is given by the endofunctor on $[\mathcal{K}, \mathbf{Set}]^2$ sending each morphism $f: C \rightarrow D$ to the morphism $\langle \text{id}_D, f \rangle: C + D \rightarrow D$, and the unit and multiplication are defined by (left) coprojections and codiagonals for coproducts respectively (see [BG16a, 4.1]). Then by the (strict version of) Beck's theorem and the pullback square (1.3.9.1) for the case where $\mathcal{K} = \mathcal{K}'$ as in Remark 1.3.8, we conclude that the forgetful functor $U: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}^2$ strictly creates U -split coequalisers. Now it remains to show that U has a left adjoint, which follows from [Bir84, Theorem 2.18] by observing that the bottom horizontal nerve functor in (1.3.9.1) is right adjoint and the right vertical forgetful functor in (1.3.9.1) is an isofibration. $\lrcorner$

⁸If $\mathbf{D}$ has pullbacks, the codomain functor $\mathbf{SpEpi}(\mathbf{D}) \rightarrow \mathbf{D}$ is a Grothendieck fibration, which is called the *fibration of points* in [nLa25] or the *fibration of pointed objects* in [Bou91, Section 3].

Definition 1.3.11 ([BG16a, (2.10)]). Suppose that we have a composable pair of algebraically $\mathcal{K}$ -injective morphisms

$$C \xrightarrow{(g, \kappa^g)} D \xrightarrow{(h, \kappa^h)} E.$$

Then we define the *composite* $(g, \kappa^g) \circ (h, \kappa^h)$ to be the algebraically $\mathcal{K}$ -injective morphism $(g \circ h, \kappa^{g \circ h})$ where the $\mathcal{K}$ -lifting structure $\kappa^{g \circ h}$ is defined by

$$\kappa^{g \circ h}(j; x, y) = \kappa^g(j; x, \kappa^h(j; x \circ g, y))$$

for each $j \in \mathcal{K}$ and each $(x, y): |j| \rightarrow g \circ h$ in $\mathbf{C}^2$. Diagrammatically, the right-hand side is the canonical morphism fitting into the following commutative diagram:

$$\begin{array}{ccc} \cdot & \xrightarrow{x} & C \\ \downarrow |j| & \nearrow \kappa^g(j; x, \kappa^h(j; x \circ g, y)) & \downarrow g \\ & \nearrow \kappa^h(j; x \circ g, y) & D \\ \cdot & \xrightarrow{y} & E \\ & & \downarrow h \end{array}$$

Definition 1.3.12 ([BG16a, 5.2]). The composite (Definition 1.3.11) defines a (strict) double category $\mathcal{K}\text{-AIMor}$ as follows.

- Objects are objects in $\mathbf{C}$, and the horizontal category is $\mathbf{C}$.
- A vertical morphism is an algebraically $\mathcal{K}$ -injective morphism in $\mathbf{C}$.
- A configuration of the following form

$$\begin{array}{ccc} C & \xrightarrow{k} & C' \\ \mathbf{g} \downarrow & & \downarrow \mathbf{g}' \\ D & \xrightarrow{l} & D' \end{array}$$

is filled by a unique cell if and only if the pair $(k, l): \mathbf{g} \rightarrow \mathbf{g}'$ is a morphism in $\mathcal{K}\text{-AIMor}$: i.e., the square commutes in $\mathbf{C}$ and the pair (k, l) preserves the $\mathcal{K}$ -lifting structure as a morphism in $\mathbf{C}^2$ between algebraically $\mathcal{K}$ -injective morphisms.

- Vertical composition is defined by the composite of algebraically $\mathcal{K}$ -injective morphisms, and vertical identity (Id_X, ω^X) on X is the identity morphism equipped with the unique $\mathcal{K}$ -lifting structure ω^X defined by $\omega^X(j; u, v) = v$.

This double category $\mathcal{K}\text{-AIMor}$ is precisely the same as $\mathcal{J}^\sharp$ defined in [BG16a, 5.2] where $\mathcal{J} = \mathcal{K}$.

Remark 1.3.13 (cf. [BG16a, 3.4]). If $\mathbf{C}$ has pullbacks along $\mathcal{K}$ -injective morphisms, $\text{cod}: \mathbf{C}^2 \rightarrow \mathbf{C}$ is a Grothendieck fibration, then it is straightforward to check that the double category $\mathcal{K}\text{-AIMor}$ is *right-connected*, and hence the forgetful functor $\mathcal{K}\text{-AIMor} \rightarrow \mathbf{C}^2$ is a *discrete pullback-fibration* [BG16a, 3.4]. Consequently, the codomain functor cod observed in (1.3.6.1) is a Grothendieck fibration. Moreover, the forgetful functor $\mathcal{K}\text{-AIMor} \rightarrow \mathbf{C}^2$ sends cartesian morphisms to pullback squares in $\mathbf{C}$.

Note that by the pullback square (1.3.9.1), the forgetful functor $\mathcal{K}\text{-AInj} \rightarrow \mathbf{C}$ satisfies (strict) Beck's monadicity condition, since $\mathbf{SpEpi}(\mathbf{D}) \rightarrow \mathbf{D}$ does so for any category $\mathbf{D}$.

Definition 1.3.14 ([FHM25b, Section 5]). Suppose that the forgetful functor $\mathcal{K}\text{-AInj} \rightarrow \mathbf{C}$ has a left adjoint. Then the induced monad on $\mathbf{C}$ is called the *algebraically $\mathcal{K}$ -injective replacement monad*, and given an object C in $\mathbf{C}$, its image under the left adjoint is called the *algebraically $\mathcal{K}$ -injective replacement* of C .

Remark 1.3.10 implies that the situation there is sufficient for the existence of the algebraically $\mathcal{K}$ -injective replacement.

Proposition 1.3.15 (cf. remarks before [FHM25b, Definition 5.0.2]). If $\mathbf{C}$ is locally presentable and $\mathcal{K}$ is small, the algebraically $\mathcal{K}$ -injective replacement exists.

Remark 1.3.16 ([FHM25b, Lemma 5.0.3]). In the situation of Proposition 1.3.15, the induced monad on $\mathbf{C}$ is precisely the restriction of the monad $\mathbf{R}$ in Remark 1.3.10 on the slice category $\mathbf{C}/1 \cong \mathbf{C}$, where 1 is the terminal object in $\mathbf{C}$. Such a restriction is possible since $\mathbf{R}$ does not change the codomain. This monad is precisely the *fibrant replacement monad* with respect to the AWFS $(\mathbf{L}, \mathbf{R})$ in the sense of [BG16b, 3.1]. By the definition of AWFS, the unit at $C \in \mathbf{C}$ of the induced monad is the image of $!_C: C \rightarrow 1$ under $\mathbf{L}$; see e.g. [BG16a, 2.1]. In

particular, it lies in the left class of the *underlying weak factorisation system* (in the sense of [BG16a, 2.6]) of the AWFS $(\mathbf{L}, \mathbf{R})$, which in this case coincides with the class $\mathcal{K}\text{-}\mathbf{cof}$ of $\mathcal{K}$ -cofibrations. $\lrcorner$

As explained in the preceding remark, in a suitably good situation, the algebraically $\mathcal{K}$ -injective replacement monad is precisely the fibrant replacement monad associated with the cofibrantly generated AWFS $(\mathbf{L}, \mathbf{R})$ on $\mathbf{C}$ by $\mathcal{K}$. Thus the notion of algebraically $\mathcal{K}$ -injective replacement captures the same structure without requiring the full data of an AWFS. Dually, the *cofibrant replacement comonad* ([BG16b, 3.1]) associated with $(\mathbf{L}, \mathbf{R})$ can likewise be characterised by a construction that does not assume the whole AWFS structure, as follows.

Construction 1.3.17 ([FHM25a, Definition 4.1.1.(3)]). Suppose that the functor $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ as in (1.3.6.1) has a *left adjoint right inverse (lari)*, i.e., a left adjoint with identity unit. Explicitly, this means the following: for any object $X \in \mathbf{C}$, there exists an algebraically $\mathcal{K}$ -injective morphism $(\varepsilon_X: QX \rightarrow X, \kappa_X)$ over X such that, for any algebraically $\mathcal{K}$ -injective morphism $(g: Z \rightarrow W, \kappa^g)$ and any morphism $k: X \rightarrow W$ in $\mathbf{C}$, there exists a unique morphism $h: QX \rightarrow Z$ in $\mathbf{C}$ such that (h, k) is a morphism of algebraically $\mathcal{K}$ -injective morphisms $(\varepsilon_X, \kappa_X) \rightarrow (g, \kappa^g)$.⁹ Under this situation, we can make the following definitions.

- For each $X \in \mathbf{C}$, let $\delta_X: QX \rightarrow QQX$ be the unique morphism in $\mathbf{C}$ such that the following configuration

$$\begin{array}{ccc} QX & \xrightarrow{\delta_X} & QQX \\ \downarrow (\varepsilon_X, \kappa_X) & & \downarrow (\varepsilon_{QX}, \kappa_{QX}) \\ & & QX \\ & & \downarrow (\varepsilon_X, \kappa_X) \\ X & \xlongequal{\quad} & X \end{array} \quad (\text{D})$$

has a filler in $\mathcal{K}\text{-}\mathbf{AIMor}$. In other words, the pair (δ_X, id_X) is a morphism of algebraically $\mathcal{K}$ -injective morphisms of above type.

- For each morphism $f: X \rightarrow Y$ in $\mathbf{C}$, let $Qf: QX \rightarrow QY$ be the unique morphism in $\mathbf{C}$ such that the following configuration

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ (\varepsilon_X, \kappa_X) \downarrow & & \downarrow (\varepsilon_Y, \kappa_Y) \\ X & \xrightarrow{f} & Y \end{array} \quad (\text{Q})$$

has a filler in $\mathcal{K}\text{-}\mathbf{AIMor}$. In other words, the pair (Qf, f) is a morphism of algebraically $\mathcal{K}$ -injective morphisms of above type. $\lrcorner$

Remark 1.3.18. Notice that the vertical identity $(\text{id}_X: X \rightarrow X, \omega^X)$ is the terminal object in the category $\mathcal{K}\text{-}\mathbf{AIMor}_X$. For any $(f, \kappa) \in \mathcal{K}\text{-}\mathbf{AIMor}_X$, the unique map is given as the square

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & X \\ (f, \kappa) \downarrow & & \parallel (\text{id}_X, \omega^X) \\ X & \xlongequal{\quad} & X. \end{array} \quad (\text{I})$$

Remark 1.3.19. If $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ has a lari, then the value $(\varepsilon_X: QX \rightarrow X, \kappa_X)$ of the lari at $X \in \mathbf{C}$ is an initial object in $\mathcal{K}\text{-}\mathbf{AIMor}_X$. In particular, if $\mathbf{C}$ has a terminal object 1 , then $Q1$ is an initial object of $\mathcal{K}\text{-}\mathbf{Inj}$, and, if $\mathbf{C}$ has an initial object, $Q1$ is its algebraically $\mathcal{K}$ -injective replacement.

The converse holds if the functor $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ is a (Grothendieck) fibration; that is, the existence of an initial object in each fibre $\mathcal{K}\text{-}\mathbf{AIMor}_X$ implies that $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ has a lari in that case. A sufficient condition for $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ to be a fibration is that $\mathbf{C}$ has pullbacks along $\mathcal{K}$ -injective morphisms as is observed in Remark 1.3.13. $\lrcorner$

Proposition 1.3.20 ([FHM25a, Proposition 4.1.4]). In the situation of Construction 1.3.17, the data (Q, ε, δ) give rise to a comonad on $\mathbf{C}$. $\lrcorner$

We call this comonad the *algebraically $\mathcal{K}$ -cellular replacement comonad*.¹⁰

⁹This is equivalent to the condition that the unique morphism in the slice 2-category $\mathcal{CAT}/\mathbf{C}$ from the object $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ to the terminal object $\text{id}_{\mathbf{C}}: \mathbf{C} \rightarrow \mathbf{C}$ has a left adjoint in $\mathcal{CAT}/\mathbf{C}$.

¹⁰This terminology follows Riehl's use of the term *cellular* for certain objects and morphisms; see [Rie11b; Rie11a; Rie14].

Proof. In short, one can verify the functoriality of Q , the naturality of ε and δ , and the comonad axioms, using the universality of $(\varepsilon_X, \kappa_X)$. The functoriality of Q and the naturality of ε are immediate from the definitions.

To show the naturality of δ , it suffices to show the following: for any morphism $f: X \rightarrow Y$ in $\mathbf{C}$, both $(\delta_Y \circ Qf, f)$ and $(QQf \circ \delta_X, f)$ are morphisms $(\varepsilon_X, \kappa_X) \rightarrow (\varepsilon_Y, \kappa_Y) \circ (\varepsilon_Y, \kappa_Y)$ in $\mathcal{K}\text{-}\mathbf{AIMor}$. This can be done by forming the following pasting composites in $\mathcal{K}\text{-}\mathbf{AIMor}$:

$$\begin{array}{ccc}
 QX & \xrightarrow{Qf} & QY \xrightarrow{\delta_Y} & QQY \\
 \downarrow (\varepsilon_X, \kappa_X) & & \downarrow (\varepsilon_Y, \kappa_Y) & \downarrow (\varepsilon_{QY}, \kappa_{QY}) \\
 & (Q) & (\varepsilon_Y, \kappa_Y) & (D) \\
 & \downarrow & & \downarrow \\
 X & \xrightarrow{f} & Y & \xlongequal{\quad} Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 QX & \xrightarrow{\delta_X} & QQX \xrightarrow{QQf} & QQY \\
 \downarrow (\varepsilon_X, \kappa_X) & & \downarrow (\varepsilon_{QX}, \kappa_{QX}) & \downarrow (\varepsilon_{QY}, \kappa_{QY}) \\
 & (D) & (Q) & \downarrow \\
 & \downarrow & & \downarrow \\
 X & \xlongequal{\quad} X & \xrightarrow{f} & Y
 \end{array}$$

For the axiom $\delta_X \circ \varepsilon_{QX} = \text{Id}_{QX} = \delta_X \circ Q\varepsilon_X$, it suffices to show that both $(\varepsilon_{QX}, \text{Id}_X)$ and $(Q\varepsilon_X, \text{Id}_X)$ are morphisms $(\varepsilon_{QX}, \kappa_{QX}) \circ (\varepsilon_X, \kappa_X) \rightarrow (\varepsilon_X, \kappa_X)$ in $\mathcal{K}\text{-}\mathbf{AIMor}$ (then $(\varepsilon_{QX} \circ \delta_X, \text{Id}_X)$, $(\text{Id}_{QX}, \text{Id}_X)$, and $(Q\varepsilon_X \circ \delta_X, \text{Id}_X)$ are parallel morphisms with domain $(\varepsilon_X, \kappa_X)$ over the same morphism Id_X , and hence have to coincide). These can be seen by the following composites in $\mathcal{K}\text{-}\mathbf{AIMor}$:

$$\begin{array}{ccc}
 QQX & \xrightarrow{\varepsilon_{QX}} & QX \\
 \downarrow (\varepsilon_{QX}, \kappa_{QX}) & (I) & \parallel (\text{Id}_X, \omega^X) \\
 QX & \xlongequal{\quad} & QX \\
 \downarrow (\varepsilon_X, \kappa_X) & = & \downarrow (\varepsilon_X, \kappa_X) \\
 X & \xlongequal{\quad} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 QQX & \xrightarrow{Q\varepsilon_X} & QX \\
 \downarrow (\varepsilon_{QX}, \kappa_{QX}) & (Q) & \downarrow (\varepsilon_X, \kappa_X) \\
 QX & \xrightarrow{\varepsilon_X} & X \\
 \downarrow (\varepsilon_X, \kappa_X) & (I) & \parallel (\text{Id}_X, \omega^X) \\
 X & \xlongequal{\quad} & X
 \end{array}$$

Finally, for the axiom $\delta_X \circ \delta_{QX} = \delta_X \circ Q\delta_X$, it suffices to show that $(\delta_{QX}, \text{Id}_X)$ and $(Q\delta_X, \text{Id}_X)$ are parallel morphisms in $\mathcal{K}\text{-}\mathbf{AIMor}$, which can be seen as follows:

$$\begin{array}{ccc}
 QQX & \xrightarrow{\delta_{QX}} & QQQX \\
 \downarrow (\varepsilon_{QX}, \kappa_{QX}) & (D) & \downarrow (\varepsilon_{QQX}, \kappa_{QQX}) \\
 & & \downarrow \\
 QX & \xlongequal{\quad} & QX \\
 \downarrow (\varepsilon_X, \kappa_X) & = & \downarrow (\varepsilon_X, \kappa_X) \\
 X & \xlongequal{\quad} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 QQX & \xrightarrow{Q\delta_X} & QQQX \\
 \downarrow (\varepsilon_{QX}, \kappa_{QX}) & (Q) & \downarrow (\varepsilon_{QQX}, \kappa_{QQX}) \\
 QX & \xrightarrow{\delta_X} & QQX \\
 \downarrow (\varepsilon_X, \kappa_X) & (D) & \downarrow (\varepsilon_{QX}, \kappa_{QX}) \\
 X & \xlongequal{\quad} & X
 \end{array}$$

□

As indicated before Construction 1.3.17, a sufficient condition for $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ to have a lari (and hence for the algebraically $\mathcal{K}$ -cellular replacement Q to be defined) is given by the following.

Proposition 1.3.21 ([FHM25a, Proposition 4.1.6 and Remark 4.1.7]). If $\mathbf{C}$ is a locally presentable category and $\mathcal{K}$ is small, the functor $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ has a lari. Moreover, the algebraically $\mathcal{K}$ -cellular replacement comonad coincides with the cofibrant replacement comonad with respect to the AWFS cofibrantly generated by $\mathcal{K}$. $\square$

Proof. The existence of a lari follows from the left adjoint of the forgetful functor $\mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}^2$ observed in Remark 1.3.10; the lari sends an object $X \in \mathbf{C}$ to the free algebraically $\mathcal{K}$ -injective morphism on the unique morphism $!^X: \emptyset \rightarrow X$ from the initial object $\emptyset$ in $\mathbf{C}$. For the second part, let us write $(Q', \varepsilon', \delta')$ for the cofibrant replacement comonad, which is just the dual of the fibrant replacement monad observed in Remark 1.3.16. As observed there, the counit ε'_X is obtained by applying the monad R to the object $!^X: \emptyset \rightarrow X$ in $\mathbf{C}^2$, which agrees with the above construction of the image $\varepsilon_X: QX \rightarrow X$ under the lari. On the other hand, [Gar10, Proposition 2.8] (or [BG16a, (2.11)]) implies that the comultiplication δ'_X is determined as in (D). Finally, given a morphism $f: X \rightarrow Y$, the morphism $Q'f: Q'X \rightarrow Q'Y$ is determined as in (Q), since $(Q'f, f)$ is obtained by applying R to the morphism $(\text{Id}_\emptyset, f): !^X \rightarrow !^Y$ in $\mathbf{C}^2$, and hence is a morphism in $R\text{-}\mathbf{Alg} \cong \mathcal{K}\text{-}\mathbf{AIMor}$. $\square$

Example 1.3.22 (cf. [Gar09a]). For a cartesian monad P over $\mathbf{T}$ (see Definition 1.2.2.4), a contraction $\bar{\kappa}^P$ on P (see Definition 1.2.2.5) is exactly a $\widetilde{\mathcal{I}}_o^\times$ -lifting structure on the morphism $\text{ar}_1^P: P1 \rightarrow \mathbf{T}1$ in $\mathbf{GSet}$. Moreover, if we write $\mathbf{OC}$ for the category of cartesian monads with contractions (Definition 1.2.2.5 and Proposition 1.2.2.7), then it fits into the left-hand pullback square below:

$$\begin{array}{ccccc} \mathbf{OC} & \longrightarrow & \mathcal{I}_o^\times\text{-}\mathbf{AIMor}_{\mathbf{T}1} & \longrightarrow & \mathbf{SpEpi}\left(\left[\widetilde{\mathcal{I}}_o^\times, \mathbf{Set}\right]\right) \\ \downarrow & \lrcorner & \downarrow \text{forget} & \lrcorner & \downarrow \text{forget} \\ \mathbf{CartMnd}/\mathbf{T} & \xrightarrow{\text{ev}_1} & \mathbf{GSet}/\mathbf{T}1 & \longrightarrow & \left[\widetilde{\mathcal{I}}_o^\times, \mathbf{Set}\right]^2 \end{array}$$

Here, we write $\mathbf{CartMnd}/\mathbf{T}$ for the category of cartesian monads over $\mathbf{T}$. The pullback square on the right is (1.3.9.1) in the case where $\mathbf{C} = \mathbf{GSet}$ and $\mathcal{K} = \widetilde{\mathcal{I}}_o^\times := (\widetilde{\mathcal{I}}_o^\times, |\widetilde{-}|)$, where $\widetilde{\mathcal{I}}_o^\times$ is defined as follows: the underlying set is

$$\widetilde{\mathcal{I}}_o^\times = \{(n, \mathbf{k}) \mid n \geq 1 \text{ and } \mathbf{k}: G^n \rightarrow \mathbf{T}1\}$$

and we set $|\widetilde{(n, \mathbf{k})}|$ to be the morphism

$$\iota^n: (\partial G^n, \langle \partial_{n-1}^{\mathbf{T}1} \mathbf{k}, \partial_{n-1}^{\mathbf{T}1} \mathbf{k} \rangle) \rightarrow (G^n, \mathbf{k})$$

in $\mathbf{GSet}/\mathbf{T}1$.

The functor labelled ev_1 is the evaluation at the terminal object. Using the observation made in Section A.2.1 and Fact A.2.6, one sees that it is equivalent to a right adjoint functor between locally presentable categories. Therefore, by [Bir84, Theorem 2.18], $\mathbf{OC}$ is locally presentable and the left vertical functor above has a left adjoint. This in particular gives another proof of Proposition 1.2.2.7.

Moreover, post-composing the left adjoint of ev_1 with $|\widetilde{-}|$, we obtain another $\widetilde{\mathcal{I}}_o^\times \in \mathbf{SET} \downarrow (\mathbf{CartMnd}/\mathbf{T})^2$. Again by (1.3.9.1) in the case where $\mathbf{C} = \mathbf{CartMnd}/\mathbf{T}$ and $\mathcal{K} = \widetilde{\mathcal{I}}_o^\times$, we conclude that $\mathbf{OC}$ is equivalent to the category $\widetilde{\mathcal{I}}_o^\times\text{-}\mathbf{AInj}$. Therefore, by Remark 1.3.19, the initial object $(\mathbf{L}, \bar{\kappa})$ is characterised as the algebraically $\widetilde{\mathcal{I}}_o^\times$ -cellular replacement of the terminal object in $\mathbf{CartMnd}/\mathbf{T}$, which is simply $\mathbf{T}$. This gives another homotopy-theoretic characterisation of $\mathbf{L}$ that is parallel to the characterisation in [Gar09a]; however, note that this is different from the one given there, since we are working in the category of cartesian monads over $\mathbf{T}$ here, whereas [Gar09a] works in the category of *normalised globular operads* (see also Section A.2.2). $\square$

Finally, we describe a ‘change of base’ result for algebraic injectivity and for algebraically cellular replacement. For the case Remark 1.3.10 of locally presentable categories and small indexing classes, the same thing can also be achieved by more directly applying some change of base results for AWFS established in [BG16a]; see Remark 1.3.26. This will be used in Section 3.3.3.

Definition 1.3.23 ([FHM25a, Definition 4.3.1]). Let $F: \mathbf{D} \xleftarrow{\perp} \mathbf{C}: U$ be an adjunction between categories, and let $\mathcal{K}_o = (\mathcal{K}_o, |\mathcal{K}_o|) \in \mathbf{Set} \downarrow \mathbf{D}^2$ be an indexed family of morphisms in $\mathbf{D}$. Now we obtain an indexed family $\mathcal{K}$ of morphisms in $\mathbf{C}$ by post-composing $F^2: \mathbf{D}^2 \rightarrow \mathbf{C}^2$ with $|\mathcal{K}_o|: \mathcal{K}_o \rightarrow \mathbf{D}^2$; i.e., we set $\mathcal{K} = (\mathcal{K}_o, j \mapsto F[j])$.

- (i) We have a functor $U\text{-}\mathbf{AIMor}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathcal{K}_o\text{-}\mathbf{AIMor}$ defined by mapping each algebraically $\mathcal{K}$ -injective morphism $(f: X \rightarrow Y, \kappa)$ to the algebraically $\mathcal{K}_o$ -injective morphism $(Uf: UX \rightarrow UY, \kappa^\dagger)$, where the $\mathcal{K}_o$ -lifting operation $\kappa^\dagger$ on Uf is determined by the following correspondence under the adjunction $F \dashv U$.

$$\begin{array}{ccc} \bullet & \xrightarrow{u^\dagger} & UX \\ \downarrow |j| & \nearrow \kappa^\dagger(j; u^\dagger, v^\dagger) & \downarrow Uf \\ \bullet & \xrightarrow{v^\dagger} & UY \end{array} \qquad \begin{array}{ccc} \bullet & \xrightarrow{u} & X \\ \downarrow F[j] & \nearrow \kappa(j; u, v) & \downarrow f \\ \bullet & \xrightarrow{v} & Y \end{array}$$

Notice that the following square is a pullback in $\mathbf{CAT}$:

$$\begin{array}{ccc} \mathcal{K}\text{-}\mathbf{AIMor} & \xrightarrow{U\text{-}\mathbf{AIMor}} & \mathcal{K}_o\text{-}\mathbf{AIMor} \\ \downarrow \text{forget} & \lrcorner & \downarrow \text{forget} \\ \mathbf{C}^2 & \xrightarrow{U^2} & \mathbf{D}^2. \end{array} \tag{1.3.23.1}$$

- (ii) Now suppose that both functors $\text{cod}: \mathcal{K}\text{-}\mathbf{AIMor} \rightarrow \mathbf{C}$ and $\text{cod}: \mathcal{K}_o\text{-}\mathbf{AIMor} \rightarrow \mathbf{D}$ have laris. As in Construction 1.3.17, we write $(\varepsilon_X: QX \rightarrow X, \kappa_X)$ and $(\varepsilon'_A: Q'A \rightarrow A, \kappa'_A)$ for the values under these laris of objects $X \in \mathbf{C}$ and $A \in \mathbf{D}$, respectively. For each object $X \in \mathbf{C}$, we define $\sigma_X: Q'UX \rightarrow UQX$ as the unique morphism in $\mathbf{D}$ such that the following configuration

$$\begin{array}{ccc} Q'UX & \xrightarrow{\sigma_X} & UQX \\ \downarrow (\varepsilon'_{UX}, \kappa'_{UX}) & & \downarrow (U\varepsilon_X, \kappa_X^\dagger) \\ UX & \xlongequal{\quad} & UX \end{array} \quad (\text{S})$$

is filled with a unique cell in $\mathcal{K}_o\text{-}\mathbf{AIMor}$. $\lrcorner$

Lemma 1.3.24 (cf. [BG16a, Proposition 21]). In the situation of Definition 1.3.23, $U\text{-}\mathbf{AIMor}$ preserves vertical composition of algebraically injective morphisms, and hence defines a double functor

$$\mathcal{K}\text{-}\mathbf{AIMor} \xrightarrow{U\text{-}\mathbf{AIMor}} \mathcal{K}_o\text{-}\mathbf{AIMor}. \quad (1.3.24.1)$$

Proof. Let $(g, \kappa): C \rightarrow C'$ and $(h, \lambda): C' \rightarrow C''$ be algebraically $\mathcal{K}$ -injective morphisms in $\mathbf{C}$, and write $(Ug, \kappa^\dagger)$ and $(Uh, \lambda^\dagger)$ for their images under $U\text{-}\mathbf{AIMor}$. To show that $U\text{-}\mathbf{AIMor}$ preserves this composite, observe the following diagrams commute:

$$\begin{array}{ccc} \bullet & \xrightarrow{u^\dagger} & UC \\ \downarrow |j| & \nearrow \kappa^\dagger(j; u^\dagger, \lambda^\dagger(j; u^\dagger \circ Ug, v)) & \downarrow Ug \\ & & UC' \\ & \nearrow \lambda^\dagger(j; u^\dagger \circ Ug, v^\dagger) & \downarrow Uh \\ \bullet & \xrightarrow{v^\dagger} & UC'' \end{array} \quad \begin{array}{ccc} \bullet & \xrightarrow{u} & C \\ \downarrow F|j| & \nearrow \kappa(j; u, \lambda(j; u \circ g, v)) & \downarrow g \\ & & C' \\ & \nearrow \lambda(j; u \circ g, v) & \downarrow h \\ \bullet & \xrightarrow{v} & C'' \end{array}$$

for any $j \in \mathcal{K}_o$, a commutative square $(u, v): F|j| \rightarrow g \circ h$, and its adjunct $(u^\dagger, v^\dagger): |j| \rightarrow Ug \circ Uh$. Since $u^\dagger \circ Ug$ is the adjunct of $u \circ g$ with respect to the adjunction $F \dashv U$, by definition of $\kappa^\dagger$ and $\lambda^\dagger$, the resulting diagonal towards UC on the left is the adjunct of the corresponding diagonal towards C on the right. This shows that the composite $(Ug, \kappa^\dagger) \circ (Uh, \lambda^\dagger)$ is equal to the image of $(g, \kappa) \circ (h, \lambda)$ under $U\text{-}\mathbf{AIMor}$. $\square$

Proposition 1.3.25 ([FHM25a, Proposition 4.3.3]). In the situation of Definition 1.3.23.(ii), the family $(\sigma_X)_{X \in \mathbf{C}}$ is a natural transformation

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{Q} & \mathbf{C} \\ U \downarrow & \sigma \nearrow & \downarrow U \\ \mathbf{D} & \xrightarrow{Q'} & \mathbf{D} \end{array}$$

such that the pair (U, σ) is a comonad opfunctor [Str72, Section 4] from $(\mathbf{C}, Q)$ to $(\mathbf{D}, Q')$. $\lrcorner$

Proof. As in the proof of Proposition 1.3.20, one can verify the naturality of σ and the compatibility of σ with the comonads Q and Q' using the universality of $(\varepsilon'_{UX}, \kappa'_{UX})$; below are the details.

The compatibility of σ with the counits (i.e., that $\varepsilon'_{UX} = U\varepsilon_X \circ \sigma_X$ holds for any $X \in \mathbf{C}$) is immediate from the definition of σ_X .

To show the naturality of σ , it suffices to show that for any morphism $f: X \rightarrow Y$ in $\mathbf{C}$, both $(\sigma_Y \circ Q'Uf, Uf)$ and $(UQf \circ \sigma_X, Uf)$ are morphisms $(\varepsilon'_{UX}, \kappa'_{UX}) \rightarrow (U\varepsilon_Y, \kappa_Y^\dagger)$ in $\mathcal{K}_o\text{-}\mathbf{AIMor}$. These can be seen by the following diagrams, which can be regarded either as composition of morphisms in the category $\mathcal{K}_o\text{-}\mathbf{AIMor}$ or horizontal composition of squares in the double category $\mathcal{K}_o\text{-}\mathbf{AIMor}$.

$$\begin{array}{ccc} Q'UX & \xrightarrow{Q'Uf} & Q'UY & \xrightarrow{\sigma_Y} & UQY \\ \downarrow (\varepsilon'_{UX}, \kappa'_{UX}) & \downarrow (\text{Q}) & \downarrow (\varepsilon'_{UY}, \kappa'_{UY}) & \downarrow (\text{S}) & \downarrow (U\varepsilon_Y, \kappa_Y^\dagger) \\ UX & \xrightarrow{Uf} & UY & \xlongequal{\quad} & UY \end{array} \quad \begin{array}{ccc} Q'UX & \xrightarrow{\sigma_X} & UQX & \xrightarrow{UQf} & UQY \\ \downarrow (\varepsilon'_{UX}, \kappa'_{UX}) & \downarrow (\text{S}) & \downarrow (U\varepsilon_X, \kappa_X^\dagger) & \downarrow (\text{Q}) & \downarrow (U\varepsilon_Y, \kappa_Y^\dagger) \\ UX & \xlongequal{\quad} & UX & \xrightarrow{Uf} & UY \end{array}$$

Finally, the compatibility of σ with the comultiplications means that for any object $X \in \mathbf{C}$, the diagram

$$\begin{array}{ccc} Q'UX & \xrightarrow{\sigma_X} & UQX \\ \delta'_{UX} \downarrow & & \downarrow U\delta_X \\ Q'Q'UX & \xrightarrow{Q'\sigma_X} Q'UQX & \xrightarrow{\sigma_{QX}} UQQX \end{array} \quad (1.3.25.1)$$

commutes. To see this, observe that the two composites in (1.3.25.1) lift to the top horizontal composites in the following diagrams in the double category $\mathcal{K}_o\text{-}\mathbf{AIMor}$:

$$\begin{array}{ccccc} Q'UX & \xrightarrow{\sigma_X} & UQX & \xrightarrow{U\delta_X} & UQQX \\ \downarrow (\varepsilon'_{UX}, \kappa'_{UX}) & & \downarrow (U\varepsilon_X, \kappa_X^\dagger) & & \downarrow (U\varepsilon_{QX}, \kappa_{QX}^\dagger) \\ & (S) & UQX & & \\ & \downarrow (U\varepsilon_X, \kappa_X^\dagger) & & & \\ UX & \xlongequal{\quad} & UX & \xlongequal{\quad} & UX \end{array}$$

$$\begin{array}{ccccccc} Q'UX & \xrightarrow{\delta'_{UX}} & Q'Q'UX & \xrightarrow{Q'\sigma_X} & Q'UQX & \xrightarrow{\sigma_{QX}} & UQQX \\ \downarrow (\varepsilon'_{UX}, \kappa'_{UX}) & & \downarrow (\varepsilon'_{Q'UX}, \kappa'_{Q'UX}) & (Q) & \downarrow (\varepsilon'_{UQX}, \kappa'_{UQX}) & (S) & \downarrow (U\varepsilon_{QX}, \kappa_{QX}^\dagger) \\ & (D) & Q'UX & \xrightarrow{\sigma_X} & UX & \xlongequal{\quad} & UQX \\ & & \downarrow (\varepsilon'_{UX}, \kappa'_{UX}) & (S) & \downarrow (U\varepsilon_X, \kappa_X^\dagger) & = & \downarrow (U\varepsilon_X, \kappa_X^\dagger) \\ UX & \xlongequal{\quad} & UX & \xlongequal{\quad} & UX & \xlongequal{\quad} & UX \end{array}$$

The square labelled by $U(D)$ in the first diagram is the image of (D) in $\mathcal{K}\text{-}\mathbf{AIMor}$ under $U\text{-}\mathbf{AIMor}$ (1.3.24.1). This means that the two composites of (1.3.25.1) define parallel morphisms in $\mathcal{K}_o\text{-}\mathbf{AIMor}_{UX}$ from the initial object (see Remark 1.3.19), and hence they must be equal. $\square$

Remark 1.3.26 ([FHM25a, Remark 4.3.5]). Let us consider the situation of Definition 1.3.23, and assume moreover that the categories $\mathbf{C}$ and $\mathbf{D}$ are locally presentable and the indexing set $\mathcal{K}_o$ is small. Then by Proposition 1.3.21, the assumptions in Definition 1.3.23.(ii) are satisfied. We now explain that in this case, the result of Proposition 1.3.25 can also be obtained via the theory of AWFS.

As explained in Remark 1.3.10, there exist AWFSs (L, R) on $\mathbf{C}$ cofibrantly generated by $\mathcal{K}$ and (L_o, R_o) on $\mathbf{D}$ cofibrantly generated by $\mathcal{K}_o$. (The AWFS (L_o, R_o) coincides with the one obtained by *projectively lifting* (L, R) along $U: \mathbf{C} \rightarrow \mathbf{D}$ [BG16a, Section 4.5], i.e., the pullback square (1.3.23.1) lifts to a suitable pullback of double categories.) By [BG16a, Proposition 22], the universality of (L_o, R_o) induces a canonical oplax morphism of AWFS $(L_o, R_o) \rightarrow (L, R)$ whose functor part is F . This involves a comonad functor from $(\mathbf{D}^2, L_o)$ to $(\mathbf{C}^2, L)$ whose functor part is F^2 . Since F preserves the initial object, we can restrict it and obtain a comonad functor

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{Q} & \mathbf{C} \\ F \uparrow & \rightsquigarrow \tau & \uparrow F \\ \mathbf{D} & \xrightarrow{Q'} & \mathbf{D} \end{array}$$

from $(\mathbf{D}, Q')$ to $(\mathbf{C}, Q)$. The comonad opfunctor (U, σ) in Proposition 1.3.25 and (F, τ) form a (doctrinal) adjunction [Kel74], i.e., σ is the mate of τ under $F \dashv U$. (To see this, observe that we also have a canonical lax morphism of AWFS $(L, R) \rightarrow (L_o, R_o)$ whose functor part is U by [BG16a, Section 2.10]. One can also restrict this to obtain a comonad opfunctor from Q to Q' , which coincides with (U, σ) and is also the right adjoint of (F, τ) .) $\lrcorner$

A comonad opfunctor (such as the one in Proposition 1.3.25) induces a functor between the respective Kleisli categories, and in Section 3.3.3, this gives the notion of *underlying globular map* of a weak ω -functor in a functorial way.

1.4. Coinduction

We work with invertible cells in a weak ω -category throughout this thesis, and in particular, they are discussed in Chapters 2 and 4. Our definition of invertible cells is a *coinductive* one, which is also discussed in [Che07] for the weak case and in [LMW10] for the strict case. In this subsection, we observe some basic results on a lattice-theoretic formalisation of the notion of coinduction, which will be used in Chapters 2 and 4. The substance of this subsection is largely based on [FHM25b, Section 3.1], which formalises the technique employed in the proof of the main theorem of [FHM24].

A *complete lattice* A is a poset such that every subset $S \subseteq A$ has a join $\bigvee S \in A$, or equivalently, every subset $S \subseteq A$ has a meet $\bigwedge S \in A$. Let $F: A \rightarrow A$ be an monotone endomap on a complete lattice A . For each element $x \in A$, we write $\downarrow x = \{a \in A \mid a \leq x\}$ for the downward closure of x in A , and write $\uparrow x = \{a \in A \mid x \leq a\}$ for the upward closure of x in A .

Definition 1.4.1. A *pre-fixed point* of F is an element $x \in A$ such that $F(x) \leq x$. Dually, a *post-fixed point* of F is an element $x \in A$ such that $x \leq F(x)$. We write $\text{Pre}(F)$ and $\text{Post}(F)$ for the full subposets of A spanned by pre-fixed points and post-fixed points of F , respectively. $\lrcorner$

Definition 1.4.2. Given an element $x \in A$, we write

$$\text{Pre}_x(F) = \{y \in A \mid x \leq y \text{ and } F(y) \leq y\} = \text{Pre}(F) \cap \uparrow x$$

for the set of pre-fixed points of F above x . $\lrcorner$

Proposition 1.4.3. $\text{Pre}(F)$ is reflective in A and the reflection is given by $x \mapsto \bigwedge(\text{Pre}_x(F))$. Dually, $\text{Post}(F)$ is coreflective. In particular, those posets $\text{Post}(F)$ and $\text{Pre}(F)$ are complete lattices. $\lrcorner$

Proof. By the universality of meets, for each $x \in A$, we have $F(\bigwedge(\text{Pre}_x(F))) \leq \bigwedge(\text{Pre}_x(F))$ and hence $\bigwedge(\text{Pre}_x(F))$ is the minimum element in $\text{Pre}_x(F)$, which shows that $\bigwedge(\text{Pre}_x(F))$ is the reflection of x into $\text{Pre}(F)$. The remaining part follows immediately. $\square$

We write νF for the greatest element in $\text{Post}(F)$, which is called the *greatest post-fixed point* [Tar55] of F . This exists by the above proposition, and coincides with $\bigvee(\text{Post}(F))$.

The following lemma shows that, when we are considering the greatest post-fixed point of F , we may replace F with a deflationary monotone map.

Lemma 1.4.4. Consider another monotone endomorphism $F \wedge \text{Id}_A: A \rightarrow A$ defined by $(F \wedge \text{Id}_A)(x) = F(x) \wedge x$ for each $x \in A$. Then we have $\text{Post}(F \wedge \text{Id}_A) = \text{Post}(F)$, and hence $\nu(F \wedge \text{Id}_A) = \nu F$. $\lrcorner$

Now the following¹¹ is the main proposition of this subsection, which is a formalisation of the technique used in [FHM24].

Proposition 1.4.5. Let $F, G: A \rightarrow A$ be monotone maps. If $\text{Pre}_{\nu F}(G)$ is closed under F , then we have $G(\nu F) \leq \nu F$. $\lrcorner$

Proof. Take the reflection $b = \bigwedge(\text{Pre}_{\nu F}(G))$ of νF into $\text{Pre}(G)$, which is the least element in $\text{Pre}_{\nu F}(G)$. By assumption, we have $Fb \in \text{Pre}_{\nu F}(G)$. The minimality of b then implies $b \leq Fb$, i.e., that b is a post-fixed point of F . It follows that we have $b \leq \nu F$. Therefore $\nu F = b \in \text{Pre}_{\nu F}(G)$ and hence in particular $G(\nu F) \leq \nu F$. $\square$

Remark 1.4.6. In order to check that the assumption of Proposition 1.4.5 is satisfied, we only need to show that for any $b \in \text{Pre}_{\nu F}(G)$, we have $GFb \leq Fb$. The condition $\nu F \leq Fb$ follows from $\nu F \leq b$ since we have $\nu F = F(\nu F)$. $\lrcorner$

Applying Proposition 1.4.5 to $F = F \wedge \text{Id}_A$ and using Lemma 1.4.4, we obtain the following variant¹².

Corollary 1.4.7. Let $F, G: A \rightarrow A$ be monotone maps. If $\text{Pre}_{\nu F}(G)$ is closed under $F \wedge \text{Id}_A$, then we have $G(\nu F) \leq \nu F$. $\lrcorner$

¹¹After my PhD defence, Galal Zeinab pointed out a similarity between this proposition and the (constructive) proof of Pataia's fixed point theorem [nLa26]. While I agree that there is a genuine analogy, I was not able to make the relationship precise; we leave this as future work.

¹²Galal Zeinab also pointed out to me that this corollary might be seen as an example of (co)induction up-to techniques [Bon+16]. Concretely, to show that νF is a pre-fixed point of G , it suffices to prove $G \circ (F \wedge \text{Id}_A) \leq F$ on $\text{Pre}_{\nu F}(G)$. If we regard $G \circ -$ as an operator on the lattice $[\text{Pre}_{\nu F}(G), A]$, then this statement can be seen as an instance of induction up to the operator $H \mapsto H \wedge \text{Id}_A$.

Weakly invertible cells in a weak ω -category

2.1. Introduction

In (higher-dimensional) category theory, the ‘correct’ notion of equivalence is often something weaker than that of equality. When working inside an ordinary category, it is the notion of isomorphism; when working on the totality of all (small) categories, or more generally in a 2-category, the word ‘equivalence’ already has a well-established meaning. For general n -categories where $n \in \mathbb{N} \setminus \{0\}$, one can define such a notion of equivalence, or more generally the notion of equivalence k -cell, or weakly invertible k -cell¹, by induction: equivalence n -cells in an n -category X are the same as strictly invertible n -cells, and a k -cell $u: x \rightarrow y$ for $0 < k < n$ is an equivalence cell if there exists a k -cell $v: y \rightarrow x$ and equivalence $(k+1)$ -cells $vu \rightarrow 1_x$ and $uv \rightarrow 1_y$. Unravelling the induction, a witness for weak invertibility of a k -cell $u: x \rightarrow y$ in an n -category involves a k -cell $v: y \rightarrow x$ together with 2^{m-k+1} m -cells of suitable types for each $k < m \leq n$, which are subject to 2^{n-k+1} equations at the top dimension n (cf. *exponential wedge* in [Str87, Section 1]).

In an ω -category, one can still define equivalence cells in the same manner, using the unravelled description: a k -cell $u: x \rightarrow y$ in an ω -category X is an equivalence cell if there exist a k -cell $v: y \rightarrow x$ and 2^{m-k+1} m -cells in X for each $m > k$, of suitable types. (This time, we do not demand any equations between these cells.) This definition can be stated more succinctly as: a k -cell $u: x \rightarrow y$ in an ω -category X is an *equivalence* if there exist a k -cell $v: y \rightarrow x$ and *equivalence* $(k+1)$ -cells $vu \rightarrow 1_x$ and $uv \rightarrow 1_y$ in X . To remove ambiguity in this seemingly circular definition, we should also add that here we are defining weak invertibility *coinductively* (see Remark 2.2.2 and Section 1.4 for details of coinduction). This notion of equivalence cell in an ω -category plays a key role in e.g. the definition of a model structure on the category of strict ω -categories [LMW10] (see also [Che07]).

In this chapter, we study equivalence cells in a *weak ω -category* in the sense of Leinster [Lei04] (which is based on an earlier definition by Batanin [Bat98]). Our main theorem (Theorem 2.4.7) states that the set of equivalence cells in a weak ω -category is closed under the operations of (globular) pasting.² For example, if we are given a 2-dimensional pasting diagram

$$\begin{array}{c}
 \begin{array}{ccc}
 & f & \\
 a & \xrightarrow{\quad} & b \\
 & \downarrow \alpha & \\
 & g &
 \end{array}
 \xrightarrow{h}
 \begin{array}{ccccc}
 & & i & & \\
 & \searrow & \downarrow \beta & \searrow & \\
 c & \xrightarrow{\quad} & j & \xrightarrow{\quad} & d \\
 & \swarrow & \downarrow \gamma & \swarrow & \\
 & & k & &
 \end{array}
 \end{array}
 \quad (2.1.0.1)$$

in a weak ω -category, in which all 2-cells α, β , and γ appearing in the diagram are equivalences, then so is their composite, regardless of the way of ‘bracketing’ (Recall that in a weak ω -category, composition is neither unital nor associative, so for example the 1-dimensional domain of the above composite could be taken either as $(ih)f$ or $i(hf)$, and these might be different). We remark that for the composite 2-cell of (2.1.0.1) to be an equivalence, the 1-cells in the diagram (such as f) need not be invertible.

The special case of our main theorem, where there is no n -cell in an n -dimensional pasting diagram, can be regarded as a *coherence* result for weak ω -categories. For example, one can regard (2.1.0.1) not as a 2-dimensional pasting diagram but as a 3-dimensional one in which no 3-cell appears. Then, the assumption of the main theorem (i.e., all 3-cells in (2.1.0.1) should be equivalences) is vacuously satisfied. Consequently, we see that any two composite 2-cells δ and δ' of (2.1.0.1) are connected by an equivalence 3-cell (provided that δ and δ' are parallel). See Proposition 2.3.7 and Remark 2.3.9 for more details.

¹Although the title of this chapter follows [FHM24] in using the term *weakly invertible cell*, for the sake of consistency of terminology throughout this thesis we shall also freely use the term *equivalence* for this notion.

²We prove this only for globular pasting operations since Leinster’s definition of weak ω -category is based on them. Of course, the result trivially extends to whatever notion of pasting operation, as long as it is expressible as a repeated application of globular pasting operations.

A typical application of our main theorem would proceed as follows. Firstly, one takes some elementary fact about strict ω -categories (starting either an equality between cells or existence of an equivalence cell) which one wishes to generalise to the weak setting. Then one inserts a coherence cell whenever one sees an equality in the proof of the strict case. This yields a chain of whiskerings of equivalence cells, which can be composed to a single cell. Finally, using the main theorem, one deduces that the resulting cell is itself invertible. We give a couple of examples in Corollaries 2.4.15 and 2.4.16.

Another immediate consequence of the main theorem is that any weak ω -category X has a largest weak ω -subgroupoid $\mathbf{k}(X)$ which we call the *core* of X . Indeed, let $\mathbf{k}(X)$ consists of all (hereditary) equivalence cells of X . Then the main theorem ensures that $\mathbf{k}(X)$ is a weak ω -subcategory of X . We obtain a functor $k: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Gpd}_s$ from the category of weak ω -categories to its full subcategory consisting of all weak ω -groupoids, which is characterised as the right adjoint of the inclusion functor $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Gpd}_s \hookrightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. This generalises the well-known universal property of the core groupoid functor $k: \mathbf{Cat} \rightarrow \mathbf{Gpd}$.

Related work. After presenting our main theorem at Category Theory 2023 in Louvain-la-Neuve, we learnt from Emily Riehl that tsilil clingman had independently proved a similar result. More precisely, Leinster gives a ‘non-algebraic’ variant of his definition of weak ω -category in [Lei02, Definition L’] and [Lei04, Section 10.2]. In [cli22, Section 3.2], clingman shows that the equivalence cells in a slight variant of these non-algebraic weak ω -categories (called *proof-relevant categories* in [cli22]) are closed under compositions.

Outline of this chapter. After recalling the coinductive definition of equivalence cells in a weak ω -category in Section 2.2, and establishing a few facts about coherence in Section 2.3, we state and prove the main theorem in Section 2.4, and discuss a few applications including core weak ω -groupoids of weak ω -categories, which is in Section 2.5.

2.2. The definition of invertible cells in a weak ω -category

In order to define (weakly) invertible cells (or equivalence cells) in a weak ω -category X , we only need a small part of its structure. Specifically, we use the following operations, where $n \geq 1$: the operation mapping each $(n-1)$ -cell x of X to the n -cell $\text{id}_n(x)$ of X , and the operation mapping each pair of n -cells u and v of X such that $t_{n-1}(u) = s_{n-1}(v)$ to the n -cell $u *_{n-1} v$ of X ; see Definition 1.2.3.8. (Globular sets equipped with such operations are called ω -precategories in [Che07].)

Definition 2.2.1 ([Che07; LMW10]). An n -cell $u: x \rightarrow y$ (with $n \geq 1$) in a weak ω -category X is an *equivalence cell* or a *weakly invertible cell* if there exist

- an n -cell $v: y \rightarrow x$,
- an equivalence $(n+1)$ -cell $p: u *_{n-1} v \rightarrow \text{id}_n(x)$, and
- an equivalence $(n+1)$ -cell $q: v *_{n-1} u \rightarrow \text{id}_n(y)$

in X . In this situation, we say v is a *pseudo inverse* of u . For n -cells x and y (with $n \geq 0$), we write $x \sim y$ if there exists an equivalence $(n+1)$ -cell $u: x \rightarrow y$ in X . $\lrcorner$

Since in this thesis the notion of (*strictly*) *invertible cell* in a strict ω -category seldom appears, we shall abbreviate ‘weakly invertible cell’ to ‘invertible’ and ‘pseudo inverse’ to ‘inverse’ in what follows.

Remark 2.2.2. In Definition 2.2.1, the notion of equivalence n -cell is defined *coinductively*. In this block, we formalise this coinductive definition using the lattice-theoretic framework prepared in Section 1.4. First, we define an endo-monotone function Φ^X on the poset $\wp(\coprod_{n \in \mathbb{N}} X_n)$ of all sets of cells in X as follows.

$$\begin{aligned} \Phi^X(S) = \{ (u: x \rightarrow y) \in X_n \mid n \geq 1, \exists (v: y \rightarrow x) \in X_n, \\ \exists (p: u *_{n-1} v \rightarrow \text{id}_n(x)) \in S \cap X_{n+1}, \exists (q: v *_{n-1} u \rightarrow \text{id}_n(y)) \in S \cap X_{n+1} \} \end{aligned}$$

In words, $\Phi^X(S)$ is the set of all cells in X which are ‘invertible up to S ’. Since $\wp(\coprod_{n \in \mathbb{N}} X_n)$ is a complete lattice, Φ^X has a greatest post-fixed point $\nu\Phi^X$, and Definition 2.2.1 says that we define this $\nu\Phi^X$ to be the set of all equivalence cells of X . Observe that the characterisation of $\nu\Phi^X$ as the greatest post-fixed point of Φ^X yields the following proof method to show that a certain cell in X is invertible: in order to show that a cell u in X is an equivalence cell, it suffices to find a set $W \subseteq \coprod_{n \in \mathbb{N}} X_n$ with $W \subseteq \Phi^X(W)$ and $u \in W$.

We occasionally omit the superscript X when the weak ω -category is clear from context; this is done in particular in Section 2.4. $\lrcorner$

Remark 2.2.3. Let X be a weak ω -category. By Corollary 1.2.4.3, for $n \geq 2$, an n -cell u of X is an equivalence cell if and only if it is invertible as an $(n-1)$ -cell in the hom weak ω -category $X(s_0^X(u), t_0^X(u))$. (This is also commented in [CF22, Section 6].) $\lrcorner$

2.3. Coherence

We now prove elementary properties of equivalence cells, and establish the existence of enough *coherence* cells in a weak ω -category, which plays an important role throughout this paper.

Proposition 2.3.1 (cf. [LMW10, Lemma 4.3]). Any strict ω -functor preserves equivalence cells. $\lrcorner$

Proof. Let $f: X \rightarrow Y$ be a strict ω -functor between weak ω -categories. Define the set $W \subseteq \coprod_{n \in \mathbb{N}} Y_n$ as

$$W = \{ u' \in Y_n \mid n \geq 1 \text{ and there exists an equivalence } n\text{-cell } u \text{ of } X \text{ with } u' = fu \},$$

which is exactly the image of $\nu\Phi^X$ under f . We show that $W \subseteq \Phi^Y(W)$, which concludes $W \subseteq \nu\Phi^Y$. If $u' \in W \cap Y_n$, then there exists an equivalence n -cell $u: x \rightarrow x'$ of X with $u' = fu$. So there exist an n -cell $v: x' \rightarrow x$ and equivalence $(n+1)$ -cells $p: u *_{n-1}^X v \rightarrow \text{id}_n^X(x)$ and $q: v *_{n-1}^X u \rightarrow \text{id}_n^X(x')$ in X . Then $fp, fq \in W$ and we have $fp: fu *_{n-1}^Y fv \rightarrow \text{id}_n^Y(fx)$ and $fq: fv *_{n-1}^Y fu \rightarrow \text{id}_n^Y(fx')$ in Y by Lemma 1.2.3.3, showing $u' = fu \in \Phi^Y(W)$. $\square$

Definition 2.3.2 (Example 1.3.4). A strict ω -functor $f: X \rightarrow Y$ between weak ω -categories is *contractible* (or *locally a trivial fibration*) [Lei04, Section 9.1] if its underlying morphism of globular sets has the right lifting property with respect to $\iota^n: \partial G^n \rightarrow G^n$ for each $n \geq 1$. As is observed in Example 1.3.4, contractible strict ω -functors are precisely $\mathcal{I}^\times$ -injective morphisms. In more elementary terms, f is contractible if for any n -cell $v: fx \rightarrow fx'$ in Y , there exists an n -cell $\bar{v}: x \rightarrow x'$ in X with $f\bar{v} = v$. $\lrcorner$

Proposition 2.3.3 (cf. [LMW10, Lemma 4.9]). Any contractible strict ω -functor reflects equivalence cells. $\lrcorner$

Proof. Let $f: X \rightarrow Y$ be a contractible strict ω -functor between weak ω -categories. Define the set $W \subseteq \coprod_{n \in \mathbb{N}} X_n$ by

$$W = \{ u \in X_n \mid n \geq 1 \text{ and } fu \text{ is invertible in } Y \},$$

which is exactly the inverse image of $\nu\Phi^Y$ under f . We show that $W \subseteq \Phi^X(W)$. If $(u: x \rightarrow x') \in W \cap X_n$, then $fu: fx \rightarrow fx'$ is invertible in Y and hence we can take its inverse $v: fx' \rightarrow fx$. By contractibility of f , there exists $\bar{v}: x' \rightarrow x$ in X with $f\bar{v} = v$. We also have equivalence cells $p: fu *_{n-1}^Y \bar{v} \rightarrow \text{id}_n^Y(fx)$ and $q: v *_{n-1}^Y fu \rightarrow \text{id}_n^Y(fx')$. By Lemma 1.2.3.3, we have $p: f(u *_{n-1}^X \bar{v}) \rightarrow f(\text{id}_n^X(x))$ and $q: f(\bar{v} *_{n-1}^X u) \rightarrow f(\text{id}_n^X(x'))$ in Y , and hence by contractibility of f again, we obtain $\bar{p}: u *_{n-1}^X \bar{v} \rightarrow \text{id}_n^X(x)$ and $\bar{q}: \bar{v} *_{n-1}^X u \rightarrow \text{id}_n^X(x')$ in X with $f\bar{p} = p$ and $f\bar{q} = q$. Since $\bar{p}$ and $\bar{q}$ are in W , we have $u \in \Phi^X(W)$. $\square$

Recall from Proposition 1.2.2.9 that any strict ω -category can be regarded as a weak ω -category.

Proposition 2.3.4 ([LMW10, Proposition 4.4. (1)]). Any identity cell in a strict ω -category is invertible. $\lrcorner$

Proof. Given a strict ω -category X , consider the set W of all identity cells in X . $\square$

It follows that any strictly invertible cell in a strict ω -category is invertible (in the sense of Definition 2.2.1). Note that in the free strict ω -category $\mathbf{T1} = (\mathbf{T1}, \mu_1^{\mathbf{T}})$, an n -cell $\mathbf{k}$ is an identity n -cell if and only if it is degenerate in the sense of Notation 1.1.2.9 (see Remark 1.2.3.12). Similarly, for any globular set X , an n -cell $\mathbf{u}$ of the strict ω -category $\mathbf{TX}$ is an identity n -cell if and only if its shape is degenerate.

Proposition 2.3.5. Let $X = (X, \xi)$ be a weak ω -category, $n \geq 1$, and $(\phi, \mathbf{u}) \in (\mathbf{LX})_n$. If $\text{ar}(\phi) \in (\mathbf{T1})_n$ is degenerate, then the n -cell $\xi(\phi, \mathbf{u})$ in X is invertible. $\lrcorner$

Proof. Since ξ is a strict ω -functor from $(\mathbf{LX}, \mu_X^{\mathbf{T}})$ to X , by Proposition 2.3.1 it suffices to show that the n -cell $(\phi, \mathbf{u})$ in $(\mathbf{LX}, \mu_X^{\mathbf{T}})$ is invertible. This follows from the facts that $\text{ar}_X: \mathbf{LX} \rightarrow \mathbf{TX}$ is a contractible strict ω -functor, and that $\mathbf{u}$ is an identity n -cell in $\mathbf{TX}$ by Proposition 2.3.4. $\square$

Applying Proposition 2.3.5 to the case where $\phi = \text{spi}_{n+1}(\lceil n \rceil)$ (see Definition 1.2.3.8.(i)) and $\mathbf{u} = [x]^{(n+1)}$, we can now generalise Proposition 2.3.4 to weak ω -categories.

Proposition 2.3.6. If (X, ξ) is a weak ω -category, $n \geq 0$, and $x \in X_n$, then $\text{id}_{n+1}^X(x) \in X_{n+1}$ is invertible. $\lrcorner$

More generally, notice that if $n \geq 0$ and $\phi, \phi' \in (\mathbf{L1})_n$ are parallel n -cells with $\text{ar}(\phi) = \text{ar}(\phi') = \mathbf{k} \in (\mathbf{T1})_n$, then we can regard $\mathbf{k}$ as an $(n+1)$ -cell $\mathbf{k}^{(n+1)}: \mathbf{k}^{(n)} \rightarrow \mathbf{k}^{(n)}$ in $\mathbf{T1}$ and hence obtain an $(n+1)$ -cell

$$\bar{\kappa}(\langle \phi, \phi' \rangle, \mathbf{k}^{(n+1)}) : \phi \rightarrow \phi'$$

of $\mathbf{L1}$. Since $\mathbf{k}^{(n+1)}$ is degenerate, by Proposition 2.3.5 we have the following coherence result.

Proposition 2.3.7 (Coherence, cf. [FHM25a, Proposition 2.4.6]). Let $n \geq 0$ and $\phi, \phi' \in (\mathbf{L1})_n$ be parallel n -cells with $\text{ar}(\phi) = \text{ar}(\phi') = \mathbf{k} \in (\mathbf{T1})_n$. Then for any weak ω -category (X, ξ) and any pasting diagram $\mathbf{u}$ in X of shape $\mathbf{k}$, the $(n+1)$ -cell

$$\xi \left(\bar{\kappa} \left(\langle \phi, \phi' \rangle, \mathbf{k}^{(n+1)} \right), \mathbf{u}^{(n+1)} \right) : \xi(\phi, \mathbf{u}) \rightarrow \xi(\phi', \mathbf{u})$$

in X is invertible. In particular, we have $\xi(\phi, \mathbf{u}) \sim \xi(\phi', \mathbf{u})$ in X . By Remark 1.2.3.12, we can rewrite the above coherence cell as

$$\xi \left(\bar{\kappa} \left(\langle \phi, \phi' \rangle, \mathbf{k}^{(n+1)} \right), \mathbf{u}^{(n+1)} \right) = \xi \left(\bar{\kappa} \left(\langle \phi, \phi' \rangle, \text{id}_{n+1}^{\mathbf{T1}}(\mathbf{k}) \right), \text{id}_{n+1}^{\mathbf{TX}}(\mathbf{u}) \right)$$

┘

When we take

$$\mathbf{k} = \begin{bmatrix} n & & n & & n \\ & n-1 & & n-1 & \\ & & n & & n \\ & & & n-1 & \\ & & & & n \end{bmatrix} \quad \text{and} \quad \phi = (\tilde{e}_n *_{n-1}^{\mathbf{L1}} \tilde{e}_n) *_{n-1}^{\mathbf{L1}} \tilde{e}_n, \quad \phi' = \tilde{e}_n *_{n-1}^{\mathbf{L1}} (\tilde{e}_n *_{n-1}^{\mathbf{L1}} \tilde{e}_n), \quad (2.3.7.1)$$

by Remark 1.2.3.2.(v) and $\xi \circ \mu_X^{\mathbf{L}} = \xi \circ \mathbf{L}\xi$, we have the following *associativity* result:

Corollary 2.3.8 ([FHM25b, Proposition 2.2.9]). Let X be a weak ω -category and $p: x \rightarrow y$, $q: y \rightarrow z$, and $r: z \rightarrow w$ be n -cells of X with $n \geq 1$. Then we have $(p *_{n-1}^X q) *_{n-1}^X r \sim p *_{n-1}^X (q *_{n-1}^X r)$. ┘

Note that Proposition 2.3.7 shows that any two parallel (in the sense of being induced from parallel n -cells of $\mathbf{L1}$) composites of a pasting diagram in a weak ω -category are equivalent up to an equivalence cell.

Remark 2.3.9. Proposition 2.3.7 is the uniqueness part of the pasting theorem promised in Section 1.2.1; it states precisely that, in a weak ω -category, any two ways of composing a given n -dimensional pasting diagram yield the same composite up to an equivalence $(n+1)$ -cell, as long as they agree on how to compose the boundary. However, it is possible to give a different formulation of the uniqueness, namely as the contractibility of a suitable ‘space’ of composites.

Recall that, given a weak ω -category X and 0-cells $x, y \in X_0$, we may construct the hom weak ω -category $X(x, y)$ (Fact 1.2.4.1). It is easy to see that if (the underlying globular map of) a strict ω -functor $f: X \rightarrow Y$ has the right lifting property with respect to $\iota^k: \partial G^k \rightarrow G^k$ for some $k \geq 1$, then the induced map $f_{x,y}: X(x, y) \rightarrow Y(fx, fy)$ has the right lifting property with respect to $\iota^{k-1}: \partial G^{k-1} \rightarrow G^{k-1}$.

Now, let X be a weak ω -category, $n \geq 1$, $\mathbf{k}$ be an n -dimensional pasting scheme, $\mathbf{u}$ a pasting diagram of shape $\mathbf{k}$ in X , and $\phi, \phi' \in (\mathbf{L1})_{n-1}$ parallel cells of arity $\partial_{n-1}^{\mathbf{T1}}(\mathbf{k})$. Then we can construct the following *trivial fibration* in the sense of Example 1.3.3 (that is, a strict ω -functor whose underlying globular map has the right lifting property with respect to $\iota^k: \partial G^k \rightarrow G^k$ for all $k \geq 0$) by repeatedly taking the hom weak ω -categories:

$$\mathbf{LX} \left((\phi, s_{n-1}^{\mathbf{TX}}(\mathbf{u})), (\phi', t_{n-1}^{\mathbf{TX}}(\mathbf{u})) \right) \rightarrow \mathbf{TX} \left(s_{n-1}^{\mathbf{TX}}(\mathbf{u}), t_{n-1}^{\mathbf{TX}}(\mathbf{u}) \right).$$

The codomain is a strict ω -category (by Fact 1.2.4.1 and Remark 1.2.3.11), so there is a strict ω -functor from the terminal weak ω -category into it that picks out the 0-cell $\mathbf{u}$. The pullback in $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$ of the resulting cospan is then the weak ω -category of possible composites of $\mathbf{u}$ satisfying the boundary conditions specified by ϕ and ϕ' . Since it admits a trivial fibration to the terminal weak ω -category, one can reasonably call it a *contractible space*; at least it is a weak ω -groupoid (i.e., a weak ω -category in which every cell of dimension ≥ 1 is an equivalence cell) by Propositions 2.3.3 and 2.3.4. ┘

2.4. The main theorem

The main theorem of this chapter is the following: if (X, ξ) is a weak ω -category and $(\phi, \mathbf{u})$ is an n -cell of $\mathbf{LX}$ such that all n -cells of X appearing in the pasting diagram $\mathbf{u}$ are equivalences in X , then the n -cell $\xi(\phi, \mathbf{u})$ is also an equivalence in X . Notice that the special case of this claim where $\mathbf{u}$ does not contain any n -cell of X , is precisely Proposition 2.3.5. However, since the n -cell $(\phi, \mathbf{u})$ is not an equivalence in $\mathbf{LX}$ whenever $\mathbf{u}$ is non-degenerate, we will need more discussion in the general case.

Definition 2.4.1. Let $\mathbf{k}$ be a pasting scheme of dimension n and rank r . Let $0 \leq i \leq r$ and suppose $k_i = n$:

$$\mathbf{k} = \begin{bmatrix} k_0 & \dots & n & \dots & k_r \\ & \underline{k}_1 & \dots & \underline{k}_i & \dots & \underline{k}_r \end{bmatrix}$$

We write $\delta^i(\mathbf{k})$ for the n -dimensional pasting scheme defined by

$$\delta^i(\mathbf{k}) = \mu_1^{\mathbf{T}} \left(\begin{bmatrix} [k_0] & \dots & [n-1]^{(n)} & \dots & [k_r] \\ & [\underline{k}_1] & \dots & [\underline{k}_i] & \dots & [\underline{k}_r] \end{bmatrix} \right).$$

Explicitly, we have the following description of $\delta^i(\mathbf{k})$.

- If $i > 0$ and $\underline{k}_i = n - 1$, then $\delta^i(\mathbf{k})$ is obtained from $\mathbf{k}$ by removing k_i and $\underline{k}_i$.
- If $i < r$ and $\underline{k}_{i+1} = n - 1$, then $\delta^i(\mathbf{k})$ is obtained from $\mathbf{k}$ by removing k_i and $\underline{k}_{i+1}$.
- Otherwise, $\delta^i(\mathbf{k})$ is obtained from $\mathbf{k}$ by replacing $k_i = n$ by $n - 1$.

(Note that, although it is possible for $\mathbf{k}$ to satisfy the premises of both the first and second clauses, the two definitions of $\delta^i(\mathbf{k})$ agree in that case.)

Let $\mathbf{u}$ be a pasting diagram of shape $\mathbf{k}$ in a weak ω -category X . Suppose that there exists $x \in X_{n-1}$ such that $u_i = \text{id}_n^X(x)$:

$$\mathbf{u} = \begin{bmatrix} u_0 & \dots & \text{id}_n^X(x) & \dots & u_r \\ \underline{u}_1 & \dots & \underline{u}_i & \dots & \underline{u}_r \end{bmatrix}.$$

We write $\delta^i(\mathbf{u})$ for the pasting diagram of shape $\delta^i(\mathbf{k})$ in X defined by

$$\delta^i(\mathbf{u}) = \mu_X^T \left(\begin{bmatrix} [u_0] & \dots & [x]^{(n)} & \dots & [u_r] \\ [\underline{u}_1] & \dots & [\underline{u}_i] & \dots & [\underline{u}_r] \end{bmatrix} \right)$$

Explicitly, we have the following description of $\delta^i(\mathbf{u})$.

- If $i > 0$ and $\underline{k}_i = n - 1$, then $\delta^i(\mathbf{u})$ is obtained from $\mathbf{u}$ by removing u_i and $\underline{u}_i = x$.
- If $i < r$ and $\underline{k}_{i+1} = n - 1$, then $\delta^i(\mathbf{u})$ is obtained from $\mathbf{u}$ by removing u_i and $\underline{u}_{i+1} = x$.
- Otherwise, $\delta^i(\mathbf{u})$ is obtained from $\mathbf{u}$ by replacing u_i by x .

(Note that, although it is possible for $\mathbf{u}$ to satisfy the premises of both the first and second clauses, the two definitions of $\delta^i(\mathbf{u})$ agree in that case.)

Let ϕ be an n -cell of $\mathbf{L1}$ with $\text{ar}(\phi) = \mathbf{k}$ as above. We write $\delta^i(\phi)$ for the n -cell of $\mathbf{L1}$ defined as $\delta^i(\phi) = \bar{\kappa}(\langle s_{n-1}^{\mathbf{L1}}(\phi), t_{n-1}^{\mathbf{L1}}(\phi) \rangle, \delta^i(\mathbf{k}))$. $\lrcorner$

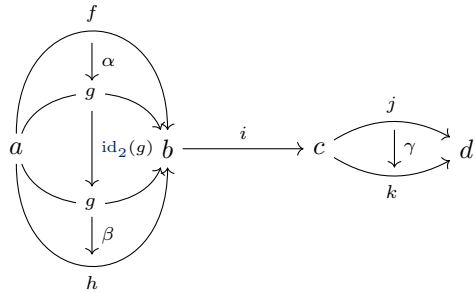
Example 2.4.2. Let

$$\mathbf{k} = \begin{bmatrix} 2 & & 2 & & 2 & & 1 & & 2 \\ & 1 & & 1 & & 0 & & 0 & \end{bmatrix},$$

and suppose we have a 2-dimensional pasting diagram

$$\mathbf{u} = \begin{bmatrix} \alpha & & \text{id}_2(g) & & \beta & & i & & \gamma \\ & g & & g & & b & & c \end{bmatrix}$$

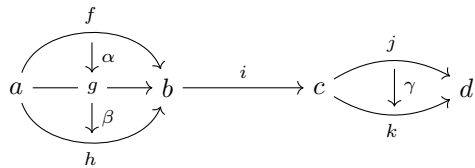
of shape $\mathbf{k}$ in a weak ω -category X , which may be depicted as


(2.4.2.1)

Then we have

$$\delta^1(\mathbf{u}) = \begin{bmatrix} \alpha & & \beta & & i & & \gamma \\ & g & & b & & c \end{bmatrix},$$

which may be depicted as


(2.4.2.2) $\lrcorner$

The following lemma formalises the idea that composites of diagrams such as (2.4.2.1) and (2.4.2.2) in a weak ω -category should be equivalent up to an equivalence cell.

Proposition 2.4.3. Let $n \geq 1$, $\mathbf{k}$ be a pasting scheme of dimension n and rank r , $X = (X, \xi)$ be a weak ω -category, and $(\phi, \mathbf{u}) \in (\mathbf{L}X)_n$ be an n -cell such that $\text{ar}(\phi) = \mathbf{k}$. Then we have

$$\xi(\phi, \mathbf{u}) \sim \xi(\delta^i(\phi), \delta^i(\mathbf{u}))$$

in X . ┘

Proof. By the assumption, we have

$$\mathbf{k} = \begin{bmatrix} k_0 & \dots & n & \dots & k_r \\ & \underline{k}_1 & \dots & \underline{k}_i & \underline{k}_{i+1} & \dots & \underline{k}_r \end{bmatrix} \in (\mathbf{T}1)_n$$

and

$$\mathbf{u} = \begin{bmatrix} u_0 & \dots & \text{id}_n^X(x) & \dots & u_r \\ & \underline{u}_1 & \dots & \underline{u}_i & \underline{u}_{i+1} & \dots & \underline{u}_r \end{bmatrix} \in (\mathbf{T}X)_n.$$

Define a pasting diagram $\tilde{\mathbf{u}}$ in $\mathbf{L}X$ by

$$\tilde{\mathbf{u}} = \begin{bmatrix} (\tilde{e}_{k_0}, [u_0]) & \dots & (\text{id}_n^{\mathbf{L}1}(\tilde{e}_{n-1}), [x]) & \dots & (\tilde{e}_{k_r}, [u_r]) \\ (\tilde{e}_{k_1}, [\underline{u}_1]) & \dots & (\tilde{e}_{k_i}, [\underline{u}_i]) & \dots & (\tilde{e}_{k_{i+1}}, [\underline{u}_{i+1}]) & \dots & (\tilde{e}_{k_r}, [\underline{u}_r]) \end{bmatrix}.$$

Since $\tilde{\mathbf{u}}$ is also of shape $\mathbf{k}$, we obtain $(\phi, \tilde{\mathbf{u}}) \in (\mathbf{L}X)_n$. By $\xi(\tilde{e}_{k_j}, [u_j]) = u_j$, $\xi(\tilde{e}_{k_j}, [\underline{u}_j]) = \underline{u}_j$, and $\xi(\text{id}_n^{\mathbf{L}1}(\tilde{e}_{n-1}), [x]) = \text{id}_n^X(x)$, we have

$$(\mathbf{L}\xi)(\phi, \tilde{\mathbf{u}}) = (\phi, (\mathbf{T}\xi)(\tilde{\mathbf{u}})) = (\phi, \mathbf{u}) \in (\mathbf{L}X)_n.$$

Let us calculate $\mu_X^{\mathbf{L}}(\phi, \tilde{\mathbf{u}}) \in (\mathbf{L}X)_n$. To this end, we decompose $\tilde{\mathbf{u}} \in (\mathbf{T}\mathbf{L}X)_n$ into

$$\chi = \begin{bmatrix} \tilde{e}_{k_0} & \dots & \text{id}_n^{\mathbf{L}1}(\tilde{e}_{n-1}) & \dots & \tilde{e}_{k_r} \\ \tilde{e}_{k_1} & \dots & \tilde{e}_{k_i} & \dots & \tilde{e}_{k_{i+1}} & \dots & \tilde{e}_{k_r} \end{bmatrix} \in (\mathbf{T}\mathbf{L}X)_n$$

and

$$\bar{\mathbf{u}} = \begin{bmatrix} [u_0] & \dots & [x]^{(n)} & \dots & [u_r] \\ [\underline{u}_1] & \dots & [\underline{u}_i] & \dots & [\underline{u}_{i+1}] & \dots & [\underline{u}_r] \end{bmatrix} \in (\mathbf{T}\mathbf{T}X)_n.$$

(See also Remark 1.2.3.2.(v).) Then we have

$$\mu_X^{\mathbf{L}}(\phi, \tilde{\mathbf{u}}) = (\mu_1^{\mathbf{L}}(\phi, \chi), \mu_X^{\mathbf{T}}(\bar{\mathbf{u}})) = (\mu_1^{\mathbf{L}}(\phi, \chi), \delta^i(\mathbf{u}))$$

On the other hand, the n -cells $\mu_1^{\mathbf{L}}(\phi, \chi)$ and $\delta^i(\phi) = \bar{\kappa}(\langle s_{n-1}^{\mathbf{L}1}(\phi), t_{n-1}^{\mathbf{L}1}(\phi) \rangle, \delta^i(\mathbf{k}))$ of $\mathbf{L}1$ are parallel. This is because

$$\begin{aligned} s_{n-1}^{\mathbf{L}1}(\mu_1^{\mathbf{L}}(\phi, \chi)) &= \mu_1^{\mathbf{L}}(s_{n-1}^{\mathbf{L}1}(\phi), s_{n-1}^{\mathbf{T}\mathbf{L}1}(\chi)) \\ &= \mu_1^{\mathbf{L}}(s_{n-1}^{\mathbf{L}1}(\phi), (\mathbf{T}\eta_1^{\mathbf{L}})(\text{ar}(s_{n-1}^{\mathbf{L}1}(\phi)))) \\ &= \mu_1^{\mathbf{L}} \circ (\mathbf{L}\eta_1^{\mathbf{L}})(s_{n-1}^{\mathbf{L}1}(\phi)) \\ &= s_{n-1}^{\mathbf{L}1}(\phi) \end{aligned}$$

and similarly $t_{n-1}^{\mathbf{L}1}(\mu_1^{\mathbf{L}}(\phi, \chi)) = t_{n-1}^{\mathbf{L}1}(\phi)$. Hence by Proposition 2.3.7, we have

$$\begin{aligned} \xi(\phi, \mathbf{u}) &= \xi \circ (\mathbf{L}\xi)(\phi, \tilde{\mathbf{u}}) \\ &= \xi \circ \mu_X^{\mathbf{L}}(\phi, \tilde{\mathbf{u}}) \\ &= \xi(\mu_1^{\mathbf{L}}(\phi, \chi), \delta^i(\mathbf{u})) \\ &\sim \xi(\delta^i(\phi), \delta^i(\mathbf{u})). \end{aligned} \quad \square$$

As a special case of Proposition 2.4.3, we have the following ‘ordinary’ unit law.

Corollary 2.4.4 ([FHM25b, Proposition 2.2.9]). Let X be a weak ω -category and $u: x \rightarrow y$ be an n -cell of X with $n \geq 1$. Then we have $\text{id}_n^X(x) *_{n-1}^X u \sim u \sim u *_{n-1}^X \text{id}_n^X(y)$. ┘

Definition 2.4.5. Let $\mathbf{k}$ be a pasting scheme of dimension n and rank r , and let $\mathbf{u}$ be a pasting diagram of shape $\mathbf{k}$ in a globular set X . By the set of full-dimensional labels in $\mathbf{u}$, we mean the set

$$\text{fdl}_n(\mathbf{u}) = \{u_i \mid 0 \leq i \leq r, \quad k_i = n\}$$

We sometimes omit the subscript n when the dimension specified on $\mathbf{u}$ is clear from the context. ┘

Definition 2.4.6. Let $S \subseteq X$ be a set of cells of a weak ω -category X (i.e., it consists of $S_n \subseteq X_n$ for each $n \geq 0$ subject to no conditions). By the *set of S -labelled pastings*, we mean the set

$$\text{pst}^X(S) = \{ \xi(\phi, \mathbf{u}) \mid n \geq 0, \quad (\phi, \mathbf{u}) \in (\mathbf{L}X)_n, \quad \text{fdl}_n(\mathbf{u}) \subseteq S \}$$

of cells of X . We often omit the superscript X when the weak ω -category is clear from the context. $\lrcorner$

Note that the set $\text{pst}(S)$ may contain n -cells in S whiskered with m -cells not in S with $m < n$. Using pst , we can now state our main theorem as follows.

Theorem 2.4.7. Let $X = (X, \xi)$ be a weak ω -category and let $\nu\Phi$ be the set of all equivalence cells in X . Then we have $\text{pst}(\nu\Phi) \subseteq \nu\Phi$. $\lrcorner$

Remark 2.4.8 (cf. [FHM25a, Theorem 2.4.8]). In more explicit terms, Theorem 2.4.7 is stated as follows. Let $X = (X, \xi)$ be a weak ω -category and let $(\phi, \mathbf{u}) \in (\mathbf{L}X)_n$ with $n \geq 1$ and

$$\text{ar}(\phi) = \begin{bmatrix} k_0 & \cdots & k_r \\ \underline{k}_1 & \cdots & \underline{k}_r \end{bmatrix}.$$

Suppose that, for each $i \leq r$ with $k_i = n$, the n -cell u_i appearing in $\mathbf{u}$ is invertible. Then $\xi(\phi, \mathbf{u})$ is invertible. $\lrcorner$

We need some definitions for the proof of Theorem 2.4.7.

Definition 2.4.9. For each $n \geq 0$ and each n -dimensional pasting scheme $\mathbf{k}$ of rank r , define a natural number

$$\|\mathbf{k}\|^{(n)} = |\{ i \leq r \mid k_i = n \}|,$$

which is the number of occurrences of n in the upper row of $\mathbf{k}$. $\lrcorner$

Notice that $\mathbf{k}$ is non-degenerate as an n -cell of $\mathbf{T1}$ if and only if $\|\mathbf{k}\|^{(n)} > 0$.

Definition 2.4.10. Let X be a weak ω -category and $S \subseteq X$ be a set of cells of X . Given any $n \geq 1$ and any n -cell $u: x \rightarrow y$ of X , an n -cell $v: y \rightarrow x$ of X is called an *S -inverse* of u if there exist $(n+1)$ -cells $p: u *_{n-1} v \rightarrow \text{id}_n(x)$ and $q: v *_{n-1} u \rightarrow \text{id}_n(y)$ in S .

Let $n \geq 1$, $(\phi, \mathbf{u}) \in (\mathbf{L}X)_n$, and $\mathbf{k} = \text{ar}(\phi)$. An *S -inverse instruction* of $(\phi, \mathbf{u})$ is an n -cell $(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$ satisfying the following conditions.

- $s_{n-1}^{\text{L1}}(\phi^{\text{inv}}) = t_{n-1}^{\text{L1}}(\phi)$,
- $t_{n-1}^{\text{L1}}(\phi^{\text{inv}}) = s_{n-1}^{\text{L1}}(\phi)$, and
- $\mathbf{u}^{\text{inv}} \in (\mathbf{T}X)_n$ is obtained from $\mathbf{u}$ by replacing, for each $(n-1)$ -transversal component $0 \leq i \leq j \leq r$ of $\mathbf{k}$, the corresponding segment

$$\begin{bmatrix} u_i & \cdots & u_j \\ \underline{u}_{i+1} & \cdots & \underline{u}_j \end{bmatrix},$$

with

$$\begin{bmatrix} v_i & \cdots & v_j \\ \underline{u}_{i+1} & \cdots & \underline{u}_j \end{bmatrix},$$

where v_l is an S -inverse of u_l for each $i \leq l \leq j$. $\lrcorner$

Note that u admits an S -inverse if and only if $u \in \Phi(S)$, and $(\phi, \mathbf{u})$ admits an S -inverse instruction if and only if $\text{fdl}(\mathbf{u}) \subseteq \Phi(S)$. Also note that the shape of $\mathbf{u}^{\text{inv}}$ is the same as that of $\mathbf{u}$ (recall that the cell $\underline{u}_l$ is of dimension $(n-1)$ for each $i+1 \leq l \leq j$), and that the types of $(\phi, \mathbf{u})$ and $(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$ in $\mathbf{L}X$ are related by

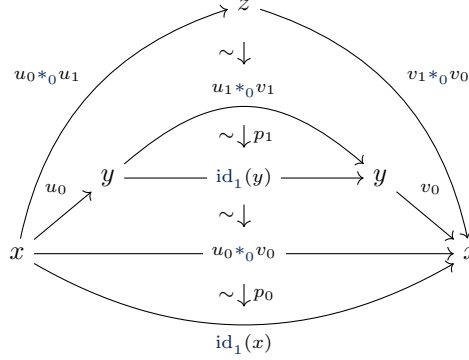
$$\begin{array}{ccc} s_{n-1}^{\text{LX}}(\phi, \mathbf{u}) & \xrightarrow{(\phi, \mathbf{u})} & t_{n-1}^{\text{LX}}(\phi, \mathbf{u}) \\ \parallel & & \parallel \\ t_{n-1}^{\text{LX}}(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) & \xleftarrow{(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})} & s_{n-1}^{\text{LX}}(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}). \end{array} \quad (2.4.10.1)$$

(Here we are not claiming that this diagram ‘commutes’.)

Although the proof of Theorem 2.4.7 is rather long, the underlying idea is simple; we show that any n -cell of the form $\xi(\phi, \mathbf{u})$ with $\text{fdl}(\mathbf{u}) \subseteq \nu\Phi$ admits an $(\nu\Phi)$ -inverse, namely $\xi(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$, where $(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$ is an $\nu\Phi$ -inverse instruction of $(\phi, \mathbf{u})$ (whose existence follows from $\text{fdl}(\mathbf{u}) \subseteq \nu\Phi = \Phi(\nu\Phi)$). The non-trivial part is constructing equivalence $(n+1)$ -cells witnessing the invertibility of $\xi(\phi, \mathbf{u})$. Even in the simplest case of $u_0 *_{\mathbf{0}} u_1$, where the 1-cells $u_0: x \rightarrow y$ and $u_1: y \rightarrow z$ admit inverses $v_0: y \rightarrow x$ and $v_1: z \rightarrow y$ respectively, connecting the composite $(u_0 *_{\mathbf{0}} u_1) *_{\mathbf{0}} (v_1 *_{\mathbf{0}} v_0)$ to $\text{id}_1(x)$ requires

- rebracketing the expression (using coherence) so that we are composing u_1 and v_1 first,
- whiskering with u_0 and v_0 the 2-cell p_1 witnessing the invertibility of u_1 ,
- applying the unit law (Proposition 2.4.3) to obtain $u_0 *_0 v_0$ (getting rid of an extra identity in the middle), and
- using the 2-cell p_0 witnessing the invertibility of u_0 .

The following diagram illustrates the situation.



Moreover, we must show that the resulting 2-cell is itself invertible. In the actual proof, this last part is treated by considering certain (pre-)fixed points of pst .

When connecting $(u_0 *_0 u_1) *_0 (v_1 *_0 v_0)$ to $\text{id}_1(x)$ in this example, we first cancelled out just one of the inverse pairs (namely u_1 and v_1) and obtained $u_0 *_0 v_0$. In the general case, writing out such intermediate composites can be rather cumbersome. This is our motivation for introducing the following notation, which allows us to write e.g.

$$\delta_+^1 \left(\begin{bmatrix} u_0 & & u_1 \\ & y & \\ & & \end{bmatrix} \right) = \begin{bmatrix} u_0 \\ & & \end{bmatrix} \quad \text{and} \quad \delta_-^0 \left(\begin{bmatrix} v_1 & & v_0 \\ & y & \\ & & \end{bmatrix} \right) = \begin{bmatrix} v_0 \\ & & \end{bmatrix}.$$

Definition 2.4.11. Let $\mathbf{k}$ be a pasting scheme of dimension n and rank r . Let $0 \leq i \leq r$ and suppose $k_i = n$:

$$\mathbf{k} = \begin{bmatrix} k_0 & \dots & n & \dots & k_r \\ & \underline{k}_1 & \dots & \underline{k}_i & \underline{k}_{i+1} & \dots & \underline{k}_r \end{bmatrix}.$$

Recall the pasting scheme $\delta^i(\mathbf{k})$ of Definition 2.4.1. Let $\mathbf{u}$ be a pasting diagram of shape $\mathbf{k}$ in a weak ω -category X .

- (i) Suppose that either $i = r$ or $\underline{k}_{i+1} < n - 1$ holds. We write $\delta_+^i(\mathbf{u})$ for the following pasting diagram of shape $\delta^i(\mathbf{k})$ in X .
 - If $i > 0$ and $\underline{k}_i = n - 1$, then $\delta_+^i(\mathbf{u})$ is obtained from $\mathbf{u}$ by removing u_i and $\underline{u}_i$.
 - Otherwise, $\delta_+^i(\mathbf{u})$ is obtained from $\mathbf{u}$ by replacing u_i by $s_{n-1}^X(u_i)$.
- (ii) Suppose that either $i = 0$ or $\underline{k}_i < n - 1$ holds. We write $\delta_-^i(\mathbf{u})$ for the following pasting diagram of shape $\delta^i(\mathbf{k})$ in X .
 - If $i < r$ and $\underline{k}_{i+1} = n - 1$, then $\delta_-^i(\mathbf{u})$ is obtained from $\mathbf{u}$ by removing u_i and $\underline{u}_{i+1}$.
 - Otherwise, $\delta_-^i(\mathbf{u})$ is obtained from $\mathbf{u}$ by replacing u_i by $t_{n-1}^X(u_i)$. ⌋

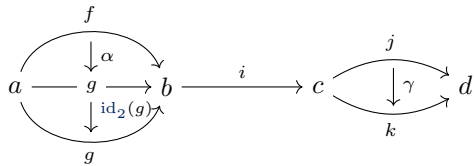
Example 2.4.12. For

$$\mathbf{k} = \begin{bmatrix} 2 & & 2 & & 2 & & 1 & & 2 \\ & 1 & & 1 & & 0 & & 0 & \\ & & & & & & & & \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} \alpha & & \text{id}_2(g) & & \beta & & i & & \gamma \\ & g & & & g & & b & & c \end{bmatrix}$$

of Example 2.4.2, we have

$$\delta_+^2(\mathbf{u}) = \begin{bmatrix} \alpha & & \text{id}_2(g) & & i & & c & & \gamma \\ & g & & & b & & & & \end{bmatrix},$$

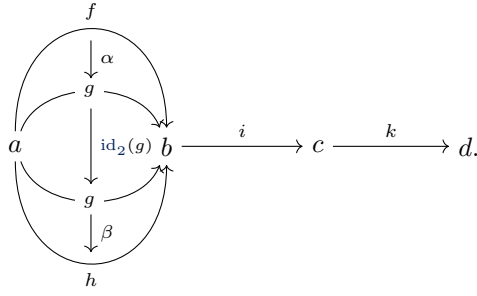
which may be depicted as



and

$$\delta_-^4(\mathbf{u}) = \begin{bmatrix} \alpha & & \text{id}_2(g) & & \beta & & i & & k \\ & g & & & g & & b & & c \end{bmatrix},$$

which may be depicted as



Proof of Theorem 2.4.7. Observe that pst defines a monotone map on $\wp(\coprod_{n \geq 0} X_n)$, and by Proposition 1.4.5, it suffices to show that $\text{Pre}_{\nu\Phi}(\text{pst})$ is closed under Φ . Note that since $S \subseteq \text{pst}(S)$ holds for any $S \subseteq \coprod_{n \geq 0} X_n$, all pre-fixed points of pst are in fact fixed points.

To this end, take $S \in \text{Pre}_{\nu\Phi}(\text{pst})$. Then $\Phi(S)$ contains all equivalence cells since $\nu\Phi = \Phi(\nu\Phi) \subseteq \Phi(S)$. Thus it suffices to show that $\text{pst}(\Phi(S)) \subseteq \Phi(S)$ holds. Since every cell in $\text{pst}(\Phi(S))$ is of the form $\xi(\phi, \mathbf{u})$ for some pasting instruction $(\phi, \mathbf{u})$ (of dimension ≥ 1) that admits an S -inverse instruction, we would want to prove that:

$$\left\{ \begin{array}{l} \text{for each } n \geq 1, \text{ each } (\phi, \mathbf{u}) \in (\text{LX})_n, \text{ and each } S\text{-inverse instruction } (\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) \text{ of } (\phi, \mathbf{u}), \text{ there} \\ \text{exist } (n+1)\text{-cells} \\ \xi(\phi, \mathbf{u}) *_{n-1}^X \xi(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) \rightarrow \text{id}_n^X (\text{s}_{n-1}^X \xi(\phi, \mathbf{u})) \\ \text{and} \\ \xi(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) *_{n-1}^X \xi(\phi, \mathbf{u}) \rightarrow \text{id}_n^X (\text{t}_{n-1}^X \xi(\phi, \mathbf{u})) \\ \text{in } S \text{ (or equivalently in } \text{pst}(S)). \end{array} \right. \quad (2.4.12.1)$$

We shall do so by induction on $\|\text{ar}(\phi)\|^{(n)}$.

The base case $\|\text{ar}(\phi)\|^{(n)} = 0$ is covered by Proposition 2.3.5; recall that $\nu\Phi \subseteq S$. So fix $N \geq 0$ and suppose that (2.4.12.1) holds for each $n \geq 1$ and $(\phi, \mathbf{u}), (\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) \in (\text{LX})_n$ with $\|\text{ar}(\phi)\|^{(n)} \leq N$. Take $(\phi, \mathbf{u}) \in (\text{LX})_n$ with $\|\text{ar}(\phi)\|^{(n)} = N+1$ and its S -inverse instruction $(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$.

A sketch of the rest of the proof is as follows. We shall show that there exist

- a sequence of $(n+1)$ -cells in S from $\xi(\phi, \mathbf{u}) *_{n-1}^X \xi(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$ to $\text{id}_n^X (\text{s}_{n-1}^X \xi(\phi, \mathbf{u}))$, and
- a sequence of $(n+1)$ -cells in S from $\xi(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) *_{n-1}^X \xi(\phi, \mathbf{u})$ to $\text{id}_n^X (\text{t}_{n-1}^X \xi(\phi, \mathbf{u}))$.

Notice that this suffices because $S = \text{pst}(S)$ is closed under compositions. We only carry out the construction of the former sequence. We eventually obtain the sequence

$$\begin{aligned} & \xi(\phi, \mathbf{u}) *_{n-1}^X \xi(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) \\ & \parallel \\ & \xi((\phi, \mathbf{u}) *_{n-1}^{\text{LX}} (\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})) \\ & \sim \downarrow w_1 \\ & \xi(\phi', \mathbf{u}^*) \\ & \downarrow w_2 = \xi(\phi'', \mathbf{u}^{\text{p}}) \\ & \xi(\phi', \mathbf{u}^{\text{id}}) \\ & \sim \downarrow w_3 \\ & \xi(\delta^{\bar{j}}(\phi'), \delta^{\bar{j}}(\mathbf{u}^{\text{id}})) \\ & \sim \downarrow w_4 \\ & \xi((\delta^{\bar{j}}(\phi), \delta^{\bar{j}}_+(\mathbf{u})) *_{n-1}^{\text{LX}} (\delta^{\bar{i}}(\phi^{\text{inv}}), \delta^{\bar{i}}_-(\mathbf{u}^{\text{inv}}))) \\ & \parallel \\ & \xi(\delta^{\bar{j}}(\phi), \delta^{\bar{j}}_+(\mathbf{u})) *_{n-1}^X \xi(\delta^{\bar{i}}(\phi^{\text{inv}}), \delta^{\bar{i}}_-(\mathbf{u}^{\text{inv}})) \\ & \downarrow w_5 \\ & \text{id}_n^X (\text{s}_{n-1}^X \xi(\phi, \mathbf{u})). \end{aligned} \quad (2.4.12.2)$$

Here, $\bar{i}$ and $\bar{j}$ are suitable integers for which the cells indicated (such as $\delta^{\bar{j}}_+(\mathbf{u})$) are well-defined, and the two equations hold because ξ is a strict ω -functor from LX to X . The $(n+1)$ -cells w_1 , w_3 , and w_4 are equivalence cells induced by coherence (Proposition 2.3.7), and hence they are in $\nu\Phi \subseteq S$. The $(n+1)$ -cells w_2 and w_5 , on

the other hand, are cells in $S = \text{pst}(S)$, where the latter is produced by the induction hypothesis (2.4.12.1) with respect to N .

Let

$$\text{ar}(\phi) = \mathbf{k} = \begin{bmatrix} k_0 & \cdots & k_r \\ & \underline{k}_1 & \cdots & \underline{k}_r \end{bmatrix}$$

Since $(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$ is an S -inverse instruction of $(\phi, \mathbf{u})$, for each $0 \leq i \leq r$ with $k_i = n$, we have $(n+1)$ -cells

$$p_i: u_i *_{n-1}^X v_i \rightarrow \text{id}_n^X(s_{n-1}^X(u_i)) \quad \text{and} \quad q_i: v_i *_{n-1}^X u_i \rightarrow \text{id}_n^X(t_{n-1}^X(u_i)) \quad (2.4.12.3)$$

in S . Also notice that $(\phi, \mathbf{u})$ and $(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$ are n -cells in LX composable along the $(n-1)$ -dimensional boundary (see (2.4.10.1)), and hence the first equality

$$\xi(\phi, \mathbf{u}) *_{n-1}^X \xi(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}) = \xi((\phi, \mathbf{u}) *_{n-1}^{LX} (\phi^{\text{inv}}, \mathbf{u}^{\text{inv}}))$$

in (2.4.12.2) indeed makes sense (and it holds by Lemma 1.2.3.3 because ξ is a strict ω -functor).

The n -cell $(\phi, \mathbf{u}) *_{n-1}^{LX} (\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})$ of LX is equal to $(\phi *_{n-1}^{L1} \phi^{\text{inv}}, \mathbf{u} *_{n-1}^{TX} \mathbf{u}^{\text{inv}})$, since both $L! : LX \rightarrow L1$ and $\text{ar}_X : LX \rightarrow TX$ are strict ω -functors.

As a pasting diagram, $\mathbf{u} *_{n-1}^{TX} \mathbf{u}^{\text{inv}}$ is obtained from $\mathbf{u}$ by replacing, for each $(n-1)$ -transversal component $0 \leq i \leq j \leq r$ of $\mathbf{k}$, the corresponding subsequence

$$\begin{bmatrix} u_i & \cdots & u_j \\ & \underline{u}_{i+1} & \cdots & \underline{u}_j \end{bmatrix}$$

with

$$\begin{bmatrix} u_i & \cdots & u_j & v_j & \cdots & v_i \\ & \underline{u}_{i+1} & \cdots & \underline{u}_j & t_{n-1}(u_j) & \underline{u}_j & \cdots & \underline{u}_{i+1} \end{bmatrix}.$$

Now let $0 \leq \bar{i} \leq \bar{j} \leq r$ be the leftmost (or first)³ $(n-1)$ -transversal component of $\mathbf{k}$. Let $\mathbf{k}' = \delta^{\bar{j}}(\mathbf{k} *_{n-1}^{T1} \mathbf{k})$; that is, $\mathbf{k}'$ is the pasting scheme obtained from $\mathbf{k} *_{n-1}^{T1} \mathbf{k}$ by removing one n in the top row and one $n-1$ in the bottom row from the segment corresponding to the first $(n-1)$ -transversal component of $\mathbf{k}$. Let $\phi' = \bar{\kappa}(\langle s_{n-1}^{L1}(\phi), s_{n-1}^{L1}(\phi) \rangle, \mathbf{k}')$ and let $\mathbf{u}^*$ be the pasting diagram of shape $\mathbf{k}'$ obtained from

$$\mathbf{u} *_{n-1}^{TX} \mathbf{u}^{\text{inv}} = \begin{bmatrix} u_0 & \cdots & u_{\bar{j}} & v_{\bar{j}} & \cdots \\ & \underline{u}_1 & \cdots & \underline{u}_{\bar{j}} & t_{n-1}^X(u_{\bar{j}}) & \underline{u}_{\bar{j}} & \cdots \end{bmatrix}$$

(see Remark 1.2.3.12) by replacing the segment

$$\begin{bmatrix} u_{\bar{j}} & v_{\bar{j}} \\ & t_{n-1}^X(u_{\bar{j}}) \end{bmatrix}$$

with

$$\begin{bmatrix} u_{\bar{j}} *_{n-1}^X v_{\bar{j}} \end{bmatrix};$$

that is,

$$\mathbf{u}^* = \begin{bmatrix} u_0 & \cdots & u_{\bar{j}} *_{n-1}^X v_{\bar{j}} & \cdots \\ & \underline{u}_1 & \cdots & \underline{u}_{\bar{j}} & \underline{u}_{\bar{j}} & \cdots \end{bmatrix}.$$

Let $\tilde{\mathbf{u}}$ be the pasting diagram of shape $\mathbf{k}'$ in LX defined as

$$\tilde{\mathbf{u}} = \begin{bmatrix} (\tilde{e}_{k_0}, [u_0]) & \cdots & (\tilde{e}_n *_{n-1}^{L1} \tilde{e}_n, [u_{\bar{j}}] *_{n-1}^{TX} [v_{\bar{j}}]) & \cdots \\ (\tilde{e}_{k_1}, [\underline{u}_1]) & \cdots & (\tilde{e}_{k_{\bar{j}}}, [\underline{u}_{\bar{j}}]) & \cdots \end{bmatrix},$$

where

$$\begin{bmatrix} u_{\bar{j}} \end{bmatrix} *_{n-1}^{TX} \begin{bmatrix} v_{\bar{j}} \end{bmatrix} = \begin{bmatrix} u_{\bar{j}} & v_{\bar{j}} \\ & t_{n-1}^X(u_{\bar{j}}) \end{bmatrix} \in (TX)_n.$$

Notice that $(\phi', \tilde{\mathbf{u}})$ is an n -cell of LLX , and that we have

$$(L\xi)(\phi', \tilde{\mathbf{u}}) = (\phi', (T\xi)(\tilde{\mathbf{u}})) = (\phi', \mathbf{u}^*).$$

Next we calculate $\mu_X^L(\phi', \tilde{\mathbf{u}})$. To this end, we decompose $\tilde{\mathbf{u}}$ into

$$\chi = \begin{bmatrix} \tilde{e}_{k_0} & \cdots & \tilde{e}_n *_{n-1}^{L1} \tilde{e}_n & \cdots \\ & \tilde{e}_{k_1} & \cdots & \tilde{e}_{k_{\bar{j}}} & \tilde{e}_{k_{\bar{j}}} & \cdots \end{bmatrix} \in (TL1)_n$$

and

$$\bar{\mathbf{u}} = \begin{bmatrix} [u_0] & \cdots & [u_{\bar{j}}] *_{n-1}^{TX} [v_{\bar{j}}] & \cdots \\ & [\underline{u}_1] & \cdots & [\underline{u}_{\bar{j}}] & [\underline{u}_{\bar{j}}] & \cdots \end{bmatrix} \in (TTX)_n.$$

³One can choose any $(n-1)$ -transversal component here, but this choice simplifies the indices involved.

Then we have

$$\begin{aligned}\mu_X^L(\phi', \tilde{\mathbf{u}}) &= (\mu_1^L(\phi', \chi), \mu_X^T(\bar{\mathbf{u}})) \\ &= (\mu_1^L(\phi', \chi), \mathbf{u}_{n-1}^{TX} \mathbf{u}^{\text{inv}}) \in (LX)_n.\end{aligned}$$

Now, the n -cells $\mu_1^L(\phi', \chi)$ and $\phi_{n-1}^{L1} \phi^{\text{inv}}$ of $L1$ are parallel. This is because

$$\begin{aligned}s_{n-1}^{L1}(\mu_1^L(\phi', \chi)) &= \mu_1^L(s_{n-1}^{L1}(\phi'), s_{n-1}^{TL1}(\chi)) \\ &= \mu_1^L(s_{n-1}^{L1}(\phi), (T\eta_1^L)(\text{ar}(s_{n-1}^{L1}(\phi)))) \\ &= \mu_1^L \circ (L\eta_1^L)(s_{n-1}^{L1}(\phi)) \\ &= s_{n-1}^{L1}(\phi) \\ &= s_{n-1}^{L1}(\phi_{n-1}^{L1} \phi^{\text{inv}})\end{aligned}\tag{Proposition 1.2.3.10}$$

and similarly

$$\begin{aligned}t_{n-1}^{L1}(\mu_1^L(\phi', \chi)) &= \mu_1^L(t_{n-1}^{L1}(\phi'), t_{n-1}^{TL1}(\chi)) \\ &= \mu_1^L(s_{n-1}^{L1}(\phi), (T\eta_1^L)(\text{ar}(s_{n-1}^{L1}(\phi)))) \\ &= \mu_1^L \circ (L\eta_1^L)(s_{n-1}^{L1}(\phi)) \\ &= s_{n-1}^{L1}(\phi) \\ &= t_{n-1}^{L1}(\phi^{\text{inv}}) \\ &= t_{n-1}^{L1}(\phi_{n-1}^{L1} \phi^{\text{inv}}).\end{aligned}\tag{Proposition 1.2.3.10}$$

Thus by Proposition 2.3.7, we have an equivalence $(n+1)$ -cell

$$w_1: \xi(\phi_{n-1}^{L1} \phi^{\text{inv}}, \mathbf{u}_{n-1}^{TX} \mathbf{u}^{\text{inv}}) \rightarrow \xi(\mu_1^L(\phi', \chi), \mathbf{u}_{n-1}^{TX} \mathbf{u}^{\text{inv}})$$

in X . Here, the n -dimensional domain of w_1 is

$$\xi(\phi_{n-1}^{L1} \phi^{\text{inv}}, \mathbf{u}_{n-1}^{TX} \mathbf{u}^{\text{inv}}) = \xi((\phi, \mathbf{u}) *_{n-1} LX(\phi^{\text{inv}}, \mathbf{u}^{\text{inv}})),$$

whereas the n -dimensional codomain of w_1 is

$$\begin{aligned}\xi(\mu_1^L(\phi', \chi), \mathbf{u}_{n-1}^{TX} \mathbf{u}^{\text{inv}}) &= \xi \circ \mu_X^L(\phi', \tilde{\mathbf{u}}) \\ &= \xi \circ (L\xi)(\phi', \tilde{\mathbf{u}}) \\ &= \xi(\phi', \mathbf{u}^*),\end{aligned}$$

as indicated in (2.4.12.2).

To define w_2 , let $\mathbf{k}''$ be the pasting scheme obtained from $\mathbf{k}'$ by replacing the middle n in the first $(n-1)$ -transversal component by $n+1$, and let $\phi'' = \kappa(\langle \phi', \phi' \rangle, \mathbf{k}'')$. Combining this ϕ'' with the pasting diagram

$$\mathbf{u}^{\mathbf{p}} = \begin{bmatrix} u_0 & \cdots & p_{\bar{j}} & \cdots \\ & \underline{u}_1 & \cdots & \underline{u}_{\bar{j}} & \underline{u}_{\bar{j}} & \cdots \end{bmatrix},$$

which is obtained from $\mathbf{u}^*$ by replacing $u_{\bar{j}} *_{n-1}^X v_{\bar{j}}$ with the $(n+1)$ -cell

$$p_{\bar{j}}: u_{\bar{j}} *_{n-1}^X v_{\bar{j}} \rightarrow \text{id}_n^X(s_{n-1}^X(u_{\bar{j}}))$$

in S as in (2.4.12.3), we obtain

$$w_2 = \xi(\phi'', \mathbf{u}^{\mathbf{p}}): \xi(\phi', \mathbf{u}^*) \rightarrow \xi(\phi', \mathbf{u}^{\text{id}})$$

in $\text{pst}(S)$, where

$$\mathbf{u}^{\text{id}} = \begin{bmatrix} u_0 & \cdots & \text{id}_n^X(s_{n-1}^X(u_{\bar{j}})) & \cdots \\ & \underline{u}_1 & \cdots & \underline{u}_{\bar{j}} & \underline{u}_{\bar{j}} & \cdots \end{bmatrix}.$$

By Proposition 2.4.3, we obtain an equivalence $(n+1)$ -cell

$$w_3: \xi(\phi', \mathbf{u}^{\text{id}}) \rightarrow \xi(\delta^{\bar{j}}(\phi'), \delta^{\bar{j}}(\mathbf{u}^{\text{id}}))$$

in X .

Next, to define w_4 , we show that the n -cells $\delta^{\bar{J}}(\phi')$ and $\delta^{\bar{J}}(\phi) *_{n-1}^{\text{L1}} \delta^{\bar{J}}(\phi^{\text{inv}})$ of L1 are parallel. Indeed, we have

$$\begin{aligned} s_{n-1}^{\text{L1}}(\delta^{\bar{J}}(\phi')) &= s_{n-1}^{\text{L1}}(\phi') \\ &= s_{n-1}^{\text{L1}}(\phi) \\ &= s_{n-1}^{\text{L1}}(\delta^{\bar{J}}(\phi)) \\ &= s_{n-1}^{\text{L1}}(\delta^{\bar{J}}(\phi) *_{n-1}^{\text{L1}} \delta^{\bar{J}}(\phi^{\text{inv}})) \end{aligned} \quad (\text{Proposition 1.2.3.10})$$

and

$$\begin{aligned} t_{n-1}^{\text{L1}}(\delta^{\bar{J}}(\phi')) &= t_{n-1}^{\text{L1}}(\phi') \\ &= s_{n-1}^{\text{L1}}(\phi) \\ &= t_{n-1}^{\text{L1}}(\phi^{\text{inv}}) \\ &= t_{n-1}^{\text{L1}}(\delta^{\bar{J}}(\phi^{\text{inv}})) \\ &= t_{n-1}^{\text{L1}}(\delta^{\bar{J}}(\phi) *_{n-1}^{\text{L1}} \delta^{\bar{J}}(\phi^{\text{inv}})). \end{aligned} \quad (\text{Proposition 1.2.3.10})$$

Moreover, we have

$$\delta^{\bar{J}}(\mathbf{u}^{\text{id}}) = \delta_+^{\bar{J}}(\mathbf{u}) *_{n-1}^{\text{TX}} \delta_-^{\bar{J}}(\mathbf{u}^{\text{inv}}).$$

Therefore by Proposition 2.3.7, we obtain an equivalence $(n+1)$ -cell

$$w_4: \xi(\delta^{\bar{J}}(\phi'), \delta^{\bar{J}}(\mathbf{u}^{\text{id}})) \rightarrow \xi(\delta^{\bar{J}}(\phi) *_{n-1}^{\text{L1}} \delta^{\bar{J}}(\phi^{\text{inv}}), \delta_+^{\bar{J}}(\mathbf{u}) *_{n-1}^{\text{TX}} \delta_-^{\bar{J}}(\mathbf{u}^{\text{inv}}))$$

in X . The n -dimensional codomain of w_4 is

$$\xi(\delta^{\bar{J}}(\phi) *_{n-1}^{\text{L1}} \delta^{\bar{J}}(\phi^{\text{inv}}), \delta_+^{\bar{J}}(\mathbf{u}) *_{n-1}^{\text{TX}} \delta_-^{\bar{J}}(\mathbf{u}^{\text{inv}})) = \xi((\delta^{\bar{J}}(\phi), \delta_+^{\bar{J}}(\mathbf{u})) *_{n-1}^{\text{LX}} (\delta^{\bar{J}}(\phi^{\text{inv}}), \delta_-^{\bar{J}}(\mathbf{u}^{\text{inv}}))),$$

as indicated in (2.4.12.2).

The second equation in (2.4.12.2) holds since ξ is a strict ω -functor.

Since $\|\text{ar}(\delta^{\bar{J}}(\phi))\|^{(n)} = N$ and $(\delta^{\bar{J}}(\phi^{\text{inv}}), \delta_-^{\bar{J}}(\mathbf{u}^{\text{inv}}))$ is an S -inverse instruction of $(\delta^{\bar{J}}(\phi), \delta_+^{\bar{J}}(\mathbf{u}))$, we obtain an $(n+1)$ -cell

$$w_5: \xi(\delta^{\bar{J}}(\phi), \delta_+^{\bar{J}}(\mathbf{u})) *_{n-1}^X \xi(\delta^{\bar{J}}(\phi^{\text{inv}}), \delta_-^{\bar{J}}(\mathbf{u}^{\text{inv}})) \rightarrow \text{id}_n^X(s_{n-1}^X \xi(\delta^{\bar{J}}(\phi), \delta_+^{\bar{J}}(\mathbf{u})))$$

in S , by the induction hypothesis (2.4.12.1). Because we have

$$s_{n-1}^{\text{L1}}(\delta^{\bar{J}}(\phi)) = s_{n-1}^{\text{L1}}(\phi) \quad \text{and} \quad s_{n-1}^{\text{TX}}(\delta_+^{\bar{J}}(\mathbf{u})) = s_{n-1}^{\text{TX}}(\mathbf{u}),$$

the n -dimensional codomain of w_5 is $\text{id}_n^X(s_{n-1}^X \xi(\phi, \mathbf{u}))$, as indicated in (2.4.12.2). $\square$

Applying Remark 2.4.8 to the case of $\phi = \tilde{e}_n *_{n-1}^{\text{L1}} \tilde{e}_n$, we conclude $\sim$ is transitive. Together with Proposition 2.3.4, we obtain the following corollary.

Corollary 2.4.13. Let X be a weak ω -category. Then $\sim$ is an equivalence relation on the set of cells of X . $\lrcorner$

Moreover, if we consider the case when $\phi = \tilde{e}_{n+1} *_{n-1}^{\text{L1}} \tilde{e}_{n+1}$, we concludes that $\sim$ is a *congruence* with respect to the composition operation:

Corollary 2.4.14 (see [LMW10, Proposition 4.4] for the strict case). Let X be a weak ω -category. Suppose that $u, u': x \rightarrow y$ and $v, v': y \rightarrow z$ are n -cells of X with $n \geq 1$. If $u \sim u'$ and $v \sim v'$, then we have $u *_{n-1}^X v \sim u' *_{n-1}^X v'$. $\lrcorner$

Inverses are unique up to $\sim$:

Corollary 2.4.15. Let X be a weak ω -category, $n \geq 1$, $u: x \rightarrow y$ be an equivalence n -cell in X , and $v, v': y \rightarrow x$ be inverses of u . Then we have $v \sim v'$. $\lrcorner$

Proof. By Corollary 2.4.4, we have $v \sim v *_{n-1} \text{id}_n(x)$. Thus we have

$$\begin{aligned} v &\sim v *_{n-1} \text{id}_n(x) \\ &\sim v *_{n-1} (u *_{n-1} v') \\ &\sim (v *_{n-1} u) *_{n-1} v' \\ &\sim \text{id}_n(y) *_{n-1} v' \\ &\sim v'. \end{aligned} \quad (\text{Corollary 2.3.8})$$

$\square$

Finally for this section, we show that invertibility is invariant under $\sim$:

Corollary 2.4.16. Let X be a weak ω -category, $n \geq 1$, and $u, v: x \rightarrow y$ be a parallel pair of n -cells in X such that $u \sim v$. Suppose that u is invertible. Then v is invertible too. $\lrcorner$

Proof. Since u is invertible, it has an inverse w with $u *_{n-1} w \sim \text{id}_n(x)$ and $w *_{n-1} u \sim \text{id}_n(y)$. Thus we have

$$v *_{n-1} w \sim u *_{n-1} w \sim \text{id}_n(x)$$

and similarly $w *_{n-1} v \sim \text{id}_n(y)$. $\square$

2.5. The core weak ω -groupoid of a weak ω -category

We conclude this chapter with the construction of the *core weak ω -groupoid* of a weak ω -category.

Definition 2.5.1. Let X be a weak ω -category and $n \geq 0$. An n -cell $x \in X_n$ is *hereditarily invertible* if either

- $n = 0$, or
- $n \geq 1$, x is invertible, and $s_{n-1}^X(x)$ and $t_{n-1}^X(x)$ are hereditarily invertible.

By definition, the set of all hereditarily invertible cells of X is closed under s_n^X and t_n^X , and hence forms a globular subset $\mathbf{k}(X)$ of X . $\lrcorner$

Moreover, $\mathbf{k}(X)$ is closed under pasting:

Proposition 2.5.2. Let $X = (X, \xi)$ be a weak ω -category, $n \in \mathbb{N}$, and $(\phi, \mathbf{u}) \in (\mathbf{L}(\mathbf{k}(X)))_n \subseteq (\mathbf{L}X)_n$. Then $\xi(\phi, \mathbf{u}) \in \mathbf{k}(X)_n$. Therefore, $\mathbf{k}(X)$ is also a weak ω -category, which is a weak ω -subcategory (i.e, subobject in $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$) of X . $\lrcorner$

Note that, since $\mathbf{L}$ preserves pullbacks, it also preserves monomorphisms. Therefore it makes sense to regard $\mathbf{L}(\mathbf{k}(X))$ as a globular subset of $\mathbf{L}X$.

Proof. We prove this by induction on n . The base case $n = 0$ is clear, so let $m \geq 1$ and suppose that the claim holds when $n = m - 1$. Take any $(\phi, \mathbf{u}) \in (\mathbf{L}(\mathbf{k}(X)))_{m-1}$ and let $x = \xi(\phi, \mathbf{u})$. Then x is invertible by Theorem 2.4.7. Moreover, $s_{m-1}^X(x) = \xi(s_{m-1}^{\mathbf{L}\mathbf{1}}(\phi), s_{m-1}^{\mathbf{T}X}(\mathbf{u}))$ is hereditarily invertible by the induction hypothesis since $(s_{m-1}^{\mathbf{L}\mathbf{1}}(\phi), s_{m-1}^{\mathbf{T}X}(\mathbf{u})) \in (\mathbf{L}(\mathbf{k}(X)))_{m-1}$. Similarly, $t_{m-1}^X(x)$ is hereditarily invertible. $\square$

We define a *weak ω -groupoid* to be a weak ω -category in which every cell of dimension ≥ 1 is invertible, or equivalently, every cell is hereditarily invertible.

Let $\mathbf{Wk}\text{-}\omega\text{-Gpd}_s$ be the full subcategory of $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$ consisting of all weak ω -groupoids. Since every strict ω -functor preserves (hereditarily) invertible cells, we see that any strict ω -functor $X \rightarrow Y$ from a weak ω -groupoid X to a weak ω -category Y factors through the inclusion $\mathbf{k}(Y) \rightarrow Y$. Since $\mathbf{k}(Y)$ is a weak ω -groupoid, $\mathbf{k}: \mathbf{Wk}\text{-}\omega\text{-Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-Gpd}_s$ is the right adjoint of the inclusion $\mathbf{Wk}\text{-}\omega\text{-Gpd}_s \hookrightarrow \mathbf{Wk}\text{-}\omega\text{-Cat}_s$.

Remark 2.5.3. We remark that $\mathbf{k}$ also gives rise to the right adjoint of the inclusion functor $\mathbf{Wk}\text{-}\omega\text{-Gpd}_w \rightarrow \mathbf{Wk}\text{-}\omega\text{-Cat}_w$, where $\mathbf{Wk}\text{-}\omega\text{-Cat}_w$ is the category (defined in [Gar10] and Definition 3.3.1.4) of weak ω -categories and *weak ω -functors*, and $\mathbf{Wk}\text{-}\omega\text{-Gpd}_w$ is the full subcategory of $\mathbf{Wk}\text{-}\omega\text{-Cat}_w$ consisting of all weak ω -groupoids. To show this, it is essentially enough to observe that weak ω -functors ‘preserve’ invertible cells (and hence also hereditarily invertible ones). However, the latter statement perhaps needs some clarification.

We recall that if X and Y are weak ω -categories, then a weak ω -functor $X \rightsquigarrow Y$ defined in [Gar10] and later in Definition 3.3.1.4 comes equipped with a span

$$\begin{array}{ccc} & \mathbf{Q}X & \\ \varepsilon_X \swarrow & & \searrow f \\ X & & Y \end{array} \quad (2.5.3.1)$$

of strict ω -functors, where ε_X is moreover a *trivial fibration* (Example 1.3.3) in the sense that its underlying map between globular sets has the right lifting property with respect to $\iota^n: \partial G^n \rightarrow G^n$ for all $n \geq 0$. ($\mathbf{Q}$ is in fact a comonad on $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$, and $\mathbf{Wk}\text{-}\omega\text{-Cat}_w$ is defined as the Kleisli category of $\mathbf{Q}$.) Since every trivial fibration is contractible, we see by Propositions 2.3.1 and 2.3.3 that for any equivalence n -cell u of X , every n -cell $\bar{u}$ of $\mathbf{Q}X$ such that $\varepsilon_X(\bar{u}) = u$ is invertible, and so is the n -cell $f(\bar{u})$ of Y . (Note also that for any cell u of X there exists some cell $\bar{u}$ of $\mathbf{Q}X$ with $\varepsilon_X(\bar{u}) = u$, since a trivial fibration is surjective.) This is what we mean by ‘weak ω -functors preserve equivalence cells’.

In order to show that k is the right adjoint of the inclusion $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Gpd}_w \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w$, observe that if X is a weak ω -groupoid, then so is QX (by the presence of a trivial fibration ε_X), and hence the right leg f of the span (2.5.3.1) factors through $k(Y)$. Therefore the weak ω -functors $X \rightarrow Y$ correspond to the weak ω -functors $X \rightarrow k(Y)$. $\lrcorner$

A completely parallel argument works more generally for (∞, n) -categories with $n \in \mathbb{N}$ in place of weak ω -groupoids, where we say that a weak ω -category is an (∞, n) -category if every cell of dimension $> n$ is invertible. Thus we can also obtain the *core* (∞, n) -category of any weak ω -category.

CHAPTER 3

ω -weak equivalences between weak ω -categories

3.1. Introduction

In the previous chapter (which is based on [FHM24]), we studied equivalence cells (or weakly invertible cells) in a weak ω -category. The purpose of this chapter is to investigate equivalence ω -functors *between* weak ω -categories.

Although one might be tempted to define the equivalence ω -functors between weak ω -categories as the weakly invertible 1-cells in a putative ‘weak ω -category of all (small) weak ω -categories,’ the latter has not yet been defined; hence, we adopt a more hands-on definition in this chapter. We first define *ω -weak equivalences* as the strict ω -functors between weak ω -categories which are *essentially surjective* in every dimension. One of the main purposes of this chapter is to prove that these ω -weak equivalences enjoy the *2-out-of-3 property* (Theorem 3.2.3.3); that is, given a composable pair $f: X \rightarrow Y$, $g: Y \rightarrow Z$ of strict ω -functors between weak ω -categories, if any two of f , g , and gf are ω -weak equivalences then so is the third. We believe that this result is a necessary step towards the construction of a (left semi-)model structure on the category of weak ω -categories and strict ω -functors, in which the weak equivalences are the ω -weak equivalences.

In the case of *strict* ω -categories, such a model structure was constructed by Lafont, Métayer, and Worytkiewicz [LMW10]. In particular, they established the 2-out-of-3 property of the ω -weak equivalences between *strict* ω -categories; our result is its generalisation to the context of *weak* ω -categories. Although our proof is partly inspired by theirs, we also face a brand new kind of complication that arises from the fact that we are dealing with *weak* ω -categories. For example, there is a seemingly innocuous step in their argument which uses the fact that a 2-cell w (or more generally an n -cell w , with $n \geq 2$, whose 0-source is x) in a strict ω -category, as in the left diagram below, is the same as the composite $\text{id}_1(x) * w$, as in the right diagram below.

$$\begin{array}{ccc}
 x & \begin{array}{c} \xrightarrow{u} \\ \Downarrow w \\ \xrightarrow{v} \end{array} & y \\
 & & \\
 x & \xrightarrow{\text{id}_1(x)} x & \begin{array}{c} \xrightarrow{u} \\ \Downarrow w \\ \xrightarrow{v} \end{array} y
 \end{array}$$

When generalising such an argument to the context of *weak* ω -categories, it is natural to attempt to formalise the intuition that $\text{id}_1(x) * w$ is still ‘essentially just w ’. However, in doing so, one readily realises that $w: u \rightarrow v$ and $\text{id}_1(x) * w: \text{id}_1(x) * u \rightarrow \text{id}_1(x) * v$ may not even be parallel, which makes unclear what the formal statement should be.

In order to deal with this sort of problem, we introduce the *padding* construction (Construction 3.2.2.4). Armed with a kind of naturality (Lemma 3.2.2.8), this construction allows us to go back and forth between non-parallel composites arising from the same pasting diagram. Since ‘applying’ the unit, associativity, or interchange law to compositions of high codimension (i.e., composition of n -cells along k -cells for $k < n - 1$) can always result in such non-parallel composites, we believe the padding construction to be of interest even beyond the scope of this chapter.

After establishing the 2-out-of-3 property, we extend the scope of study to the more general class of *weak ω -weak equivalences*, where the first ‘weak’ indicates that these maps are not necessarily strictly ω -functorial. The notion of weak ω -functor that we adopt is due to Garner [Gar10], which is essentially a span of strict ω -functors with the left-pointing leg a trivial fibration. In addition to generalising the 2-out-of-3 property to the larger class (Proposition 3.3.2.5), we characterise the weak ω -weak equivalences in terms of the essential ω -surjectivity of their ‘underlying’ globular maps (Proposition 3.3.3.3), establishing an analogy with the accepted notions of ‘equivalence’ in low-dimensional cases such as biequivalences and triequivalences.

Related work. At the conference *(∞, n)-Categories and Their Applications* held in Utrecht in April 2024, where Yuki Maehara presented this work, we learnt from Clémence Chanavat that a similar technique to our padding construction was being developed independently in the setting of diagrammatic sets. This work later appeared in her joint paper with Amar Hadzihanovic [CH24a].

Outline of the chapter. We introduce the notion of ω -weak equivalence and prove that they enjoy the 2-out-of-3 property in Section 3.2. This result is extended to weakly ω -functorial maps in Section 3.3.

Remark 3.1.0.1 ([FHM25a, Remark 2.4.11]). There are two kinds of approaches to infinite-dimensional categorical structures, namely *inductive* and *coinductive* ones. The purpose of this remark is to address why we adopt the latter instead of the former in this chapter; if the reader is not already familiar with this dichotomy, they may safely skip to the next section.

The main difference between the two kinds of approaches is that, in an inductive setting, the notion of ‘invertible cell’ is a primitive one. Implemented in our setting (analogously to the case of *strict* ω -categories treated in [HL25]), this translates to considering pairs (X, E) where X is a weak ω -category in the sense of Definition 1.2.2.8 and $E \subseteq \coprod_{n \geq 1} X_n$ is a set of *marked* cells (see also Section 4.3.2), which are to be thought of as ‘abstract invertible cells’. Of course the set E must satisfy some conditions in order to present a reasonable notion of invertibility (cf. Remark 4.3.2.2), and our best guess for the ‘correct’ conditions (cf. [HL25, Definition 2.15 and Theorem 3.38]) is that

- (i) E is ‘closed under all pastings’ in the sense analogous to Theorem 2.4.7, and
- (ii) $\Phi^X(E) = E$ holds.

Note that (3.1.0.1.(i)) in particular implies that all identities and coherence cells (as in Proposition 2.3.7) are always marked. Both directions of (3.1.0.1.(ii)) are non-trivial, where $E \subseteq \Phi^X(E)$ says that each marked cell is coinductively invertible (with all witness cells also marked), and the inclusion $\Phi^X(E) \subseteq E$ may be regarded as a sort of *completeness* (as in Segal spaces) or *saturation* (as in complicial sets).

In fact, we suspect that it is possible to prove (though with additional non-trivial work) marked analogues of the results in Section 3.2 and possibly those in Section 3.3 after making the following modifications.

- The relation ‘ $x \sim y$ ’ should be interpreted as ‘there exists a *marked* (as opposed to *invertible*) cell $x \rightarrow y$ ’.
- In Definition 3.2.1.3, define an ω -weak equivalence to be a marking-preserving strict ω -functor that is essentially ω -surjective (in the sense of Definition 3.2.1.1 but with the modified meaning of ‘ $x \sim y$ ’) and further reflects marked cells (cf. [HL25, Proposition 3.33] and Proposition 3.2.1.8).
- In Example 1.3.3, replace the adjunction ((1.3.3.1)) with the suitable marked variant, add to $\mathcal{I}_o$ the inclusion of the standard (unmarked) n -globe into the standard marked n -globe for $n \geq 1$, and add their images under the left adjoint to $\mathcal{I}$. (These modifications make sure that the corresponding ‘trivial fibrations’ reflect marked cells.)

The caveat is that, if one is to actually pursue the inductive approach, one must consider pairs (X, E) which do not necessarily satisfy the condition (3.1.0.1.(ii)) in order to obtain a well-behaved category so that e.g., the adjunction in the third bullet point above actually exists. (Whether one chooses to impose (3.1.0.1.(i)) does not matter too much in this respect.) It is expected that one can then single out those pairs that do satisfy ((3.1.0.1.(i)) and) (3.1.0.1.(ii)) as the fibrant objects using a left semi-model structure analogous to that in [HL25, Theorem 2.43]. Since our arguments below rely on (3.1.0.1.(ii)) (or more precisely, an analogue of *prefibrancy* in the sense of [HL25, Definition 3.18]), the best we can prove in the marked setting is the 2-out-of-3 property only for ω -weak equivalences between putative *fibrant* objects (cf. [HL25, Proposition 3.33]). In fact, for the above definition of ω -weak equivalence, all three parts of the 2-out-of-3 property ((A), (B), and (C) in the next section) can be shown to fail between not-necessarily-fibrant marked weak ω -categories; it seems to us that the ‘correct’ notion of weak equivalence should be defined, and also the 2-out-of-3 property for that class should be proved, using a more abstract means as done in [HL25]. Thus, instead of generalising our work to include such partial results, we simply leave a detailed development of the inductive approach for future work. $\lrcorner$

3.2. The 2-out-of-3 property for ω -weak equivalences

In this section, we define and study the class of ω -weak equivalences between weak ω -categories. In particular, we show that this class enjoys the *2-out-of-3 property*; that is, for strict ω -functors $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ between weak ω -categories, if two of f , g , and gf are ω -weak equivalences, then so is the third. In other words, it is the conjunction of the following three statements.

- (A) If f and g are ω -weak equivalences, then so is gf .
- (B) If g and gf are ω -weak equivalences, then so is f .
- (C) If f and gf are ω -weak equivalences, then so is g .

After defining ω -weak equivalences, we prove the easy two thirds ((A) and (B)) of the 2-out-of-3 property in Section 3.2.1. In order to prove the other third (C) of the 2-out-of-3 property, we study whiskering maps in a weak ω -category in Section 3.2.2, which culminates in a proof of a key intermediate result (Theorem 3.2.2.12). Using this, we complete our proof of the 2-out-of-3 property in Section 3.2.3. Finally, we mention some additional properties of ω -weak equivalences in Section 3.2.4.

3.2.1. Essential ω -surjectivity and ω -weak equivalences. The following definition concerns arbitrary globular maps between weak ω -categories (and not just strict ω -functors) because we shall apply it also to e.g. *whiskering maps*, which are not strict ω -functors in general.

Definition 3.2.1.1 (cf. [BS10, Definition 4]). Let X and Y be weak ω -categories and $f: X \rightarrow Y$ be a globular map between the underlying globular sets of X and Y .

- f is *essentially 0-surjective* if for each $y \in Y_0$ there exists $x \in X_0$ such that $fx \sim y$.
- f is *essentially $(n+1)$ -surjective* ($n \in \mathbb{N}$) if f is essentially 0-surjective and for each $x, x' \in X_0$, the induced globular map $f_{x,x'}: X(x, x') \rightarrow Y(fx, fx')$ is essentially n -surjective.
- f is *essentially ω -surjective* if it is essentially n -surjective for all $n \in \mathbb{N}$. $\lrcorner$

Remark 3.2.1.2. Unravelling the induction, a globular map $f: X \rightarrow Y$ between weak ω -categories is essentially n -surjective ($n \geq 0$) if and only if

- for each $y \in Y_0$, there exists $x \in X_0$ such that $fx \sim y$, and
- for any $1 \leq k \leq n$, any parallel pair (u, v) of $(k-1)$ -cells in X , and any k -cell $w: fu \rightarrow fv$ in Y , there exists a k -cell $\bar{w}: u \rightarrow v$ in X such that $f\bar{w} \sim w$. (The case $k = 2$ may be visualised as below.)

$$\begin{array}{ccc}
 X & \begin{array}{c} \xrightarrow{u} \\ \Downarrow \bar{w} \\ \xrightarrow{v} \end{array} & \\
 \downarrow f & & \\
 Y & \begin{array}{c} \xrightarrow{fu} \\ \Downarrow f\bar{w} \sim w \\ \xrightarrow{fv} \end{array} &
 \end{array} \tag{3.2.1.2.1}$$

For example, if we regard a functor $f: X \rightarrow Y$ between ordinary categories as a strict ω -functor between weak ω -categories (here we define the underlying globular set of a category X by setting $s_k^X, t_k^X: X_{k+1} \rightarrow X_k$ to be the identity function for all $k \geq 1$ (see [Lei04, Theorem 9.3.17])), then f is

- essentially 0-surjective if and only if it is essentially surjective on objects (in the usual sense),
- essentially 1-surjective if and only if it is essentially surjective on objects and full, and
- essentially 2-surjective if and only if it is essentially surjective on objects, full, and faithful (because the existence of w as above is equivalent to $fu = fv$, and similarly for $\bar{w}$), if and only if it is essentially ω -surjective.

Definition 3.2.1.3. An ω -weak equivalence is an essentially ω -surjective strict ω -functor between weak ω -categories. $\lrcorner$

It is easy to see from the above rephrasing of essential n -surjectivity that, when restricted to those between *strict* ω -categories, our definition of ω -weak equivalence recovers precisely [LMW10, Definition 4.7].

Notice that a globular map $f: X \rightarrow Y$ between weak ω -categories is essentially ω -surjective if and only if it is essentially 0-surjective and the induced globular map $f_{x,x'}: X(x, x') \rightarrow Y(fx, fx')$ is essentially ω -surjective for each $x, x' \in X_0$. The following is an immediate consequence of this fact.

Proposition 3.2.1.4. A strict ω -functor $f: X \rightarrow Y$ between weak ω -categories is an ω -weak equivalence if and only if it is essentially 0-surjective and the induced strict ω -functor $f_{x,x'}: X(x, x') \rightarrow Y(fx, fx')$ is an ω -weak equivalence for each $x, x' \in X_0$. $\lrcorner$

Remark 3.2.1.5. The statement of Proposition 3.2.1.4 can be taken as an alternative, *coinductive* definition of ω -weak equivalences (see Section 1.4 for details of coinduction). More precisely, define the monotone map

$$\Phi: \wp(\text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s)) \rightarrow \wp(\text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s))$$

on the powerset lattice $\wp(\text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s))$ of the set $\text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s)$ of all strict ω -functors between weak ω -categories as follows. It maps each $S \subseteq \text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s)$ to the set $\Phi(S)$ of all strict ω -functors which are essentially 0-surjective and locally in S , i.e.,

$$\Phi(S) = \left\{ (f: X \longrightarrow Y) \in \text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s) \mid f \text{ is essentially 0-surjective and} \right. \\ \left. \text{we have } (f_{x,x'}: X(x, x') \longrightarrow Y(fx, fx')) \in S \text{ for each } x, x' \in X_0 \right\}.$$

Then the set E of all ω -weak equivalences is the largest (post-)fixed point $\nu\Phi$ of Φ . Indeed, Proposition 3.2.1.4 says that we have $E = \Phi(E)$, which implies $E \subseteq \nu\Phi$. On the other hand, given any $S \subseteq \text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s)$ with $S \subseteq \Phi(S)$, it is easy to see that all strict ω -functors in S are essentially n -surjective, by induction on $n \in \mathbb{N}$. Thus we have $S \subseteq E$, and hence E is the largest (post-)fixed point $\nu\Phi$ of Φ . $\lrcorner$

We now prove two thirds ((A) and (B)) of the 2-out-of-3 property. We prove these under rather weak assumptions, partly to illuminate the properties we actually use, and partly because we use some of the extra generality later.

The following implies (A) of the 2-out-of-3 property by Proposition 2.3.1.

Lemma 3.2.1.6. Let $n \in \mathbb{N}$, X, Y, Z be weak ω -categories, $f: X \longrightarrow Y$ be a globular map, and $g: Y \longrightarrow Z$ be a globular map preserving equivalence cells. If f and g are essentially n -surjective, then so is gf . $\lrcorner$

Proof. We prove this by induction on n .

The base case $n = 0$ is shown as follows. Take any $z \in Z_0$. Then since g is essentially 0-surjective, there exists $y \in Y_0$ with $gy \sim z$. Since f is essentially 0-surjective, there exists $x \in X_0$ with $fx \sim y$. Then we have $gfx \sim gy \sim z$ since g preserves equivalence cells.

For the inductive step, let $n > 0$. For each $x, x' \in X_0$, we have $(gf)_{x,x'} = g_{fx,fx'} \circ f_{x,x'}$. Here, $f_{x,x'}$ and $g_{fx,fx'}$ are essentially $(n-1)$ -surjective, and $g_{fx,fx'}$ preserves equivalence cells. Hence $(gf)_{x,x'}$ is essentially $(n-1)$ -surjective by the inductive hypothesis. $\square$

The following implies (B) of the 2-out-of-3 property, by Proposition 3.2.1.8 below.

Lemma 3.2.1.7. Let $n \in \mathbb{N}$, X, Y, Z be weak ω -categories, $f: X \longrightarrow Y$ be a globular map, and $g: Y \longrightarrow Z$ be a globular map reflecting equivalence cells. If g is essentially $(n+1)$ -surjective and gf is essentially n -surjective, then f is essentially n -surjective. $\lrcorner$

Proof. We prove this by induction on n .

The base case $n = 0$ is shown as follows. Take any $y \in Y_0$. Then we obtain $gy \in Z_0$. Since gf is essentially 0-surjective, there exist $x \in X_0$ and an equivalence 1-cell $u: gfx \rightarrow gy$ in Z . Since g is essentially 1-surjective, there exists a 1-cell $\bar{u}: fx \rightarrow y$ in Y with $g\bar{u} \sim u$. By Corollary 2.4.16, $g\bar{u}$ is invertible. Since g reflects equivalence cells, $\bar{u}$ is invertible. This shows f is essentially 0-surjective.

For the inductive step, let $n > 0$. For each $x, x' \in X_0$, we have $(gf)_{x,x'} = g_{fx,fx'} \circ f_{x,x'}$. Here, $g_{fx,fx'}$ is essentially n -surjective and reflects equivalence cells, and $(gf)_{x,x'}$ is essentially $(n-1)$ -surjective. Hence $f_{x,x'}$ is essentially $(n-1)$ -surjective by the inductive hypothesis. $\square$

Proposition 3.2.1.8. Any ω -weak equivalence reflects equivalence cells. More generally, let $f: X \longrightarrow Y$ be a globular map between (the underlying globular sets of) weak ω -categories X and Y . Suppose that

- (i) f is essentially ω -surjective,
- (ii) for each n -cell x in X with $n \geq 0$, there exists an equivalence $(n+2)$ -cell $\text{id}_{n+1}^Y(fx) \sim f(\text{id}_{n+1}^X(x))$ in Y , and
- (iii) for each pair of n -cells $u: x \rightarrow x'$ and $u': x' \rightarrow x''$ in X composable along the $(n-1)$ -dimensional boundary, with $n \geq 1$, there exists an equivalence $(n+1)$ -cell $f(u *_{n-1}^X u') \sim fu *_{n-1}^Y fu'$ in Y .

Then f reflects equivalence cells. $\lrcorner$

Proof. Let $f: X \longrightarrow Y$ be a globular map between weak ω -categories satisfying (i)–(iii). Let

$$W = \left\{ u \in \coprod_{n \geq 1} X_n \mid fu \text{ is invertible in } Y \right\},$$

which is the inverse image under f of the set of all equivalence cells in Y . It suffices to show $W \subseteq \Phi^X(W)$ (cf. Remark 2.2.2 and Proposition 2.3.3).

Let $u: x \rightarrow x'$ be an n -cell in W . Then there exist an n -cell $v: fx' \rightarrow fx$ and equivalence $(n+1)$ -cells $p: fu*_{n-1}^Y v \rightarrow \text{id}_n^Y(fx)$ and $q: v*_{n-1}^Y fu \rightarrow \text{id}_n^Y(fx')$ in Y . By (i), there exists $\bar{v}: x' \rightarrow x$ in X such that $f\bar{v} \sim v$ in Y . Let p' be a composite of

$$f(u*_{n-1}^X \bar{v}) \sim fu*_{n-1}^Y f\bar{v} \sim fu*_{n-1}^Y v \xrightarrow{p} \text{id}_n^Y(fx) \sim f(\text{id}_n^X(x))$$

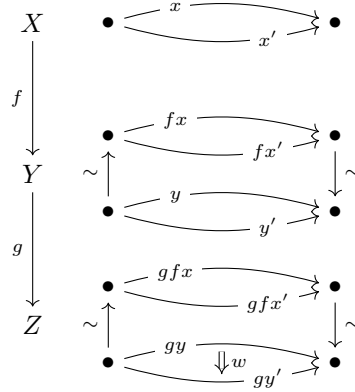
where the first and the last factors are instances of (iii) and (ii) respectively, and the second factor is invertible by Theorem 2.4.7. It follows from (i) that there exists an $(n+1)$ -cell $\bar{p}: u*_{n-1}^X \bar{v} \rightarrow \text{id}_n^X(x)$ such that $f\bar{p} \sim p'$. Note that $f\bar{p}$ is invertible in Y by Theorem 2.4.7 and Corollary 2.4.16, so $\bar{p} \in W$. Similarly, there is an $(n+1)$ -cell $\bar{q}: \bar{v}*_{n-1}^X u \rightarrow \text{id}_n^X(x')$ in W that is sent by f to a composite of

$$f(\bar{v}*_{n-1}^X u) \sim f\bar{v}*_{n-1}^Y fu \sim v*_{n-1}^Y fu \xrightarrow{q} \text{id}_n^Y(fx') \sim f(\text{id}_n^X(x'))$$

modulo the equivalence relation $\sim$. This completes the proof. $\square$

We conclude this subsection with an informal explanation of why the remaining third (C) of the 2-out-of-3 property is much more difficult to prove than (A) and (B).

Remark 3.2.1.9. The difficulty of proving (C) lies in the fact that ω -weak equivalences are defined in terms of *essential* surjectivity rather than actual surjectivity (the latter gives the definition of *trivial fibrations* (Example 1.3.3)). In particular, the ω -weak equivalence f may not be surjective on the nose, and one is led to consider the following sort of diagram in an attempt to prove (C):



That is, starting with a parallel pair of $(n-1)$ -cells (y, y') in Y and an n -cell $w: gy \rightarrow gy'$ in Z , one would hope to use the assumption that gf is an ω -weak equivalence to obtain some $v: y \rightarrow y'$ with $gv \sim w$. However, this is not immediately possible since (y, y') may not be of the form (fx, fx') for any parallel pair of $(n-1)$ -cells (x, x') in X . Instead, the assumption that f is an ω -weak equivalence yields cylinder-like diagrams as above.

In the case of *strict* ω -categories [LMW10], the authors construct a globular set ΓY of *cylinders* (see also Section A.4) in Y in order to prove (C).¹ We avoid the use of ΓY by essentially unravelling their *transport* construction [LMW10, Lemma 4.22] into a series of *whiskerings* (introduced in the next subsection) and interleaving them into a suitable inductive argument. This modification repackages the proof, but we still need similar ingredients to those used in [LMW10]. In particular, we need a weak ω -categorical counterpart of [LMW10, Lemma 4.6], which is Theorem 3.2.2.12 below. $\square$

3.2.2. Whiskering with an invertible cell. The purpose of this subsection is to establish a key intermediate result (Theorem 3.2.2.12) for proving (C), which concerns *whiskering maps* defined as follows.

Definition 3.2.2.1. For each weak ω -category X , object $z \in X_0$, and 1-cell $u: x \rightarrow y$ in X , the *whiskering map*

$$X(u, z) = u*_0^X(-): X(y, z) \longrightarrow X(x, z)$$

is defined by mapping each n -cell v of $X(y, z)$ to the n -cell $u*_0^X v$ of $X(x, z)$. Dually, for each object $x \in X_0$ and 1-cell $v: y \rightarrow z$ in X , we have the whiskering map

$$X(x, v) = (-)*_0^X v: X(x, y) \longrightarrow X(x, z).$$

¹They then endow it with a suitable strict ω -category structure, but this is not strictly necessary for proving (C).

Remark 3.2.2.2. Let X be a weak ω -category, $z \in X_0$, and $u: x \rightarrow y$ a 1-cell of X . Then $X(u, z): X(y, z) \rightarrow X(x, z)$ is a globular map by Proposition 1.2.3.10, but is not necessarily a strict ω -functor unless X is a *strict* ω -category.² It also preserves equivalence cells by Theorem 2.4.7 and Remark 2.2.3. $\lrcorner$

Theorem 3.2.2.12 below states that the whiskering map $X(u, z): X(y, z) \rightarrow X(x, z)$ is essentially ω -surjective whenever the 1-cell $u: x \rightarrow y$ is invertible in X . We note that its proof constitutes the most technical part of this chapter, and if the reader is willing to take this theorem for granted, they may safely skip the rest of this subsection.

The proof of Theorem 3.2.2.12 turns out to require the following notion of *essential ω -injectivity*.

Definition 3.2.2.3. Let X and Y be weak ω -categories and $f: X \rightarrow Y$ be a globular map between the underlying globular sets of X and Y .

- f is *essentially 0-injective* if, for each $x, x' \in X_0$, $fx \sim fx'$ implies $x \sim x'$.
- f is *essentially $(n+1)$ -injective* ($n \in \mathbb{N}$) if f is essentially 0-injective and for each $x, x' \in X_0$, the induced globular map $f_{x, x'}: X(x, x') \rightarrow Y(fx, fx')$ is essentially n -injective.
- f is *essentially ω -injective* if it is essentially n -injective for all $n \in \mathbb{N}$. $\lrcorner$

See Lemma 3.2.4.1 for a relationship between the notions of essential n -injectivity and essential n -surjectivity. Unravelling the induction, f is essentially n -injective ($n \geq 0$) if and only if, for any $0 \leq k \leq n$ and any parallel pair (u, v) of k -cells in X , $fu \sim fv$ in Y implies $u \sim v$ in X .

Our proof of Theorem 3.2.2.12 proceeds in the following four steps.

- (**Step 1**) Prove that $X(u, z): X(y, z) \rightarrow X(x, z)$ is essentially ω -injective in the special case $u = \text{id}_1^X(y)$.
- (**Step 2**) Extend (**Step 1**) to the general case of an arbitrary equivalence 1-cell u .
- (**Step 3**) Prove that $X(u, z): X(y, z) \rightarrow X(x, z)$ is essentially ω -surjective in the special case $u = \text{id}_1^X(y)$.
- (**Step 4**) Extend (**Step 3**) to the general case of an arbitrary equivalence 1-cell u .

Before embarking on the actual proof, let us discuss why even (**Step 1**) is non-trivial. For example, given a parallel pair of 2-cells

$$\begin{array}{ccc} & s & \\ y & \begin{array}{c} \Downarrow u \\ \Downarrow v \end{array} & z \\ & t & \end{array}$$

in a weak ω -category (X, ξ) with $\text{id}_1(y) *_0 u \sim \text{id}_1(y) *_0 v$, one must prove $u \sim v$. One may be tempted to resort to the ‘unit law’ and argue

$$u \sim \text{id}_1(y) *_0 u \sim \text{id}_1(y) *_0 v \sim v,$$

but in general $u: s \rightarrow t$ and $\text{id}_1(y) *_0 u: \text{id}_1(y) *_0 s \rightarrow \text{id}_1(y) *_0 t$ are not even parallel, so ‘ $u \sim \text{id}_1(y) *_0 u$ ’ does not make sense. (Note that Proposition 2.4.3 is not applicable here.)

A key observation here is that both u and $\text{id}_1(y) *_0 u$ can be obtained from the same pasting diagram $\begin{bmatrix} u \end{bmatrix} \in (\text{TX})_2$ by pasting according to different pasting instructions, namely $\chi = \tilde{e}_2 \in (\text{L1})_2$ and $\phi = \text{id}_1^{\text{L1}}(\tilde{e}_0) *_0^{\text{L1}} \tilde{e}_2 \in (\text{L1})_2$, respectively. (See the proof of Lemma 3.2.2.5.) The following construction, which transforms a cell of the form

$$w: \xi(\phi, \mathbf{u}) \rightarrow \xi(\phi, \mathbf{v})$$

(such as an equivalence 3-cell witnessing $\text{id}_1(y) *_0 u \sim \text{id}_1(y) *_0 v$) into one of the form

$$w': \xi(\chi, \mathbf{u}) \rightarrow \xi(\chi, \mathbf{v})$$

(such as an equivalence 3-cell witnessing $u \sim v$) by inserting ‘pads’ in each dimension, is designed to deal with this kind of situation. Note that if ϕ and χ were parallel cells in L1 , this could be achieved by means of suitable coherence cells (see Proposition 2.3.7); $\text{id}_1^{\text{L1}}(\tilde{e}_0) *_0^{\text{L1}} \tilde{e}_2$ and $\tilde{e}_2$ are not parallel, however.

Construction 3.2.2.4. Let $n \geq 1$ and suppose we are given $\phi, \chi \in (\text{L1})_n$ such that $\text{ar}(\phi) = \text{ar}(\chi) = \mathbf{k}$. Note that ϕ and χ are not necessarily parallel. Let $\phi_0 = \phi$. We will construct a sequence of n -cells $\phi_1, \dots, \phi_n$ in L1

²In fact, we suspect that $X(u, z)$ is the *underlying globular map* of a weak ω -functor in the sense of Section 3.3.3, but this is beyond the scope of this chapter.

where each ϕ_i is to be thought of as ‘ ϕ after inserting pads of dimension $\leq i$ so that it has the $(i-1)$ -boundary of χ ’. The following diagram describes ϕ_2 with $\phi, \chi \in (\mathbf{L1})_2$.

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & s_1^{\mathbf{L1}}(\chi) & & \\
 & & \downarrow \lambda_2 & & \\
 & & s_1^{\mathbf{L1}}(\phi) & & \\
 & & \downarrow \phi & & \\
 & & t_1^{\mathbf{L1}}(\phi) & & \\
 & & \downarrow \rho_2 & & \\
 & & t_1^{\mathbf{L1}}(\chi) & & \\
 \end{array} \\
 \begin{array}{ccccc}
 s_0^{\mathbf{L1}}(\chi) & \xrightarrow{\lambda_1} & s_0^{\mathbf{L1}}(\phi) & \xrightarrow{\rho_1} & t_0^{\mathbf{L1}}(\chi) \\
 & & \downarrow \phi & & \\
 & & t_0^{\mathbf{L1}}(\phi) & &
 \end{array}
 \end{array} \tag{3.2.2.4.1}$$

For $1 \leq i \leq n$, we set:

- $\lambda_i = \bar{\kappa} \left(\langle s_{i-1}^{\mathbf{L1}}(\chi), s_{i-1}^{\mathbf{L1}}(\phi_{i-1}) \rangle, \text{id}_i^{\mathbf{T1}}(s_{i-1}^{\mathbf{T1}}(\mathbf{k})) \right) \in (\mathbf{L1})_i$,
- $\rho_i = \bar{\kappa} \left(\langle t_{i-1}^{\mathbf{L1}}(\phi_{i-1}), t_{i-1}^{\mathbf{L1}}(\chi) \rangle, \text{id}_i^{\mathbf{T1}}(t_{i-1}^{\mathbf{T1}}(\mathbf{k})) \right) \in (\mathbf{L1})_i$, and
- $\phi_i = \lambda_i *_{i-1}^{\mathbf{L1}}(\phi_{i-1} *_{i-1}^{\mathbf{L1}} \rho_i) \in (\mathbf{L1})_n$.

To see that these expressions indeed define valid cells in $\mathbf{L1}$, we must check

- $\text{ar}(\phi_i) = \mathbf{k}$,
- $s_{i-1}^{\mathbf{L1}}(\phi_i) = s_{i-1}^{\mathbf{L1}}(\chi)$, and
- $t_{i-1}^{\mathbf{L1}}(\phi_i) = t_{i-1}^{\mathbf{L1}}(\chi)$.

The first condition holds because $\text{ar}: \mathbf{L1} \rightarrow \mathbf{T1}$, being a strict ω -functor between the weak ω -categories $(\mathbf{L1}, \mu_1^{\mathbf{L}})$ and $(\mathbf{T1}, \mu_1^{\mathbf{T}} \circ \text{ar}_{\mathbf{T1}})$ (the latter is induced by the strict ω -category $(\mathbf{T1}, \mu_1^{\mathbf{T}})$), preserves $*_{i-1}$ and sends λ_i, ρ_i to identities. The second and third hold by construction of ϕ_i and Proposition 1.2.3.10. Note that, in particular, ϕ_n and χ are parallel n -cells in $\mathbf{L1}$ with the same arity, so the above formulas for λ_i and ρ_i make sense for $i = n+1$ too:

- $\lambda_{n+1} = \bar{\kappa} \left(\langle \chi, \phi_n \rangle, \text{id}_{n+1}^{\mathbf{T1}}(\mathbf{k}) \right) \in (\mathbf{L1})_{n+1}$ and
- $\rho_{n+1} = \bar{\kappa} \left(\langle \phi_n, \chi \rangle, \text{id}_{n+1}^{\mathbf{T1}}(\mathbf{k}) \right) \in (\mathbf{L1})_{n+1}$.

Now suppose further that we are given a weak ω -category (X, ξ) , a parallel pair of pasting diagrams $\mathbf{u}, \mathbf{v} \in (\mathbf{TX})_n$ of shape $\mathbf{k}$, and an $(n+1)$ -cell $w: \xi(\phi, \mathbf{u}) \rightarrow \xi(\phi, \mathbf{v})$. Let $w_0 = w$. For $1 \leq i \leq n+1$, we set

- $\ell_i = \xi(\lambda_i, \text{id}_i^{\mathbf{TX}}(s_{i-1}^{\mathbf{TX}}(\mathbf{u}))) \in X_i$,
- $r_i = \xi(\rho_i, \text{id}_i^{\mathbf{TX}}(t_{i-1}^{\mathbf{TX}}(\mathbf{v}))) \in X_i$, and
- $w_i = \ell_i *_{i-1}^X(w_{i-1} *_{i-1}^X r_i) \in X_{n+1}$.

(When $i = n+1$, we interpret the expressions $s_n^{\mathbf{TX}}(\mathbf{u})$ and $t_n^{\mathbf{TX}}(\mathbf{v})$ in the definitions of ℓ_{n+1} and r_{n+1} as $\mathbf{u}$ and $\mathbf{v}$, respectively.) We may inductively prove that the expression in the definition of w_i is well typed by checking $s_n^X(w_i) = \xi(\phi_i, \mathbf{u})$ and $t_n^X(w_i) = \xi(\phi_i, \mathbf{v})$ for $1 \leq i \leq n$. We only spell out the former:

$$\begin{aligned}
 s_n^X(w_i) &= s_n^X(\ell_i *_{i-1}^X(w_{i-1} *_{i-1}^X r_i)) && \text{(definition of } w_i) \\
 &= \ell_i *_{i-1}^X(s_n^X(w_{i-1} *_{i-1}^X r_i)) && \text{(Proposition 1.2.3.10)} \\
 &= \xi \left(\lambda_i, \text{id}_i^{\mathbf{TX}}(s_{i-1}^{\mathbf{TX}}(\mathbf{u})) \right) *_{i-1}^X \left(\xi(\phi_{i-1}, \mathbf{u}) *_{i-1}^X \xi \left(\rho_i, \text{id}_i^{\mathbf{TX}}(t_{i-1}^{\mathbf{TX}}(\mathbf{v})) \right) \right) \\
 &\quad \text{(definitions of } \ell_i \text{ and } r_i, \text{ and inductive hypothesis)} \\
 &= \xi \left(\lambda_i, \text{id}_i^{\mathbf{TX}}(s_{i-1}^{\mathbf{TX}}(\mathbf{u})) \right) *_{i-1}^X \left(\xi(\phi_{i-1}, \mathbf{u}) *_{i-1}^X \xi \left(\rho_i, \text{id}_i^{\mathbf{TX}}(t_{i-1}^{\mathbf{TX}}(\mathbf{u})) \right) \right) && (\mathbf{u} \text{ and } \mathbf{v} \text{ are parallel)} \\
 &= \xi \left(\left(\lambda_i, \text{id}_i^{\mathbf{TX}}(s_{i-1}^{\mathbf{TX}}(\mathbf{u})) \right) *_{i-1}^{\mathbf{LX}} \left((\phi_{i-1}, \mathbf{u}) *_{i-1}^{\mathbf{LX}} \left(\rho_i, \text{id}_i^{\mathbf{TX}}(t_{i-1}^{\mathbf{TX}}(\mathbf{u})) \right) \right) \right) \\
 &\quad (\xi: (\mathbf{LX}, \mu_X^{\mathbf{L}}) \rightarrow (X, \xi) \text{ is a strict } \omega\text{-functor)} \\
 &= \xi(\lambda_i *_{i-1}^{\mathbf{L1}}(\phi_{i-1} *_{i-1}^{\mathbf{L1}} \rho_i), \mathbf{u}) && \text{(Remark 1.2.3.2.(v) and Remark 1.2.3.11)} \\
 &= \xi(\phi_i, \mathbf{u}). && \text{(definition of } \phi_i)
 \end{aligned}$$

We call the resulting $(n+1)$ -cell $w_{n+1}: \xi(\chi, \mathbf{u}) \rightarrow \xi(\chi, \mathbf{v})$ the ϕ -to- χ padding of w (with respect to $(\mathbf{u}, \mathbf{v})$). Note that all the pads $(\ell_i$ and $r_i)$ are coherence cells (Proposition 2.3.7), and hence are invertible in X . This in particular implies that, if w is invertible then so is its ϕ -to- χ padding (see Theorem 2.4.7). $\lrcorner$

Using this padding construction, (Step 1) is straightforward.

Lemma 3.2.2.5. Let (X, ξ) be a weak ω -category and $y, z \in X_0$. Then the whiskering map

$$X(\text{id}_1^X(y), z) : X(y, z) \longrightarrow X(y, z)$$

is essentially ω -injective. $\lrcorner$

Proof. Let $n \geq 1$ and $u, v \in X_n$ be parallel n -cells with $s_0^X(u) = s_0^X(v) = y$ and $t_0^X(u) = t_0^X(v) = z$. Suppose that there exists an equivalence $(n+1)$ -cell $w: \text{id}_1^X(y) *_0^X u \rightarrow \text{id}_1^X(y) *_0^X v$ in X . Observe that we have

$$\text{id}_1^X(y) *_0^X u = \xi \left(\text{id}_1^{L1}(\tilde{e}_0) *_0^{L1} \tilde{e}_n, [u] \right) \quad \text{and} \quad \text{id}_1^X(y) *_0^X v = \xi \left(\text{id}_1^{L1}(\tilde{e}_0) *_0^{L1} \tilde{e}_n, [v] \right).$$

For example, the former can be proved as follows:

$$\begin{aligned} \text{id}_1^X(y) *_0^X u &= \xi \left(\text{id}_1^{L1}(\tilde{e}_0), \text{id}_1^{TX}([y]) \right) *_0^X \xi(\tilde{e}_n, [u]) && \text{(Proposition 1.2.3.13 and Remark 1.2.3.2.(iv))} \\ &= \xi \left(\left(\text{id}_1^{L1}(\tilde{e}_0), \text{id}_1^{TX}([y]) \right) *_0^{LX} \tilde{e}_n, [u] \right) && (\xi: (LX, \mu_X^L) \rightarrow (X, \xi) \text{ is a strict } \omega\text{-functor}) \\ &= \xi(\text{id}_1^{L1}(\tilde{e}_0) *_0^{L1} \tilde{e}_n, [u]). && \text{(Remark 1.2.3.2.(v) and Remark 1.2.3.11)} \end{aligned}$$

(We shall omit similar verifications in what follows.) Thus the $(\text{id}_1^{L1}(\tilde{e}_0) *_0^{L1} \tilde{e}_n)$ -to- $\tilde{e}_n$ padding of w witnesses $u \sim v$. $\square$

The following lemma, which corresponds to (Step 2), generalises the ‘weak uniqueness’ part of [LMW10, Lemma 4.6] from strict ω -categories to weak ω -categories.

Lemma 3.2.2.6. Let X be a weak ω -category, $x, y, z \in X_0$, and $u: x \rightarrow y$ an equivalence 1-cell. Then the whiskering map

$$X(u, z): X(y, z) \longrightarrow X(x, z)$$

is essentially ω -injective. $\lrcorner$

Remark 3.2.2.7. We will prove, by induction on n , that $X(u, z)$ is essentially n -injective. Before presenting the general proof, let us illustrate how the inductive step proceeds by considering the lowest-dimensional case.

Suppose that we are given a parallel pair (v_1, v_2) of 1-cells in $X(y, z)$ which are $\sim$ -related after whiskering with u . We may visualise the situation as:

$$x \xrightarrow{u} y \begin{array}{c} \xrightarrow{s} z \\ \Downarrow v_1 \\ \xrightarrow{t} z \end{array} \sim x \xrightarrow{u} y \begin{array}{c} \xrightarrow{s} z \\ \Downarrow v_2 \\ \xrightarrow{t} z \end{array}$$

Now we can whisker both sides with an inverse $\tilde{u}$ of u , and then with one of the witness equivalence 2-cells $q: \tilde{u} *_0^X u \rightarrow \text{id}_1^X(y)$, and obtain:

$$\begin{array}{ccc} y \xrightarrow{\tilde{u}} x \xrightarrow{u} y \begin{array}{c} \xrightarrow{s} z \\ \Downarrow v_1 \\ \xrightarrow{t} z \end{array} & \sim & y \xrightarrow{\tilde{u}} x \xrightarrow{u} y \begin{array}{c} \xrightarrow{s} z \\ \Downarrow v_2 \\ \xrightarrow{t} z \end{array} \\ *_{11}^X & & *_{11}^X \\ y \xrightarrow{\tilde{u}} x \xrightarrow{u} y \xrightarrow{t} z & \sim & y \xrightarrow{\tilde{u}} x \xrightarrow{u} y \xrightarrow{t} z \\ \Downarrow q & & \Downarrow q \\ \text{id}_1^X(y) & & \text{id}_1^X(y) \end{array}$$

By padding, we can turn this equivalence 3-cell in X into one of type:

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccccc}
 & & x & & \\
 & \nearrow \tilde{u} & \Downarrow q & \searrow u & \\
 y & \xrightarrow{\text{id}_1^X(y)} & y & \xrightarrow{s} & z
 \end{array} \\
 \text{\scriptsize $*1^X$}
 \end{array}
 & \sim &
 \begin{array}{c}
 \begin{array}{ccccc}
 & & x & & \\
 & \nearrow \tilde{u} & \Downarrow q & \searrow u & \\
 y & \xrightarrow{\text{id}_1^X(y)} & y & \xrightarrow{s} & z
 \end{array} \\
 \text{\scriptsize $*1^X$}
 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccc}
 y & \xrightarrow{\text{id}_1^X(y)} & y \\
 & \searrow s & \nearrow t \\
 & \Downarrow v_1 & \\
 & y &
 \end{array}
 \end{array}
 & \sim &
 \begin{array}{c}
 \begin{array}{ccc}
 y & \xrightarrow{\text{id}_1^X(y)} & y \\
 & \searrow s & \nearrow t \\
 & \Downarrow v_2 & \\
 & y &
 \end{array}
 \end{array}
 \end{array}$$

The common upper cell in both sides may be regarded as an equivalence 1-cell in $X(y, z)$, so the essential 0-injectivity of whiskering with this cell (together with Lemma 3.2.2.5) implies $v_1 \sim v_2$. $\square$

Proof of Lemma 3.2.2.6. We prove the statement

for any weak ω -category (X, ξ) , any $x, y, z \in X_0$, and any equivalence 1-cell $u: x \rightarrow y$ in X , the whiskering map $X(u, z): X(y, z) \rightarrow X(x, z)$ is essentially n -injective (3.2.2.7.1)

by induction on $n \in \mathbb{N}$ (note the universal quantification over (X, ξ)).

For the base case $n = 0$, let $(X, \xi), x, y, z$, and u be as in (3.2.2.7.1), and let $\tilde{u}: y \rightarrow x$ be an inverse of u . Take any $v_1, v_2 \in (X(y, z))_0$ with $u*_0^X v_1 \sim u*_0^X v_2$. Since we have

$$v_1 \sim \text{id}_1^X(y)*_0^X v_1 \sim (\tilde{u}*_0^X u)*_0^X v_1 \sim \tilde{u}*_0^X (u*_0^X v_1)$$

by the unit law (Proposition 2.4.3), and similarly $v_2 \sim \tilde{u}*_0^X (u*_0^X v_2)$, we can deduce $v_1 \sim v_2$.

For the inductive step, let $n \geq 1$ and let $(X, \xi), x, y, z$, and u be as in (3.2.2.7.1), and let $\tilde{u}: y \rightarrow x$ be an inverse of u , with equivalence 2-cell $q: \tilde{u}*_0^X u \rightarrow \text{id}_1^X(y)$. Let $v_1, v_2 \in (X(y, z))_n$ be parallel n -cells. Suppose that there exists an equivalence $(n+1)$ -cell

$$w: u*_0^X v_1 \rightarrow u*_0^X v_2$$

in $X(x, z)$. Then we can whisker it with $\tilde{u}$ and obtain an equivalence $(n+1)$ -cell

$$\tilde{u}*_0^X w: \tilde{u}*_0^X (u*_0^X v_1) \rightarrow \tilde{u}*_0^X (u*_0^X v_2)$$

in $X(y, z)$. Let $\phi = \tilde{e}_1*_0^{L1}(\tilde{e}_1*_0^{L1}\tilde{e}_{n+1}) \in (L1)_{n+1}$ and $\chi = (\tilde{e}_1*_0^{L1}\tilde{e}_1)*_0^{L1}\tilde{e}_{n+1} \in (L1)_{n+1}$. Then by Remark 1.2.3.2.(v) and $\xi \circ \mu_X^L = \xi \circ L\xi$, the above cell $\tilde{u}*_0^X w$ is of type

$$\xi\left(\phi, \begin{bmatrix} \tilde{u} & & & \\ & u & & \\ & x & y & v_1 \end{bmatrix}\right) \rightarrow \xi\left(\phi, \begin{bmatrix} \tilde{u} & & & \\ & u & & \\ & x & y & v_2 \end{bmatrix}\right)$$

(note that v_1 and v_2 are $(n+1)$ -cells of X). Hence its ϕ -to- χ padding is an equivalence $(n+2)$ -cell

$$w': (\tilde{u}*_0^X u)*_0^X v_1 \rightarrow (\tilde{u}*_0^X u)*_0^X v_2$$

of X . We can now compose w' with the equivalence 2-cell $q*_0^X t_1^X(v_1) = q*_0^X t_1^X(v_2)$ of X and obtain the equivalence $(n+2)$ -cell

$$w'*_1^X(q*_0^X t_1^X(v_1)): ((\tilde{u}*_0^X u)*_0^X v_1)*_1^X(q*_0^X t_1^X(v_1)) \rightarrow ((\tilde{u}*_0^X u)*_0^X v_2)*_1^X(q*_0^X t_1^X(v_2))$$

of X . Let $\phi' = (\tilde{e}_1*_0^{L1}\tilde{e}_{n+1})*_1^{L1}(\tilde{e}_2*_0^{L1}\tilde{e}_1) \in (L1)_{n+1}$ and $\chi' = (\tilde{e}_2*_0^{L1}\tilde{e}_1)*_1^{L1}(\tilde{e}_1*_0^{L1}\tilde{e}_{n+1}) \in (L1)_{n+1}$. Then, again by Remark 1.2.3.2.(v) and $\xi \circ \mu_X^L = \xi \circ L\xi$, the cell $w'*_1^X(q*_0^X t_1^X(v_1))$ has the type

$$\xi\left(\phi', \begin{bmatrix} q & & & \\ & y & & \\ & & v_1 & \end{bmatrix}\right) \rightarrow \xi\left(\phi', \begin{bmatrix} q & & & \\ & y & & \\ & & v_2 & \end{bmatrix}\right).$$

(See Example 1.1.3.10 for the calculation of the multiplication of the shapes of ϕ' and χ' .) Therefore, its ϕ' -to- χ' padding is an equivalence $(n+2)$ -cell of type

$$(q*_0^X s_1^X(v_1))*_1^X(\text{id}_1^X(y)*_0^X v_1) \rightarrow (q*_0^X s_1^X(v_2))*_1^X(\text{id}_1^X(y)*_0^X v_2),$$

which can also be written as

$$(q*_0^X s_1^X(v_1))*_0^{X(y,z)}(\text{id}_1^X(y)*_0^X v_1) \rightarrow (q*_0^X s_1^X(v_2))*_0^{X(y,z)}(\text{id}_1^X(y)*_0^X v_2).$$

Using the inductive hypothesis to the weak ω -category $X(y, z)$ and the equivalence 1-cell $q*_0^X s_1^X(v_1)$ in $X(y, z)$, we obtain $\text{id}_1^X(y)*_0^X v_1 \sim \text{id}_1^X(y)*_0^X v_2$. By Lemma 3.2.2.5, we conclude $v_1 \sim v_2$. $\square$

Observe that, thanks to Corollary 1.2.4.3, the above lemma immediately generalises to maps of the form $u *_{k-1}^X(-)$ where u is an equivalence k -cell in X with $k \geq 1$: for any $n \geq k$ and any parallel pair (v_1, v_2) of n -cells of X such that $s_{k-1}^X(v_1) = s_{k-1}^X(v_2) = t_{k-1}^X(u)$, $u *_{k-1}^X v_1 \sim u *_{k-1}^X v_2$ implies $v_1 \sim v_2$.

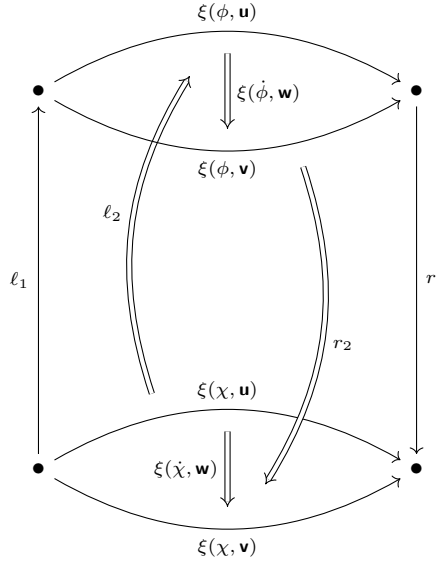
Before moving on to (Step 3), we need to prove the following lemmas.

Lemma 3.2.2.8. In the situation of Construction 3.2.2.4, suppose further that we are given:

- an $(n+1)$ -dimensional pasting scheme $\dot{\mathbf{k}}: \mathbf{k} \rightarrow \mathbf{k}$,
- $(n+1)$ -dimensional pasting instructions $\dot{\phi}: \phi \rightarrow \phi, \dot{\chi}: \chi \rightarrow \chi$ of arity $\dot{\mathbf{k}}$, and
- an $(n+1)$ -dimensional pasting diagram $\mathbf{w}: \mathbf{u} \rightarrow \mathbf{v}$ of shape $\dot{\mathbf{k}}$ in X

such that $w = \xi(\dot{\phi}, \mathbf{w})$. Then there exists an equivalence $(n+2)$ -cell in X from the ϕ -to- χ padding w_{n+1} to $\xi(\dot{\chi}, \mathbf{w})$. $\lrcorner$

Remark 3.2.2.9. We can think of this lemma as stating a kind of naturality of the padding construction as a “transformation” from “paste according to $\dot{\phi}$ ” to “paste according to $\dot{\chi}$.” For example, the case $n = 1$ may be visualised as follows:



The lemma yields an equivalence 3-cell from the ϕ -to- χ padding of $w = \xi(\dot{\phi}, \mathbf{w})$ (obtained by pasting all faces but the bottom) to $\xi(\dot{\chi}, \mathbf{w})$ (the bottom face), filling the interior of this ‘naturality’ cylinder from $\xi(\dot{\phi}, -)$ to $\xi(\dot{\chi}, -)$. $\lrcorner$

Proof of Lemma 3.2.2.8. Recall the cells $\lambda_i, \rho_i \in (\mathbf{L1})_i$, $\phi_i \in (\mathbf{L1})_n$, $\ell_i, r_i \in X_i$, and $w_i \in X_{n+1}$ ($1 \leq i \leq n+1$) defined in Construction 3.2.2.4. Let $\dot{\phi}_0 = \dot{\phi}: \phi \rightarrow \phi$. By induction on $1 \leq i \leq n+1$, we define

$$\dot{\phi}_i = \lambda_i *_{i-1}^{\mathbf{L1}} (\dot{\phi}_{i-1} *_{i-1}^{\mathbf{L1}} \rho_i);$$

the expression on the right-hand side is well typed because we have $\dot{\phi}_i: \phi_i \rightarrow \phi_i$ in $\mathbf{L1}$ for each $1 \leq i \leq n$, as can be seen by applying Proposition 1.2.3.10 inductively. When $i = n+1$, we have $\dot{\phi}_{n+1}: \chi \rightarrow \chi$. We also have $\text{ar}(\dot{\phi}_i) = \dot{\mathbf{k}}$ and $w_i = \xi(\dot{\phi}_i, \mathbf{w})$ for each $1 \leq i \leq n+1$; the latter can be checked inductively as follows:

$$\begin{aligned} w_i &= \ell_i *_{i-1}^X (w_{i-1} *_{i-1}^X r_i) \\ &= \xi \left(\lambda_i, \text{id}_i^{\text{TX}}(s_{i-1}^{\text{TX}}(\mathbf{w})) \right) *_{i-1}^X \left(\xi(\dot{\phi}_{i-1}, \mathbf{w}) *_{i-1}^X \xi \left(\rho_i, \text{id}_i^{\text{TX}}(t_{i-1}^{\text{TX}}(\mathbf{w})) \right) \right) \\ &= \xi \left(\left(\lambda_i, \text{id}_i^{\text{TX}}(s_{i-1}^{\text{TX}}(\mathbf{w})) \right) *_{i-1}^{\text{LX}} \left((\dot{\phi}_{i-1}, \mathbf{w}) *_{i-1}^{\text{LX}} \left(\rho_i, \text{id}_i^{\text{TX}}(t_{i-1}^{\text{TX}}(\mathbf{w})) \right) \right) \right) \\ &= \xi(\lambda_i *_{i-1}^{\mathbf{L1}} (\dot{\phi}_{i-1} *_{i-1}^{\mathbf{L1}} \rho_i), \mathbf{w}) \\ &= \xi(\dot{\phi}_i, \mathbf{w}). \end{aligned}$$

In particular, the ϕ -to- χ padding of w can be written as $w_{n+1} = \xi(\dot{\phi}_{n+1}, \mathbf{w})$. Since $\dot{\chi}$ is parallel to $\dot{\phi}_{n+1}$ in $\mathbf{L1}$, the desired equivalence $(n+2)$ -cell may be obtained using coherence (Proposition 2.3.7). $\square$

Lemma 3.2.2.10. In the situation of Construction 3.2.2.4, suppose we are given another $(n+1)$ -cell $w': \xi(\phi, \mathbf{u}) \rightarrow \xi(\phi, \mathbf{v})$. Then the ϕ -to- χ paddings of w and w' are connected by an equivalence $(n+2)$ -cell if and only if w and w' themselves are. $\lrcorner$

Proof. We are whiskering w and w' with exactly the same (invertible) pads, so the ‘if’ and ‘only if’ directions follow from Theorem 2.4.7 and Lemma 3.2.2.6 respectively. $\square$

Now we are ready to prove **(Step 3)**.

Lemma 3.2.2.11. Let (X, ξ) be a weak ω -category and let $y, z \in X_0$. Then the whiskering map

$$X(\text{id}_1^X(y), z) : X(y, z) \longrightarrow X(y, z)$$

is essentially ω -surjective. $\lrcorner$

Proof. The unit law (Proposition 2.4.3) implies that $X(\text{id}_1^X(y), z)$ is essentially 0-surjective. Let $n \geq 1$, $s, t \in X_n$ be parallel n -cells which are in $X(y, z)$, and suppose there exists an $(n+1)$ -cell

$$u : \text{id}_1^X(y) *_0^X s \rightarrow \text{id}_1^X(y) *_0^X t$$

in X . Define $v : s \rightarrow t$ to be the $(\text{id}_1^{L1}(\tilde{e}_0) *_0^{L1} \tilde{e}_n)$ -to- $\tilde{e}_n$ padding of u . We wish to prove

$$\text{id}_1^X(y) *_0^X v \sim u.$$

By Lemma 3.2.2.10, it suffices to provide an equivalence $(n+2)$ -cell between the $(\text{id}_1^{L1}(\tilde{e}_0) *_0^{L1} \tilde{e}_n)$ -to- $\tilde{e}_n$ paddings of both sides, but this padding of the right-hand side is precisely v . Since we can write v as $v = \xi(\tilde{e}_{n+1}, [v])$, the desired equivalence $(n+2)$ -cell can be obtained by applying Lemma 3.2.2.8 to

- $\dot{\phi} = \text{id}_1^{L1}(\tilde{e}_0) *_0^{L1} \tilde{e}_{n+1}$,
- $\dot{\chi} = \tilde{e}_{n+1}$, and
- $\mathbf{w} = [v]$.

$\square$

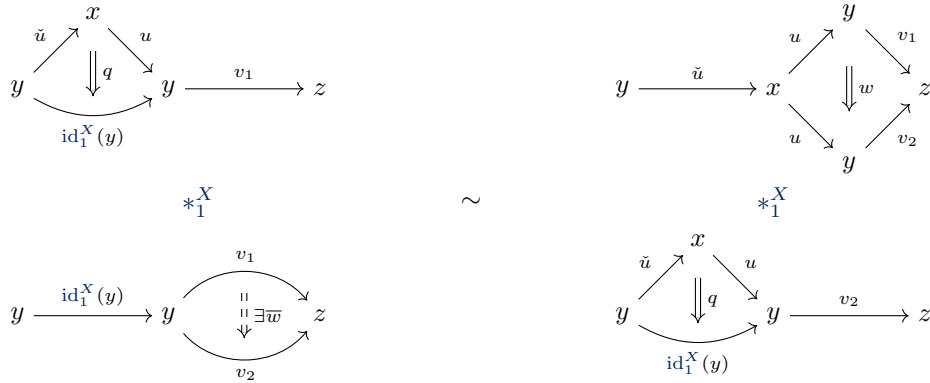
Finally we can prove **(Step 4)**. We note that combining it with Lemma 3.2.2.6 yields a generalisation of [LMW10, Lemma 4.6] from strict ω -categories to weak ω -categories.

Theorem 3.2.2.12. Let X be a weak ω -category, $x, y, z \in X_0$, and $u : x \rightarrow y$ an equivalence 1-cell. Then the whiskering map

$$X(u, z) : X(y, z) \longrightarrow X(x, z)$$

is essentially ω -surjective. $\lrcorner$

Remark 3.2.2.13. The following diagrams summarise the idea of the inductive step in the proof:



More precisely, they depict the two sides of (3.2.2.13.6) below in the case $n = 1$ with all paddings suppressed. $\lrcorner$

Proof of Theorem 3.2.2.12. We prove the statement

$$\left| \begin{array}{l} \text{for any weak } \omega\text{-category } (X, \xi), \text{ any } x, y, z \in X_0, \text{ and any equivalence 1-cell } u : x \rightarrow y \text{ in } X, \text{ the} \\ \text{whiskering map } X(u, z) : X(y, z) \longrightarrow X(x, z) \text{ is essentially } n\text{-surjective} \end{array} \right. \quad (3.2.2.13.1)$$

by induction on $n \in \mathbb{N}$.

For the base case $n = 0$, let (X, ξ) , x, y, z , and u be as in (3.2.2.13.1), and let $\tilde{u} : y \rightarrow x$ be an inverse of u . Take any $w \in (X(x, z))_0$. Then we have

$$w \sim \text{id}_1^X(x) *_0^X w \sim (u *_0^X \tilde{u}) *_0^X w \sim u *_0^X (\tilde{u} *_0^X w)$$

and this completes the base case. (Note that, for a higher dimensional w , no two of the four expressions in the above displayed line necessarily yield parallel cells. In particular, the two sides of the second ‘ $\sim$ ’ may not be

parallel even when X is a strict ω -category. Our inductive step below overcomes this problem in exactly the same way as the strict case [LMW10, Lemma 4.6].)

For the inductive step, let $n \geq 1$ and let $(X, \xi), x, y, z$, and u be as in (3.2.2.13.1), and let $\tilde{u}: y \rightarrow x$ be an inverse of u , with equivalence 2-cell $q: \tilde{u} *_0^X u \rightarrow \text{id}_1^X(y)$ of X . Let (v_1, v_2) be a parallel pair of $(n-1)$ -cells of $X(y, z)$ and let $w: u *_0^X v_1 \rightarrow u *_0^X v_2$ be an n -cell of $X(x, z)$. We wish to find an n -cell $\bar{w}: v_1 \rightarrow v_2$ of $X(y, z)$ such that

$$u *_0^X \bar{w} \sim w. \quad (3.2.2.13.2)$$

To this end, we shall consider conditions (3.2.2.13.3)–(3.2.2.13.6) on an n -cell $\bar{w}: v_1 \rightarrow v_2$ of $X(y, z)$, show that they are all equivalent to (3.2.2.13.2), and observe that there exists $\bar{w}$ satisfying the last condition (3.2.2.13.6) (hence also (3.2.2.13.2)).

By the essential ω -injectivity of whiskering with $\tilde{u}$ (Lemma 3.2.2.6), (3.2.2.13.2) is equivalent to

$$\tilde{u} *_0^X (u *_0^X \bar{w}) \sim \tilde{u} *_0^X w. \quad (3.2.2.13.3)$$

Let $\phi = \tilde{e}_1 *_0^{L1} (\tilde{e}_1 *_0^{L1} \tilde{e}_n)$ and $\chi = (\tilde{e}_1 *_0^{L1} \tilde{e}_1) *_0^{L1} \tilde{e}_n$. Then (3.2.2.13.3) relates two cells of type

$$\xi \left(\phi, \begin{bmatrix} \tilde{u} & u & v_1 \\ x & y & \end{bmatrix} \right) \rightarrow \xi \left(\phi, \begin{bmatrix} \tilde{u} & u & v_2 \\ x & y & \end{bmatrix} \right).$$

Hence, thanks to Lemma 3.2.2.10, (3.2.2.13.3) is equivalent to the existence of an equivalence $(n+2)$ -cell between their ϕ -to- χ paddings. Combining this observation with Lemma 3.2.2.8, we see that (3.2.2.13.3) is equivalent to

$$(\tilde{u} *_0^X u) *_0^X \bar{w} \sim v', \quad (3.2.2.13.4)$$

where v' is the ϕ -to- χ padding of the right-hand side of (3.2.2.13.3).

By the essential ω -injectivity of whiskering with $q *_0^X t_1^X(\bar{w})$ (Lemma 3.2.2.6 applied to $X(y, z)$), (3.2.2.13.4) is in turn equivalent to

$$((\tilde{u} *_0^X u) *_0^X \bar{w}) *_1^X (q *_0^X t_1^X(\bar{w})) \sim v' *_1^X (q *_0^X t_1^X(\bar{w})). \quad (3.2.2.13.5)$$

Note that $t_1^X(\bar{w})$ is either equal to v_2 (when $n = 1$) or to $t_1^X(v_1) = t_1^X(v_2)$ (when $n \geq 2$), and hence the right-hand side of (3.2.2.13.5) does not depend on $\bar{w}$.

Observe that the left-hand side of (3.2.2.13.5) can be written as

$$\xi \left((\tilde{e}_1 *_0^{L1} \tilde{e}_{n+1}) *_1^{L1} (\tilde{e}_2 *_0^{L1} \tilde{e}_1), \begin{bmatrix} q & \bar{w} \\ y & \end{bmatrix} \right).$$

Thus again by applying Lemmas 3.2.2.8 and 3.2.2.10, this time to the pasting instructions

$$\begin{aligned} \phi' &= (\tilde{e}_1 *_0^{L1} \tilde{e}_n) *_1^{L1} (\tilde{e}_2 *_0^{L1} \tilde{e}_1) \text{ and} \\ \chi' &= (\tilde{e}_2 *_0^{L1} \tilde{e}_1) *_1^{L1} (\tilde{e}_1 *_0^{L1} \tilde{e}_n), \end{aligned}$$

we see that (3.2.2.13.5) is equivalent to

$$(q *_0^X s_1^X(\bar{w})) *_1^X (\text{id}_1^X(y) *_0^X \bar{w}) \sim v'' \quad (3.2.2.13.6)$$

where v'' is the ϕ' -to- χ' padding of the right-hand side of (3.2.2.13.5). But $*_1^X$ in the left-hand side can be replaced by $*_0^{X(y,z)}$ thanks to Corollary 1.2.4.3, so the essential $(n-1)$ -surjectivity of whiskering with $q *_0^X s_1^X(\bar{w})$ (in $X(y, z)$; note again that $s_1^X(\bar{w})$ is either equal to v_1 or to $s_1^X(v_1) = s_1^X(v_2)$) and Lemma 3.2.2.11 imply that we can construct $\bar{w}: v_1 \rightarrow v_2$ satisfying (3.2.2.13.6). This completes the proof. $\square$

We can also prove the converse to this theorem (although we will not use this converse in what follows).

Proposition 3.2.2.14. Let (X, ξ) be a weak ω -category and $u: x \rightarrow y$ a 1-cell in X . Suppose that the whiskering map

$$X(u, z): X(y, z) \rightarrow X(x, z)$$

is essentially ω -surjective for all $z \in X_0$. Then u is invertible in X . $\dashv$

Proof. Since $X(u, x): X(y, x) \rightarrow X(x, x)$ is essentially 0-surjective, there exists $\tilde{u}: y \rightarrow x$ such that $u *_0^X \tilde{u} \sim \text{id}_1^X(x)$. We wish to show $\tilde{u} *_0^X u \sim \text{id}_1^X(y)$.

Observe that the images of $\tilde{u} *_0^X u$ and $\text{id}_1^X(y)$ under $X(u, y): X(y, y) \rightarrow X(x, y)$ are connected by the following chain of invertible 2-cells:

$$u *_0^X (\tilde{u} *_0^X u) \sim (u *_0^X \tilde{u}) *_0^X u \sim \text{id}_1^X(x) *_0^X u \sim u \sim u *_0^X \text{id}_1^X(y)$$

It thus suffices to prove that $X(u, y)$ reflects equivalence cells.

We will apply Proposition 3.2.1.8 to $X(u, y)$. Since we already know that $X(u, y)$ is essentially ω -surjective, we only need to check that it satisfies 3.2.1.8.(ii) and 3.2.1.8.(iii). They both are straightforward consequences of coherence (Proposition 2.3.7); for example, given a pair v, v' of n -cells in $X(y, y)$ composable along the $(n-1)$ -dimensional boundary, we have

$$\begin{aligned}
u *_0^X \left(v *_0^{X(y,y)} v' \right) &= u *_0^X (v *_n^X v') \\
&= \xi \left(\tilde{e}_1 *_0^{L1} (\tilde{e}_{n+1} *_n^{L1} \tilde{e}_{n+1}), \begin{bmatrix} u & v & & v' \\ & y & t_n^X(v) & \end{bmatrix} \right) \\
&\sim \xi \left((\tilde{e}_1 *_0^{L1} \tilde{e}_{n+1}) *_n^{L1} (\tilde{e}_1 *_0^{L1} \tilde{e}_{n+1}), \begin{bmatrix} u & v & & v' \\ & y & t_n^X(v) & \end{bmatrix} \right) \\
&= (u *_0^X v) *_n^X (u *_0^X v') \quad (\text{Remark 1.2.3.2.(v) and } \xi \circ \mu_X^L = \xi \circ L\xi) \\
&= (u *_0^X v) *_0^{X(y,y)} (u *_0^X v'),
\end{aligned}$$

which proves that $X(u, y)$ satisfies 3.2.1.8.(iii). The calculation for 3.2.1.8.(ii) is similar. $\square$

3.2.3. The 2-out-of-3 property. Now we are ready to complete the proof of the 2-out-of-3 property.

Lemma 3.2.3.1. Let X, Y be weak ω -categories, $f: X \rightarrow Y$ be a strict ω -functor, $u: x \rightarrow x'$ be a 1-cell of X , and x'' be a 0-cell of X . Then the following diagram of globular maps commutes:

$$\begin{array}{ccc}
X(x', x'') & \xrightarrow{f_{x', x''}} & Y(fx', fx'') \\
\downarrow X(u, x'') & & \downarrow Y(fu, fx'') \\
X(x, x'') & \xrightarrow{f_{x, x''}} & Y(fx, fx'')
\end{array}$$

┘

Proof. Given any cell v in $X(x', x'')$, we have

$$\begin{aligned}
(Y(fu, fx'') \circ f_{x', x''})(v) &= fu *_0^Y fv \\
&= f(u *_0^X v) \quad (\text{Lemma 1.2.3.3}) \\
&= (f_{x, x''} \circ X(u, x''))(v). \quad \square
\end{aligned}$$

The following implies (C) of the 2-out-of-3 property.

Lemma 3.2.3.2. Let $n \in \mathbb{N}$, X, Y, Z be weak ω -categories, $f: X \rightarrow Y$ be a globular map, and $g: Y \rightarrow Z$ be a strict ω -functor. If f and gf are essentially n -surjective, then so is g . $\square$

Proof. We prove this by induction on n .

The base case $n = 0$ is easy and only requires the essential 0-surjectivity of gf .

For the inductive step, let $n > 0$. Let $y, y' \in Y_0$. We want to show that $g_{y, y'}: Y(y, y') \rightarrow Z(gy, gy')$ is essentially $(n-1)$ -surjective. Since f is essentially 0-surjective, there exist $x, x' \in X_0$ and equivalence 1-cells $u: y \rightarrow fx$ and $u': fx' \rightarrow y'$ in Y . Consider the following diagram of globular maps.

$$\begin{array}{ccccc}
& & (gf)_{x, x'} & & \\
& \swarrow & & \searrow & \\
X(x, x') & \xrightarrow{f_{x, x'}} & Y(fx, fx') & \xrightarrow{g_{fx, fx'}} & Z(gfx, gfx') \\
& \downarrow Y(u, u') & & & \downarrow Z(gu, gu') \\
& Y(y, y') & \xrightarrow{g_{y, y'}} & & Z(gy, gy')
\end{array}$$

Here, $Y(u, u') = Y(y, u') \circ Y(u, fx')$ and $Z(gu, gu') = Z(gy, gu') \circ Z(gu, gfx')$. Then the above diagram commutes by Lemma 3.2.3.1 and its dual.

Since $f_{x, x'}$ and $(gf)_{x, x'}$ are essentially $(n-1)$ -surjective, $g_{fx, fx'}$ is essentially $(n-1)$ -surjective by the inductive hypothesis. Since $Z(gu, gu')$ is essentially ω -surjective by Theorem 3.2.2.12, $Z(gu, gu') \circ g_{fx, fx'} = g_{y, y'} \circ Y(u, u')$ is essentially $(n-1)$ -surjective by Remark 3.2.2.2 and Lemma 3.2.1.6. Since $Y(u, u')$ is essentially ω -surjective by Theorem 3.2.2.12, $g_{y, y'}$ is essentially $(n-1)$ -surjective by the inductive hypothesis. $\square$

Theorem 3.2.3.3. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be strict ω -functors between weak ω -categories. If two of f , g , and gf are ω -weak equivalences, then so is the third. $\lrcorner$

Proof. This follows from Lemmas 3.2.3.2, 3.2.1.6 and 3.2.1.7. $\square$

We can also establish the stronger 2-out-of-6 property (so we have a *homotopical category* in the sense of [Dwy+04]).

Corollary 3.2.3.4 (2-out-of-6). Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$, and $h: Z \rightarrow W$ be strict ω -functors between weak ω -categories. Suppose that gf and hg are ω -weak equivalences. Then f, g, h , and hgf are also ω -weak equivalences. $\lrcorner$

Proof. We will show that hgf is an ω -weak equivalence; the rest will follow by the 2-out-of-3 property.

To see that hgf is essentially 0-surjective, let $w \in W_0$. Since hg is essentially 0-surjective, there exists $y \in Y_0$ with $hgy \sim w$. Moreover, since gf is essentially 0-surjective, there exists $x \in X_0$ such that $gfx \sim gy$. Then we have $hgfx \sim hgy \sim w$ by Proposition 2.3.1.

By applying the above argument to the hom weak ω -categories, we can deduce that hgf is essentially n -surjective for all $n \in \mathbb{N}$. $\square$

3.2.4. Other properties of ω -weak equivalences. Here we collect miscellaneous properties of ω -weak equivalences.

Recall the notion of essential n -injectivity from Definition 3.2.2.3.

Lemma 3.2.4.1. Let $n \in \mathbb{N}$, X, Y be weak ω -categories, and $f: X \rightarrow Y$ be a globular map reflecting equivalence cells. If f is essentially $(n+1)$ -surjective, then f is essentially n -injective. $\lrcorner$

Proof. We prove this by induction on n .

The base case $n = 0$ can be proved as follows. Suppose that $f: X \rightarrow Y$ is essentially 1-surjective. Let $x, x' \in X_0$ be 0-cells such that there exists an equivalence 1-cell $u: fx \rightarrow fx'$ in Y . Then there exists a 1-cell $\bar{u}: x \rightarrow x'$ in X such that $f\bar{u} \sim u$. By Corollary 2.4.16, $f\bar{u}$ is invertible, and since f reflects equivalence cells, $\bar{u}$ is invertible. Therefore we have $x \sim x'$.

For the inductive step, let $n > 0$ and suppose that $f: X \rightarrow Y$ is essentially $(n+1)$ -surjective. Then for each $x, x' \in X_0$, $f_{x,x'}$ is essentially n -surjective. By the inductive hypothesis, $f_{x,x'}$ is essentially $(n-1)$ -injective. Hence f is essentially n -injective. $\square$

Definition 3.2.4.2. Any ω -weak equivalence is essentially ω -injective. $\lrcorner$

Proof. This follows from Proposition 3.2.1.8 and Lemma 3.2.4.1. $\square$

Proposition 3.2.4.3 (cf. [LMW10, Lemma 4.12] and [Ara+25, Proposition 20.1.18]). The class of ω -weak equivalences is closed under retracts in the arrow category $\mathbf{Wk}\text{-}\omega\text{-Cat}_s^2$. More generally, if $n \in \mathbb{N}$ and if

$$\begin{array}{ccccc}
 & & \text{Id}_X & & \\
 & \curvearrowright & & \curvearrowleft & \\
 X & \xrightarrow{i} & X' & \xrightarrow{p} & X \\
 \downarrow f & & \downarrow f' & & \downarrow f \\
 Y & \xrightarrow{j} & Y' & \xrightarrow{q} & Y \\
 & \curvearrowleft & & \curvearrowright & \\
 & & \text{Id}_Y & &
 \end{array}$$

is a commutative diagram of globular maps between (the underlying globular sets of) weak ω -categories such that q preserves equivalence cells, then f is essentially n -surjective whenever f' is. $\lrcorner$

Proof. We prove the latter statement by induction on n .

The base case $n = 0$ is shown as follows. Take any $y \in Y_0$. Then we obtain $jy \in Y'_0$. Since f' is essentially 0-surjective, there exist $x' \in X'_0$ and an equivalence 1-cell $u: f'x' \rightarrow jy$ in Y' . The 1-cell $qu: qf'x' \rightarrow qjy$ in Y is invertible by the assumption. Since $qf'x' = fp'x'$ and $qjy = y$, we have $fp'x' \sim y$ for $p'x' \in X_0$, as desired.

For the inductive step, let $n > 0$. For each $x, x' \in X_0$, we have the commutative diagram

$$\begin{array}{ccccc}
 & & \text{Id}_{X(x, x')} & & \\
 & \nearrow & & \searrow & \\
 X(x, x') & \xrightarrow{i_{x, x'}} & X'(ix, ix') & \xrightarrow{p_{ix, ix'}} & X(x, x') \\
 \downarrow f_{x, x'} & & \downarrow f'_{ix, ix'} & & \downarrow f_{x, x'} \\
 Y(fx, fx') & \xrightarrow{j_{fx, fx'}} & Y'(f'ix, f'ix') & \xrightarrow{q_{f'ix, f'ix'}} & Y(fx, fx') \\
 & \searrow & & \nearrow & \\
 & & \text{Id}_{Y(fx, fx')} & &
 \end{array}$$

of globular maps between weak ω -categories. Since $f'_{ix, ix'}$ is essentially $(n-1)$ -surjective, so is $f_{x, x'}$, by the inductive hypothesis. $\square$

Proposition 3.2.4.4 (cf. [LMW10, Lemma 4.12] and [Ara+25, Proposition 20.1.19]). The class of ω -weak equivalences is closed under filtered colimits in the arrow category $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s^2$ and transfinite compositions in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. $\dashv$

Proof. The proof is almost identical to that of [Ara+25, Proposition 20.1.19]. The only thing we must check is that the forgetful functor $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{GSet}$ preserves filtered colimits. This follows from the fact that the functor $L: \mathbf{GSet} \rightarrow \mathbf{GSet}$ preserves filtered colimits, which we established in Proposition 1.2.3.4. $\square$

3.3. Weak ω -weak equivalences

Whereas the notion of ω -weak equivalence we have studied so far is natural from the viewpoint of model category theory (cf. [LMW10]), the requirement that an ω -weak equivalence should be a *strict* ω -functor is too restrictive as a fully general notion of ‘equivalence’ between weak ω -categories, i.e., as a suitable higher-dimensional analogue of a biequivalence between bicategories or a triequivalence between tricategories. In this section, we define and study a more general class of *weak ω -weak equivalences*, which are *weak ω -functors* [Gar10] with suitable essential ω -surjectivity. In particular, we show that they also satisfy the 2-out-of-3 property. For the sake of clarity, in this section we call ω -weak equivalences in the sense of Definition 3.2.1.3 *strict ω -weak equivalences*.

3.3.1. Weak ω -functors. We first recall the definition of weak ω -functor in the sense of Garner [Gar10], which is based on the theory of AWFS [Gar09b; BG16a; BG16b]. We are, however, based on the notion of *algebraic injectivity*, which is prepared in Section 1.3. For weak functors, we are walking on the category $\mathbf{C} = \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, and for the indexed family $\mathcal{K}$, we take $\mathcal{I}$ that is defined in Example 1.3.3.

Definition 3.3.1.1. An *algebraic trivial fibration* is an algebraically $\mathcal{I}$ -injective morphism in the sense of Definition 1.3.6. $\dashv$

Definition 3.3.1.2. Let $\mathcal{I} \in \mathbf{Set} \downarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s^2$ be the set of boundary inclusions $\iota^n: \partial G^n \rightarrow G^n$ for $n \in \mathbb{N}$, defined in Example 1.3.3. We define the *weak ω -functor classifier comonad* $\mathbf{Q} = (\mathbf{Q}, \varepsilon, \delta)$ as the algebraically $\mathcal{I}$ -cellular replace comonad on $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ in the sense of Proposition 1.3.20.

We write $\kappa_X^{\mathbf{Q}}$ for the canonical $\mathcal{I}$ -lifting structure on the counit $\varepsilon_X: \mathbf{Q}X \rightarrow X$ that has the universality as the free object with respect to the codomain functor $\text{cod}: \mathcal{I}\text{-}\mathbf{AIMor} \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, which is unpacked in Construction 1.3.17. $\dashv$

Remark 3.3.1.3. As observed for the non-algebraic case in Example 1.3.3, $\mathcal{I}$ -lifting structures on a strict ω -functor $f: X \rightarrow Y$ may be identified with $\mathcal{I}_o$ -lifting structures on the underlying globular map of f . Using this identification, and writing ι^n simply as n for brevity, we regard $\kappa_X^{\mathbf{Q}}$ as a function which, to each $n \leq 0$, each parallel pair $u: \partial G^n \rightarrow \mathbf{Q}X$, and each n -cell $v: G^n \rightarrow X$ such that $(s_{n-1}^X(v), t_{n-1}^X(v)) = \varepsilon_X \circ u$, assigns an n -cell $\kappa_X^{\mathbf{Q}}(n; u, v)$ in $\mathbf{Q}X$. $\dashv$

By Corollary 1.2.3.5 and Proposition 1.3.21, such a comonad exists and coincides with the cofibrant replacement comonad in the sense of [Gar10; BG16b]. Therefore, the following notion of weak ω -functor agrees with that of [Gar10] and the notion of *weak map* in [BG16b].

Definition 3.3.1.4 ([Gar10]). Let X and Y be weak ω -categories. A *weak ω -functor* from X to Y is a strict ω -functor $\mathbf{Q}X \rightarrow Y$. The category $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w$ of weak ω -categories and weak ω -functors is defined as the (co-)Kleisli category of the comonad $\mathbf{Q}$. $\dashv$

We write a weak ω -functor p from X to Y as $p: X \rightsquigarrow Y$. By definition, it is a strict ω -functor from QX to Y ; we call the latter the *classifying strict ω -functor* of p , and denote it by $p_s: QX \rightarrow Y$. The composite of weak ω -functors $p: X \rightsquigarrow Y$ and $q: Y \rightsquigarrow Z$, denoted by $p \circ q = q \circ p: X \rightsquigarrow Z$, is defined by specifying its classifying strict ω -functor $(q \circ p)_s$ as the composite

$$QX \xrightarrow{\delta_X} QQX \xrightarrow{Qp_s} QY \xrightarrow{q_s} Z.$$

in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$.

We have the identity-on-objects right adjoint functor $J: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w$ associated with the Kleisli category $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w$. It maps each strict ω -functor $f: X \rightarrow Y$ to the weak ω -functor Jf classified by $(Jf)_s = f \circ \varepsilon_X: QX \rightarrow Y$.

3.3.2. Weak ω -weak equivalences.

Definition 3.3.2.1. Let X and Y be weak ω -categories. A *weak ω -weak equivalence* from X to Y is a weak ω -functor $p: X \rightsquigarrow Y$ whose classifying strict ω -functor $p_s: QX \rightarrow Y$ is an ω -weak equivalence in the sense of Definition 3.2.1.3. $\lrcorner$

Recall that the binary relation $\sim$ on the set of cells of a weak ω -category, defined by the existence of an equivalence cell (see Definition 2.2.1), is reflexive (see Corollary 2.4.13). This implies the following:

Lemma 3.3.2.2. Any trivial fibration between weak ω -categories is a strict ω -weak equivalence. $\lrcorner$

Therefore, the trivial fibration ε_X is a strict ω -weak equivalence, and hence Theorem 3.2.3.3 implies the following:

Lemma 3.3.2.3. A strict ω -functor $f: X \rightarrow Y$ is a strict ω -weak equivalence if and only if the corresponding weak ω -functor $Jf: X \rightsquigarrow Y$ is a weak ω -weak equivalence. $\lrcorner$

The comonad (Q, ε, δ) on $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is compatible with the strict ω -weak equivalences in the following sense.

Proposition 3.3.2.4. For each weak ω -category X , the components $\varepsilon_X: QX \rightarrow X$ and $\delta_X: QX \rightarrow QQX$ are strict ω -weak equivalences, and the functor $Q: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ preserves and reflects strict ω -weak equivalences. $\lrcorner$

Proof. We have already seen that ε_X is a strict ω -weak equivalence. One of the axioms for a comonad states that the diagram

$$\begin{array}{ccc} QX & \xrightarrow{\delta_X} & QQX \\ & \searrow & \downarrow \varepsilon_{QX} \\ & & QX \end{array}$$

commutes. Since Id_{QX} and ε_{QX} are strict ω -weak equivalences, so is δ_X by Theorem 3.2.3.3. Moreover, for any strict ω -functor $f: X \rightarrow Y$, the diagram

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ \varepsilon_X \downarrow & & \downarrow \varepsilon_Y \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Hence by Theorem 3.2.3.3, f is a strict ω -weak equivalence if and only if Qf is. $\square$

Now it is straightforward to show that the class of weak ω -weak equivalences has the 2-out-of-3 property (in the category $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w$).

Proposition 3.3.2.5 (2-out-of-3 for weak ω -weak equivalences). Let $p: X \rightsquigarrow Y$ and $q: Y \rightsquigarrow Z$ be weak ω -functors between weak ω -categories. If two of p , q , and $q \circ p$ are weak ω -weak equivalences, then so is the third. $\lrcorner$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} QX & \xrightarrow{(q \circ p)_s} & Z \\ \delta_X \downarrow & & \uparrow q_s \\ QQX & \xrightarrow{Qp_s} & QY \end{array}$$

in $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$. It is easy to see that if two of p_s , q_s , and $(q \circ p)_s$ are strict ω -weak equivalences, then so is the third, using Proposition 3.3.2.4 and Theorem 3.2.3.3. $\square$

Corollary 3.3.2.6 (2-out-of-6 for weak ω -weak equivalences). Let $p: X \rightsquigarrow Y$, $q: Y \rightsquigarrow Z$, and $r: Z \rightsquigarrow W$ be weak ω -functors between weak ω -categories. Suppose that $q \circ p$ and $r \circ q$ are weak ω -weak equivalences. Then p , q , r , and $r \circ q \circ p$ are also weak ω -weak equivalences. $\lrcorner$

Proof. We will show that $r \circ q \circ p$ is a weak ω -weak equivalence; the rest will follow by the 2-out-of-3 property for weak ω -weak equivalences.

The classifying strict ω -functor $(r \circ q \circ p)_s$ of $r \circ q \circ p$ is the composite of

$$\mathbf{Q}X \xrightarrow{\delta_X} \mathbf{Q}\mathbf{Q}X \xrightarrow{\mathbf{Q}\delta_X} \mathbf{Q}\mathbf{Q}\mathbf{Q}X \xrightarrow{\mathbf{Q}\mathbf{Q}p_s} \mathbf{Q}\mathbf{Q}Y \xrightarrow{q_s} \mathbf{Q}Z \xrightarrow{r_s} W$$

in $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$. Since δ_X and $\mathbf{Q}\delta_X$ are strict ω -weak equivalences by Proposition 3.3.2.4, it suffices to show that $r_s \circ \mathbf{Q}q_s \circ \mathbf{Q}\mathbf{Q}p_s$ is a strict ω -weak equivalence. Since $q \circ p$ is a weak ω -weak equivalence, $q_s \circ \mathbf{Q}p_s$ is a strict ω -weak equivalence. Hence so is $\mathbf{Q}q_s \circ \mathbf{Q}\mathbf{Q}p_s$, by Proposition 3.3.2.4. Since $r \circ q$ is a weak ω -weak equivalence, $r_s \circ \mathbf{Q}q_s$ is a strict ω -weak equivalence. Therefore $r_s \circ \mathbf{Q}q_s \circ \mathbf{Q}\mathbf{Q}p_s$ is a strict ω -weak equivalence by Corollary 3.2.3.4. $\square$

We also have a characterisation of weak ω -weak equivalences along the lines of Proposition 3.2.1.4. We first note the following fact. The proof is deferred to Section A.3.

Proposition 3.3.2.7. For each weak ω -category X , the strict ω -functor $\varepsilon_X: \mathbf{Q}X \rightarrow X$ is bijective on objects. $\lrcorner$

From now on, we identify the objects of $\mathbf{Q}X$ with those of X via ε_X . We recall from [CF22, Section 5] that, given any weak ω -functor $p: X \rightsquigarrow Y$ between weak ω -categories and $x, x' \in X_0$, we obtain a weak ω -functor $p_{x,x'}: X(x, x') \rightsquigarrow Y(px, px')$ between the hom weak ω -categories as follows. For each $x, x' \in X_0$, we obtain an algebraically $\mathcal{I}$ -injective morphism

$$\left((\varepsilon_X)_{x,x'}: (\mathbf{Q}X)(x, x') \rightarrow X(x, x'), (\kappa_X^{\mathbf{Q}})_{x,x'} \right)$$

by restricting $(\varepsilon_X, \kappa_X^{\mathbf{Q}})$. Indeed, a commutative square

$$\begin{array}{ccc} \partial \mathcal{C}^n & \longrightarrow & (\mathbf{Q}X)(x, x') \\ \text{\scriptsize $F\iota^n$} \downarrow & & \downarrow (\varepsilon_X)_{x,x'} \\ \mathcal{C}^n & \longrightarrow & X(x, x') \end{array}$$

can be identified with a commutative square

$$\begin{array}{ccc} \partial \mathcal{C}^{n+1} & \xrightarrow{\langle u, v \rangle} & \mathbf{Q}X \\ \text{\scriptsize ι^{n+1}} \downarrow & & \downarrow \varepsilon_X \\ \mathcal{C}^{n+1} & \xrightarrow{w} & X \end{array}$$

with $s_0^X(u) = s_0^X(v) = s_0^X(w) = x$ and $t_0^X(u) = t_0^X(v) = t_0^X(w) = x'$. The latter square admits a lift

$$k = \kappa_X^{\mathbf{Q}}(n+1; \langle u, v \rangle, w)$$

which necessarily satisfies $s_0^X(k) = x$ and $t_0^X(k) = x'$, and we can set the lift of the former square to be the map corresponding to this k .

On the other hand, the initial algebraically $\mathcal{I}$ -injective morphism over $X(x, x')$ is

$$\left(\varepsilon_{X(x,x')}: \mathbf{Q}(X(x, x')) \rightarrow X(x, x'), \kappa_{X(x,x')}^{\mathbf{Q}} \right),$$

and hence we obtain a unique strict ω -functor $(\alpha_X)_{x,x'}: \mathbf{Q}(X(x, x')) \rightarrow (\mathbf{Q}X)(x, x')$ such that

$$((\alpha_X)_{x,x'}, \text{Id}_{X(x,x')}) : \left(\varepsilon_{X(x,x')}, \kappa_{X(x,x')}^{\mathbf{Q}} \right) \longrightarrow \left((\varepsilon_X)_{x,x'}, (\kappa_X^{\mathbf{Q}})_{x,x'} \right)$$

is a morphism in $\mathcal{I}\text{-AIMor}$. In particular, $(\alpha_X)_{x,x'}$ makes the diagram

$$\begin{array}{ccc} \mathbf{Q}(X(x, x')) & \xrightarrow{(\alpha_X)_{x,x'}} & (\mathbf{Q}X)(x, x') \\ \text{\scriptsize $\varepsilon_{X(x,x')}$} \searrow & & \swarrow \text{\scriptsize $(\varepsilon_X)_{x,x'}$} \\ & X(x, x') & \end{array}$$

commute, and hence is a strict ω -weak equivalence by Proposition 3.2.1.4 and Theorem 3.2.3.3. The weak ω -functor $p_{x,x'}: X(x, x') \rightsquigarrow Y(px, px')$ is defined by setting its classifying strict ω -functor $(p_{x,x'})_s$ to be the composite of

$$Q(X(x, x')) \xrightarrow{(\alpha_X)_{x,x'}} (QX)(x, x') \xrightarrow{(p_s)_{x,x'}} Y(px, px')$$

We now give the characterisation of weak ω -weak equivalences alluded to just above Proposition 3.3.2.7. We say that a weak ω -functor is *essentially 0-surjective* if its classifying strict ω -functor is so.

Proposition 3.3.2.8. A weak ω -functor $p: X \rightsquigarrow Y$ between weak ω -categories is a weak ω -weak equivalence if and only if it is essentially 0-surjective and the induced weak ω -functor $p_{x,x'}: X(x, x') \rightsquigarrow Y(px, px')$ is a weak ω -weak equivalence for each $x, x' \in X_0$. $\lrcorner$

Proof. Since $(\alpha_X)_{x,x'}$ is a strict ω -weak equivalence, $(p_s)_{x,x'}$ is a strict ω -weak equivalence if and only if $(p_{x,x'})_s$ is, by Theorem 3.2.3.3. Therefore the claim follows from Proposition 3.2.1.4. $\square$

Remark 3.3.2.9. Just like the strict case mentioned in Remark 3.2.1.5, the statement of Proposition 3.3.2.8 can be taken as a coinductive definition of weak ω -weak equivalences. Define the monotone map

$$\Psi: \wp(\text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w)) \longrightarrow \wp(\text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w))$$

by mapping $S \subseteq \text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w)$ to the set $\Psi(S)$ of all weak ω -functors which are essentially 0-surjective and locally in S , i.e.,

$$\Psi(S) = \{ (p: X \rightsquigarrow Y) \in \text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w) \mid p \text{ is essentially 0-surjective and} \\ \text{we have } (p_{x,x'}: X(x, x') \rightsquigarrow Y(px, px')) \in S \text{ for each } x, x' \in X_0 \}.$$

Then the set E of all weak ω -weak equivalences is the greatest (post-)fixed point $\nu\Psi$ of Ψ . Indeed, Proposition 3.3.2.8 says that we have $E = \Psi(E)$, which implies $E \subseteq \nu\Psi$. On the other hand, given $S \subseteq \text{Mor}(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w)$ with $S \subseteq \Psi(S)$, one can show that the classifying strict ω -functor p_s of each $p \in S$ is essentially n -surjective, by induction on $n \in \mathbb{N}$. (In the inductive step, apply Lemma 3.2.3.2 to the commutative triangle

$$\begin{array}{ccc} & (QX)(x, x') & \\ (\alpha_X)_{x,x'} \nearrow & & \searrow (p_s)_{x,x'} \\ Q(X(x, x')) & \xrightarrow{(p_{x,x'})_s} & Y(px, px') \end{array}$$

in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$.) Therefore we have $S \subseteq E$, and hence E is the greatest (post-)fixed point $\nu\Psi$ of Ψ . $\lrcorner$

3.3.3. Weak ω -weak equivalences via underlying globular maps. Let us consider the relationship between the weak ω -weak equivalences and similar known notions in low dimensions, such as *biequivalences* between bicategories and *triequivalences* between tricategories. Recall that in low dimensions, notions of weak functors (such as *homomorphisms* of bicategories [Bén67, Section 4] or *trihomomorphisms* of tricategories [GPS95, Definition 3.2]) are commonly defined as globular maps between the underlying globular sets equipped with coherence cells. (We define the underlying globular set of a weak n -dimensional category X for $n \in \mathbb{N}$ by setting $s_k^X, t_k^X: X_{k+1} \rightarrow X_k$ to be the identity function for all $k \geq n$.) For example, a homomorphism $p: X \rightsquigarrow Y$ of bicategories is a globular map $p_g: X \rightarrow Y$ between the underlying globular sets equipped with coherence 2-cells, such as an equivalence 2-cell $p_g(u) *_0^Y p_g(v) \rightarrow p_g(u *_0^X v)$ in Y for each composable pair of 1-cells (u, v) in X . However, the definition of biequivalence only refers to the underlying globular map of a homomorphism: a homomorphism $p: X \rightsquigarrow Y$ is a biequivalence if and only if its underlying globular map $p_g: X \rightarrow Y$ is essentially 3-surjective [Str80, (1.33)], which implies that p_g is essentially ω -surjective. Similarly, a trihomomorphism $p: X \rightsquigarrow Y$ is a triequivalence if and only if its underlying globular map $p_g: X \rightarrow Y$ is essentially 4-surjective [GPS95, Definition 3.5] (or equivalently, is essentially ω -surjective).

We characterise weak ω -weak equivalences in a similar fashion: we define the *underlying globular map* $p_g: X \rightarrow Y$ of a weak ω -functor $p: X \rightsquigarrow Y$ (which is a genuine globular map $X \rightarrow Y$ and is in particular different from the classifying strict ω -functor $p_s: QX \rightarrow Y$ of p), and show that a weak ω -functor $p: X \rightsquigarrow Y$ is a weak ω -weak equivalence if and only if its underlying globular map $p_g: X \rightarrow Y$ is essentially ω -surjective.

Hence our first goal is to define the operation of taking the underlying globular map of a weak ω -functor. We would like this operation to be functorial, i.e., giving rise to a functor $V: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w \rightarrow \mathbf{GSet}$. This turns out to be a consequence of an appropriate ‘change of base’ result for the notions Definition 1.3.6 and Proposition 1.3.20, which is observed in the last part of Section 1.3.

The following proposition ensures that, in the case of our interest, the codomain of the resulting functor is indeed **GSet**.

Proposition 3.3.3.1. The cofibrant replacement comonad on **GSet** with respect to $\mathcal{I}_o$ in Example 1.3.3 is isomorphic to the identity comonad. $\lrcorner$

Proof. In view of Remark 1.3.19, it suffices to show that for each $A \in \mathbf{GSet}$, the vertical identity (Id_A, ω_A) in $\mathcal{I}_o\text{-AIMor}$ is an initial object³ in the fibre $\mathcal{I}_o\text{-AIMor}_A$. To this end, suppose that we are given an algebraic trivial fibration $(f: B \rightarrow A, \kappa)$ over A . Then a morphism $h: A \rightarrow B$ in **GSet** determines a morphism $(h, \text{Id}_A): (\text{Id}_A, \omega_A) \rightarrow (f, \kappa)$ in $\mathcal{I}_o\text{-AIMor}_A$ if and only if

- for each $a \in A_0$, we have $h(a) = \kappa(0; !^B, a)$, where $!^B: \emptyset \rightarrow B$; and
- for each $n > 0$ and each $c: a \rightarrow b$ in A_n , we have $h(c) = \kappa(n; \langle h(a), h(b) \rangle, c)$.

In fact, these conditions *define* h inductively, which means that there exists precisely one such h . $\square$

Applying Proposition 1.3.25 to the situation Example 1.3.3 of the free-forgetful adjunction $\mathbf{GSet} \xrightleftharpoons{\quad} \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ and $\mathcal{K}_o = \mathcal{I}_o$, we obtain a canonical natural transformation

$$\begin{array}{ccc} \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s & \xrightarrow{Q} & \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \\ U_L \downarrow & \sigma \nearrow & \downarrow U_L \\ \mathbf{GSet} & \xrightarrow{\text{Id}_{\mathbf{GSet}}} & \mathbf{GSet} \end{array}$$

such that (U_L, σ) is a comonad opfunctor from $(\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s, Q)$ to $(\mathbf{GSet}, \text{Id}_{\mathbf{GSet}})$. Concretely, for each weak ω -category X , the globular map $\sigma_X: X \rightarrow QX$ is obtained by applying the function κ_X^Q associated to ε_X inductively:

- A 0-cell x of X is mapped to the 0-cell $\sigma_X(x) = \kappa_X^Q(0; !^{QX}, x)$ of QX where $!^{QX}: \emptyset \rightarrow QX$.
- For $n \geq 1$, an n -cell $u: x \rightarrow y$ in X is mapped to the n -cell $\sigma_X(u) = \kappa_X^Q(n; \langle \sigma_X(x), \sigma_X(y) \rangle, u): \sigma_X(x) \rightarrow \sigma_X(y)$ in QX .

By Proposition 3.3.3.1, it follows that (U_L, σ) induces a functor $V: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w \rightarrow \mathbf{GSet}$ between the Kleisli categories making the diagram

$$\begin{array}{ccc} \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s & \xrightarrow{J} & \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w \\ & \searrow U_L & \swarrow V \\ & \mathbf{GSet} & \end{array}$$

commute, where J is the identity-on-objects functor mentioned at the end of Section 3.3.1. Explicitly, V maps each weak ω -category to its underlying globular set, and each weak ω -functor $p: X \rightsquigarrow Y$ to the composite of the globular map $\sigma_X: X \rightarrow QX$ and the classifying strict ω -functor $p_s: QX \rightarrow Y$, which we call the *underlying globular map* p_g of the weak ω -functor p . See also [Gar10, Sections 3 and 4], where a notion of weak functor between tricategories in the style of Definition 3.3.1.4 is related to the classical definition of trihomomorphism, and in which the underlying globular map of a weak functor is given by a similar construction.

Remark 3.3.3.2. One can also induce the functor $V: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w \rightarrow \mathbf{GSet}$ using the universal property of $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w$ given in [BG16b, Theorem 10], which is different from its universal property as a Kleisli category. In this approach, it suffices to construct a double functor from $\mathcal{I}\text{-AIMor}$ to $\mathbf{SpEpi}(\mathbf{GSet})$, which is possible because there exists a canonical double functor $\mathcal{I}\text{-AIMor} \rightarrow \mathbf{SpEpi}(\mathbf{GSet})$. The latter corresponds to the identity functor on **GSet** (again by [BG16b, Theorem 10]); see Proposition 3.3.3.1. $\lrcorner$

Proposition 3.3.3.3. Let $p: X \rightsquigarrow Y$ be a weak ω -functor between weak ω -categories. Then p is a weak ω -weak equivalence if and only if its underlying globular map $p_g = p_s \circ \sigma_X: X \rightarrow Y$ is essentially ω -surjective. $\lrcorner$

Proof. Since $\varepsilon_X \circ \sigma_X = \text{Id}_X$ holds in **GSet**, the map $\sigma_X: X \rightarrow QX$ is essentially ω -surjective by Lemma 3.2.1.7. Hence $p_s: QX \rightarrow Y$ is essentially ω -surjective if and only if $p_s \circ \sigma_X: X \rightarrow Y$ is, by Lemmas 3.2.3.2 and 3.2.1.6. $\square$

Lemma 3.3.3.4. For each weak ω -category X , the globular map $\sigma_X: X \rightarrow QX$ preserves and reflects equivalence cells. $\lrcorner$

³By Remark 1.3.18, it is always a terminal object, and hence a zero object in $\mathcal{I}_o\text{-AIMor}_A$. In particular, this concludes that any algebraically $\mathcal{I}_o$ -injective morphism over A has a canonical section.

Proof. This follows from the fact that the retraction ε_X of σ_X , being a strict ω -weak equivalence, reflects and preserves equivalence cells (Propositions 2.3.1 and 3.2.1.8). $\square$

Proposition 3.3.3.5. The underlying globular map of a weak ω -functor preserves equivalence cells. $\lrcorner$

Proof. By Proposition 2.3.1 and Lemma 3.3.3.4. $\square$

Proposition 3.3.3.6. The underlying globular map of a weak ω -weak equivalence reflects equivalence cells and is essentially ω -injective. $\lrcorner$

Proof. The first statement follows from Proposition 3.2.1.8 and Lemma 3.3.3.4. The second statement follows from the first, Proposition 3.3.3.3, and Lemma 3.2.4.1. $\square$

Proposition 3.3.3.7. The class of weak ω -weak equivalences is closed under retracts in the arrow category $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_w^2$. $\lrcorner$

Proof. By Proposition 3.3.3.3, we can work at the level of underlying globular maps. We can apply Proposition 3.2.4.3 thanks to Proposition 3.3.3.5. $\square$

Remark 3.3.3.8. Given a weak ω -weak equivalence $p: X \rightsquigarrow Y$, one might expect to be able to construct its (pseudo-)inverse $q: Y \rightsquigarrow X$. Although this is entirely plausible, it seems rather difficult to actually achieve it for the following two reasons.

Firstly, we do not currently have a way of constructing *any* weak ω -functor $Y \rightsquigarrow X$ from just the data of p (even assuming the axiom of choice). The most natural approach seems to be somehow factorising the strict ω -weak equivalence $p_s: \mathbf{Q}X \rightarrow Y$ as $\mathbf{Q}X \xrightarrow{s} Z \xrightarrow{f} Y$ in such a way that f is an (algebraic) trivial fibration and s is a strict ω -functor admitting a retraction r in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. The initiality of the algebraic trivial fibration $\varepsilon_Y: \mathbf{Q}Y \rightarrow Y$ then induces the first factor of the following composite, which classifies the putative inverse q :

$$q_s: \mathbf{Q}Y \longrightarrow Z \xrightarrow{r} \mathbf{Q}X \xrightarrow{\varepsilon_X} X$$

The problem is obtaining the factorisation $p_s = fs$. We can apply the small object argument to p_s with respect to $\mathcal{I}$ in Example 1.3.3, but then we do not know if i would have a retraction r . Instead of this ‘(cofibration, trivial fibration)-factorisation,’ we might want to try the ‘(trivial cofibration, fibration)-factorisation,’ but in this case we do not even know what the ‘generating trivial cofibrations’ should be.

Secondly, even if we can construct a weak ω -functor $q: Y \rightsquigarrow X$, we do not know what it means for q to be an inverse of p , let alone how to prove it. A potential approach might be to define a notion of pseudo-natural equivalence using *cylinders* (and in particular a weak ω -categorical version of ΓX in [LMW10] and Section A.4), although we currently have no concrete ideas beyond that. $\lrcorner$

CHAPTER 4

ω -equifibrations between strict and weak ω -categories

4.1. Introduction

The purpose of the present chapter is to introduce and study the notion of ω -*equifibration*, which is a weak ω -categorical analogue of isofibrations between ordinary categories.

The definition of ω -equifibration is the ‘obvious’ one, namely a strict ω -functor $f: X \rightarrow Y$ between weak ω -categories is an ω -equifibration if and only if, for any $(n-1)$ -cell x in X and for any equivalence n -cell (in the coinductive sense) $u: fx \rightarrow y$ in Y , there exists an equivalence n -cell $\bar{u}: x \rightarrow \bar{y}$ in X such that $f\bar{u} = u$. In the special case where both X and Y are ordinary categories, this recovers precisely the isofibrations, or equivalently, precisely the fibrations in the folk model structure on **Cat**. With an eye towards constructing a weak ω -categorical analogue of the folk model structure (cf. [LMW10; Lac02; Lac11]), our main result provides a characterisation of ω -equifibrations between weak ω -categories via the right lifting property.

A subtle point here is that the above definition of ω -equifibration refers to the *property*, as opposed to a *structure*, of u (or $\bar{u}$) being an equivalence. Recall (from Chapter 2, or from [LMW10] in the case of strict ω -categories) that an n -cell $u: a \rightarrow b$ in a weak ω -category is an *equivalence* if and only if there exist an n -cell $v: b \rightarrow a$ and *equivalence* $(n+1)$ -cells $p: u*v \rightarrow \text{id}(a)$ and $q: v*u \rightarrow \text{id}(b)$; this seemingly circular definition is formalised using coinduction. Although it is not difficult to construct a weak ω -category $\mathcal{E}^n$ with a specified n -cell i_n such that

an n -cell u in a weak ω -category Y is an equivalence if and only if
there exists a strict ω -functor $g: \mathcal{E}^n \rightarrow Y$ with $g(i_n) = u$,

specifying such g necessarily involves *choosing* a witness to the fact that u is an equivalence. Therefore the lifting property of the form

$$\begin{array}{ccc} \mathcal{C}^{n-1} & \xrightarrow{\quad} & X \\ s(i_n) \downarrow & \nearrow & \downarrow f \\ \mathcal{E}^n & \xrightarrow{\quad g \quad} & Y \end{array}$$

(where $\mathcal{C}^{n-1}$ is the weak ω -category freely generated by a single $(n-1)$ -cell and $s(i_n)$ is the strict ω -functor picking out the $(n-1)$ -dimensional source of i_n) encodes that we can lift not only the equivalence n -cell, but also the *chosen* witness (specified by g) through f , which is in general a stronger condition than f being an ω -equifibration. In order for this lifting property to characterise precisely the ω -equifibrations, intuitively the weak ω -category $\mathcal{E}^n$ must be ‘coherent’ so that for any equivalence n -cell u , the choice of corresponding g is essentially unique (although the precise statement of this essential uniqueness is beyond the scope of this thesis).

Inspired by various known results (and conjectures) [Lac04; Lac11; UFP13; Ric20; Had+25], we provide two solutions to this problem, or more precisely two characterisations of equivalences that lead to concrete models for such ‘coherent’ $\mathcal{E}^n$. The first of these two apparently weakens the original definition by allowing the left and right inverses to be distinct. It still captures the same notion of equivalence essentially for the familiar reason that, if a cell $u: a \rightarrow b$ has a right inverse v and a left inverse w , then we have

$$v \sim \text{id}(b)*v \sim (w*u)*v \sim w*(u*v) \sim w*\text{id}(a) \sim w$$

which implies that both v and w qualify as a two-sided inverse of u . In contrast, the second characterisation apparently strengthens the original definition by asking for an additional witness for one of the triangular identities. That this strengthening is only superficial can be deduced from the well-known special case in dimension 2, namely that any equivalence in a bicategory can be upgraded to an adjoint equivalence.

Our homotopical intuition for why these two characterisations may yield models for ‘coherent’ $\mathcal{E}^n$ is as follows. In the original definition, where we ask for the equivalence u to have a single inverse v and two

witnesses $p: u*v \rightarrow \text{id}(a)$ and $q: v*u \rightarrow \text{id}(b)$, topologically these cells form a sphere:¹

(4.1.0.0.1)

which is not contractible, and it is this homotopical non-triviality that spoils the essential uniqueness of $g: \mathcal{E}^n \rightarrow Y$ corresponding to u . The first characterisation cuts open this sphere by separating the left and right inverses:

(4.1.0.0.2)

whereas the second characterisation attaches an extra 3-dimensional cell in the interior of the sphere, both of which produce a contractible space.

We then investigate how our results relate to what is known about *strict* ω -categories. In particular, we prove that a strict ω -functor between strict ω -categories is an ω -equifibration if and only if it is a fibration in the folk model structure constructed by Lafont, Métayer, and Worytkiewicz [LMW10]. Consequently, by taking either of the two families of models for ‘coherent’ $\mathcal{E}^n$ and reflecting them to the strict ω -categories, we obtain an explicit generating set of folk trivial cofibrations. Moreover, for the version with separate left and right inverses, the strict ω -categorical reflection of (a certain universal model of) $\mathcal{E}^1$ turns out to be isomorphic to Ozornova and Rovelli’s *coherent walking ω -equivalence* $\widehat{\omega\mathcal{E}}$ [OR24]. As a corollary, we recover the main theorem of [Had+25], namely that $\widehat{\omega\mathcal{E}}$ is weakly equivalent to the terminal strict ω -category in the folk model structure.

Outline of the chapter. We show in Section 4.2 that two definitions of equivalence cells in a weak ω -category—one via a two-sided inverse and the other via separate left and right inverses—coincide. Then in Section 4.3, we define ω -equifibrations and show that they can be characterised by the right lifting property with respect to a suitable family $\mathcal{J}$ of strict ω -functors. More precisely, we give a general procedure to turn a suitable family $\mathcal{K}$ of morphisms between *marked* weak ω -categories into such a set $\mathcal{J}$, and then provide an example $\mathcal{K}_{\mathcal{F}}$ of $\mathcal{K}$ based on the definition of equivalences via separate left and right inverses. (Another example $\mathcal{K}_{\mathcal{H}}$ of $\mathcal{K}$ based on ‘half-adjoint’ equivalences is given in Section A.6.) Because this procedure involves choices of factorisations, the set $\mathcal{J}$ is not completely determined by $\mathcal{K}$. However, the *algebraic* small object argument [Gar09b] provides a universal one among all possible $\mathcal{J}$, and in particular we obtain a universal set $\mathcal{J}_{\mathcal{F}}$ of strict ω -functors from $\mathcal{K}_{\mathcal{F}}$. In Section 4.4, we give two explicit presentations of the morphisms in $\mathcal{J}_{\mathcal{F}}$. The second of these allows us to show that the strict ω -categorical reflection of the weak ω -category $\mathcal{E}_{\mathcal{F}}^1$ appearing in $\mathcal{J}_{\mathcal{F}}$ is isomorphic to Ozornova and Rovelli’s coherent walking ω -equivalence [OR24]. Finally, in Section 4.5, we show that the ω -equifibrations between strict ω -categories coincide with the fibrations in the folk model structure [LMW10].

Before getting into the main content of this chapter, we recollect some facts about (a special case of) the (higher dimensional) *whiskering* operation, which are basically observed in the previous chapter.

4.1.1. Whiskering operations. Let X be a weak ω -category. We have the following operations, which are special cases of Definition 1.2.3.8.

- Let $1 \leq n < m$ be natural numbers. Given an n -cell u and an m -cell $v: s \rightarrow t$ of X such that $t_{n-1}(u) = s_{n-1}(v)$, we have an m -cell

$$u *_{n-1} v: u *_{n-1} s \rightarrow u *_{n-1} t$$

of X .

¹The diagrams (4.1.0.0.1) and (4.1.0.0.2) are admittedly somewhat ambiguous. For example, the boundary of p would be more clearly represented as

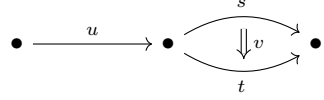
The sole purpose of (4.1.0.0.1) and (4.1.0.0.2) is to convey the geometric intuition.

- Similarly, given an m -cell $u: s \rightarrow t$ and an n -cell v of X such that $t_{n-1}(u) = s_{n-1}(v)$, we have an m -cell

$$u *_{n-1} v: s *_{n-1} v \rightarrow t *_{n-1} v$$

of X .

Intuitively, these are *whiskering* operations composing e.g.



Analogously to Lemma 1.2.3.3 and Corollary 2.4.14, strict ω -functors preserve whiskerings, and whiskering an n -cell to a pair of m -cells connected by an equivalence $(m+1)$ -cell yields again a pair connected by an equivalence $(m+1)$ -cell. Moreover, we have the following.

Proposition 4.1.1.1. Let X be a weak ω -category and $u: x \rightarrow y$ be an equivalence n -cell of X with $n \geq 1$.

- (i) For any n -cell $w: x \rightarrow z$ in X , there exists an n -cell $v: y \rightarrow z$ in X such that $w \sim u *_{n-1} v$.
- (ii) For any $m > n$, any parallel pair of $(m-1)$ -cells s and t in X with $s_{n-1}(s) = s_{n-1}(t) = y$, and any m -cell $w: u *_{n-1} s \rightarrow u *_{n-1} t$ in X , there exists an m -cell $v: s \rightarrow t$ in X such that $w \sim u *_{n-1} v$.

Moreover, in both of these situations, v is an equivalence if and only if w is. ┘

Proof. Both (i) and (ii) follow from Theorem 3.2.2.12, in view of Remark 2.2.3 (or from [LMW10, Lemma 4.6] in the strict case).

If v is an equivalence, then so is w by Theorem 2.4.7. The converse is shown in the last paragraph of the proof of Proposition 3.2.2.14, given the $\sim$ -invariance Corollary 2.4.16 of invertibility. □

The dual statements of Proposition 4.1.1.1 (concerning post-composition of an equivalence cell) also hold.

4.2. Two definitions of equivalence cells

There are two seemingly different ways to define equivalence cells in a weak ω -category X . Informally, these two definitions can be stated as follows.

- First definition [Che07; LMW10] and Definition 2.2.1: An n -cell $u: x \rightarrow y$ in X is an equivalence if there exists an n -cell $v: y \rightarrow x$ (a two-sided inverse) in X such that $u *_{n-1} v \sim \text{id}_n(x)$ and $v *_{n-1} u \sim \text{id}_n(y)$.
- Second definition [Had+25]: An n -cell $u: x \rightarrow y$ in X is an equivalence if there exist n -cells $v: y \rightarrow x$ (a right inverse) and $w: y \rightarrow x$ (a left inverse) in X such that $u *_{n-1} v \sim \text{id}_n(x)$ and $w *_{n-1} u \sim \text{id}_n(y)$.

In this section, we adapt an argument from [Ric20] and show that these two definitions in fact define the same class of cells in X . In order to avoid confusion, *in this section only* we use the terms *spherical equivalence* and *flat equivalence* to refer to the notions of equivalence defined in these two manners (see Definitions 4.2.0.2 and 4.2.0.3).

Remark 4.2.0.1. Whereas the fact that the two notions of equivalence coincide would probably follow from [Ric20, Theorem 18], we decided to include a proof for the following reasons.

- In order to apply [Ric20, Theorem 18] to weak ω -categories, one has to show that weak ω -categories *respect the graphical calculus* in the sense of [Ric20, Definition 7], which we find highly non-trivial. Although it is sufficient to only verify the special cases of [Ric20, Definition 7] that are actually used in the proof of [Ric20, Theorem 18], the reader would then have to go through that proof in order to check that we have indeed covered all the necessary cases, which seems burdensome.
- We found the proofs of [Ric20] rather hard to follow for those not familiar with type-theoretic treatment of coinduction (such as ourselves). Instead, we make use of Section 1.4, which states and justifies the necessary coinductive proof principles in ordinary mathematical (lattice-theoretic) terms.

Nevertheless, a quick inspection would reveal the similarity between our argument below and that in [Ric20]. ┘

Using the lattice-theoretic formulation of coinduction established in Section 1.4, we formalise the above two notions of equivalence cells as follows.

Definition 4.2.0.2 (Definition 2.2.1 and Remark 2.2.2 revisited). Let X be a weak ω -category. Define the monotone map $\Phi: \wp(\coprod_{n \geq 0} X_n) \rightarrow \wp(\coprod_{n \geq 0} X_n)$ by

$$\Phi(S) = \left\{ (u: x \rightarrow y) \in X_n \mid n \geq 1, \exists(v: y \rightarrow x) \in X_n, \right. \\ \left. \exists(p: u*_{n-1}v \rightarrow \text{id}_n(x)) \in S \cap X_{n+1}, \exists(q: v*_{n-1}u \rightarrow \text{id}_n(y)) \in S \cap X_{n+1} \right\}.$$

We say that a cell in X is a *spherical equivalence* if it is in the greatest (post-)fixed point $\nu\Phi \subseteq \coprod_{n \geq 0} X_n$. $\dashv$

The adjective ‘spherical’ derives from the fact that, for example, a witness for a 1-cell $u: x \rightarrow y$ in a weak ω -category X to be a spherical equivalence consists of a 1-cell $v: y \rightarrow x$ together with spherical equivalence 2-cells $p: u*_0v \rightarrow \text{id}_1(x)$ and $q: v*_0u \rightarrow \text{id}_1(y)$, which constitute a spherical shape; see (4.1.0.0.1).

Definition 4.2.0.3. Let X be a weak ω -category. Define the monotone map $\Psi: \wp(\coprod_{n \geq 0} X_n) \rightarrow \wp(\coprod_{n \geq 0} X_n)$ by

$$\Psi(S) = \left\{ (u: x \rightarrow y) \in X_n \mid n \geq 1, \exists(v: y \rightarrow x) \in X_n, \exists(w: y \rightarrow x) \in X_n, \right. \\ \left. \exists(p: u*_{n-1}v \rightarrow \text{id}_n(x)) \in S \cap X_{n+1}, \exists(q: w*_{n-1}u \rightarrow \text{id}_n(y)) \in S \cap X_{n+1} \right\}.$$

We say that a cell in X is a *flat equivalence* if it is in the greatest (post-)fixed point $\nu\Psi \subseteq \coprod_{n \geq 0} X_n$. $\dashv$

Clearly we have $\Phi(S) \subseteq \Psi(S)$ for each $S \in \wp(\coprod_{n \geq 0} X_n)$, from which it follows $\nu\Phi \subseteq \nu\Psi$, i.e., that every spherical equivalence is a flat equivalence.

In the previous chapters, we always considered spherical equivalences. In particular, various facts mentioned in Chapter 2 (including Corollaries 2.3.8 and 2.4.4 which will be used below) are to be interpreted as statements regarding spherical equivalences until we prove the two notions to coincide.

4.2.1. Closure properties of flat equivalences. Here we establish various closure properties of flat equivalences. Throughout this subsection, we fix a weak ω -category (X, ξ) .

In Theorem 2.4.7, it is shown that the class $\nu\Phi$ of all spherical equivalences in X is closed under pasting. More precisely, we defined in Definition 2.4.6 the monotone map $\text{pst}: \wp(\coprod_{n \geq 0} X_n) \rightarrow \wp(\coprod_{n \geq 0} X_n)$ by

$$\text{pst}(S) = \left\{ \xi(\phi, \mathbf{u}) \mid n \geq 0, (\phi, \mathbf{u}) \in (\text{LX})_n, \text{fdl}(\mathbf{u}) \subseteq S \right\},$$

and showed $\text{pst}(\nu\Phi) \subseteq \nu\Phi$. Recall here from Definition 2.4.5 that, for an n -dimensional pasting diagram $\mathbf{u}$ in X , we denote by $\text{fdl}(\mathbf{u})$ the set of all n -cells in X appearing in the list $\mathbf{u}$, which are called the *full-dimensional labels* of $\mathbf{u}$.

This is proved by showing that $\text{Pre}_{\nu\Phi}(\text{pst})$ is closed under Φ in light of Proposition 1.4.5. We first prove that the class $\nu\Psi$ of all flat equivalences in X is also closed under pasting, using the same strategy; Proposition 1.4.5 shows that it suffices to show the following.

Proposition 4.2.1.1. $\text{Pre}_{\nu\Psi}(\text{pst})$ is closed under Ψ . $\dashv$

Proof. We can adapt the proof of Theorem 2.4.7 to the current situation; the details (using the notation of Chapter 2) are as follows.

We first modify Definition 2.4.10. Let S be a set of cells of X .

- Given any $n \geq 1$ and any n -cell $u: x \rightarrow y$ of X , an n -cell $v: y \rightarrow x$ of X is called a *right S -inverse* of u if there exist an n -cell $w: y \rightarrow x$ and $(n+1)$ -cells $p: u*_{n-1}v \rightarrow \text{id}_n(x)$ and $q: w*_{n-1}u \rightarrow \text{id}_n(y)$ in X , with $p, q \in S$.
- Similarly, an n -cell $w: y \rightarrow x$ of X is called a *left S -inverse* of u if there exist an n -cell $v: y \rightarrow x$ and $(n+1)$ -cells $p: u*_{n-1}v \rightarrow \text{id}_n(x)$ and $q: w*_{n-1}u \rightarrow \text{id}_n(y)$ in X , with $p, q \in S$.
- Let $n \geq 1$, $(\phi, \mathbf{u}) \in (\text{LX})_n$, and $\mathbf{k} = \text{ar}(\phi)$. A *right S -inverse instruction* of $(\phi, \mathbf{u})$ is an n -cell $(\phi^{\text{inv}}, \mathbf{u}^{\text{rinv}})$ of LX satisfying the following conditions.
 - $s_{n-1}^{\text{L1}}(\phi^{\text{inv}}) = t_{n-1}^{\text{L1}}(\phi)$,
 - $t_{n-1}^{\text{L1}}(\phi^{\text{inv}}) = s_{n-1}^{\text{L1}}(\phi)$, and
 - $\mathbf{u}^{\text{rinv}} \in (\text{TX})_n$ is obtained from $\mathbf{u}$ by replacing, for each $(n-1)$ -transversal component Definition 1.1.2.6 $0 \leq i \leq j \leq r$ of $\mathbf{k}$, the corresponding segment

$$\begin{bmatrix} u_i & \dots & u_j \\ & \underline{u}_{i+1} & \dots & \underline{u}_j \end{bmatrix}$$

with

$$\begin{bmatrix} v_j & \cdots & v_i \\ & \underline{u}_j & \cdots & \underline{u}_{i+1} \end{bmatrix},$$

where v_ℓ is a right S -inverse of u_ℓ for each $i \leq \ell \leq j$.

- Similarly, define the notion of *left S -inverse instruction*.

Just like the remark immediately after Definition 2.4.10, we note that an n -cell u in X admits a right S -inverse if and only if it admits a left S -inverse, if and only if $u \in \Psi(S)$; and an n -cell $(\phi, \mathbf{u})$ in $\mathbf{LX}$ admits a right S -inverse instruction if and only if it admits a left S -inverse instruction, if and only if $\text{fdl}(\mathbf{u}) \subseteq \Psi(S)$.

Now, as in the second paragraph of the proof of Theorem 2.4.7 just before (2.4.12.1), we take $S \in \text{Pre}_{\nu\Psi}(\text{pst})$. In view of Remark 1.4.6, it suffices to show that $\text{pst}(\Psi(S)) \subseteq \Psi(S)$. To this end, instead of (2.4.12.1), we prove the following statement by induction on the number of occurrences of n -cells in $\mathbf{u}$ (denoted by $\|\text{ar}(\phi)\|^{(n)}$ in Definition 2.4.9):

$$\left| \begin{array}{l} \text{for each } n \geq 1, \text{ each } (\phi, \mathbf{u}) \in (\mathbf{LX})_n, \text{ each right } S\text{-inverse instruction } (\phi^{\text{inv}}, \mathbf{u}^{\text{rinv}}) \text{ of } (\phi, \mathbf{u}), \text{ and} \\ \text{each left } S\text{-inverse instruction } (\phi^{\text{linv}}, \mathbf{u}^{\text{linv}}) \text{ of } (\phi, \mathbf{u}), \text{ there exist } (n+1)\text{-cells} \\ \quad \xi(\phi, \mathbf{u}) *_{n-1}^X \xi(\phi^{\text{inv}}, \mathbf{u}^{\text{rinv}}) \rightarrow \text{id}_n^X (s_{n-1}^X \xi(\phi, \mathbf{u})) \\ \text{and} \\ \quad \xi(\phi^{\text{linv}}, \mathbf{u}^{\text{linv}}) *_{n-1}^X \xi(\phi, \mathbf{u}) \rightarrow \text{id}_n^X (t_{n-1}^X \xi(\phi, \mathbf{u})) \\ \text{in } S \text{ (or equivalently in } \text{pst}(S)). \end{array} \right. \quad (4.2.1.1.1)$$

This can be done exactly as in the proof of Theorem 2.4.7, bearing in mind that because we already know that spherical equivalences (called *equivalence cells* in Chapter 2) are flat equivalences, all cells induced by the coherence (Proposition 2.3.7) or the unit law (Proposition 2.4.3) are flat equivalences and hence in S . $\square$

To show that all flat equivalences are spherical equivalences, we also need to show that every right inverse of a flat equivalence is again a flat equivalence. (Note that the analogous statement for spherical equivalences is trivial.) To state this more precisely, we define the monotone map $\text{rinv}: \wp(\prod_{n \geq 0} X_n) \rightarrow \wp(\prod_{n \geq 0} X_n)$ as follows.

$$\begin{aligned} \text{rinv}(S) = \{ (v: y \rightarrow x) \in X_n \mid n \geq 1, \exists (u: x \rightarrow y) \in X_n, \exists (w: y \rightarrow x) \in X_n, \\ \exists (p: u *_{n-1} v \rightarrow \text{id}_n(x)) \in S \cap X_{n+1}, \exists (q: w *_{n-1} u \rightarrow \text{id}_n(y)) \in S \cap X_{n+1} \} \end{aligned} \quad (4.2.1.1.2)$$

In the terminology introduced in the proof of Proposition 4.2.1.1, $\text{rinv}(S)$ is the set of all cells in X which are right S -inverses of some cells in X . (One can also define the monotone map linv for left inverses similarly, but we do not need it.) The claim is that we have $\text{rinv}(\nu\Psi) \subseteq \nu\Psi$. In light of Corollary 1.4.7, it suffices to show the following.

Proposition 4.2.1.2. $\text{Pre}_{\nu\Psi}(\text{pst} \vee \text{rinv})$ is closed under $\Psi \wedge \text{id}$. $\dashv$

Proof. Let $S \in \text{Pre}_{\nu\Psi}(\text{pst} \vee \text{rinv}) = \text{Pre}_{\nu\Psi}(\text{pst}) \cap \text{Pre}_{\nu\Psi}(\text{rinv})$; we want to show $\Psi(S) \cap S \in \text{Pre}_{\nu\Psi}(\text{pst} \vee \text{rinv})$. By a reason similar to Remark 1.4.6, it suffices to show

$$\text{pst}(\Psi(S) \cap S) \cup \text{rinv}(\Psi(S) \cap S) \subseteq \Psi(S) \cap S.$$

We first show $\text{pst}(\Psi(S) \cap S) \subseteq \Psi(S) \cap S$. Since S is in $\text{Pre}_{\nu\Psi}(\text{pst})$, we have $\text{pst}(S) \subseteq S$, and hence $\text{pst}(\Psi(S) \cap S) \subseteq S$. Now observe that Proposition 4.2.1.1 implies $\Psi(S) \in \text{Pre}_{\nu\Psi}(\text{pst})$. In particular, we have $\text{pst}(\Psi(S)) \subseteq \Psi(S)$. Thus we obtain $\text{pst}(\Psi(S) \cap S) \subseteq \Psi(S) \cap S$.

Next observe that $\text{rinv}(\Psi(S) \cap S) \subseteq S$ follows from $S \in \text{Pre}_{\nu\Psi}(\text{rinv})$.

It remains to show $\text{rinv}(\Psi(S) \cap S) \subseteq \Psi(S)$. Suppose we are given an n -cell $v: y \rightarrow x$ in $\text{rinv}(\Psi(S) \cap S)$. By the definition of rinv , this means that we have

$$\begin{aligned} (u: x \rightarrow y) \in X_n, \quad (w: y \rightarrow x) \in X_n, \\ (p: u *_{n-1} v \rightarrow \text{id}_n(x)) \in \Psi(S) \cap S \cap X_{n+1}, \quad \text{and} \quad (q: w *_{n-1} u \rightarrow \text{id}_n(y)) \in \Psi(S) \cap S \cap X_{n+1}. \end{aligned}$$

Our aim is to show $v \in \Psi(S)$. Since we already have $p \in S$, it remains to construct $q': v *_{n-1} u \rightarrow \text{id}_n(x)$ in S . Thanks to $\text{pst}(S) \subseteq S$, it suffices to find $r: v \rightarrow w$ in S , since we can then set $q' = (r *_{n-1} u) *_{n-1} q$. We define r as the composite of the following string of $(n+1)$ -cells in X .

$$v \xrightarrow{(\text{unit})} \text{id}_n(y) *_{n-1} v \xrightarrow{(i)} (w *_{n-1} u) *_{n-1} v \xrightarrow{(\text{assoc})} w *_{n-1} (u *_{n-1} v) \xrightarrow{(ii)} w *_{n-1} \text{id}_n(x) \xrightarrow{(\text{unit})} w$$

Here, the cells labeled by (unit) are spherical equivalences witnessing Corollary 2.4.4, whereas the cell labeled by (assoc) is a spherical equivalence witnessing Corollary 2.3.8. They are in S because all spherical equivalences are flat equivalences and we have $\nu\Psi \subseteq S$ by the assumption. The cell (ii) is $w*_{n-1}p$, which is in S by $p \in S$ and $\text{pst}(S) \subseteq S$. Finally, to define the cell (i), observe that q admits a right S -inverse (in the sense defined in the proof of Proposition 4.2.1.1) $q_R: \text{id}_n(y) \rightarrow w*_{n-1}u$ since $q \in \Psi(S)$. We have $q_R \in \text{rinv}(S) \subseteq S$. The cell (i) is defined to be $q_R*_{n-1}v$, which is in S by $\text{pst}(S) \subseteq S$. Using $\text{pst}(S) \subseteq S$ once again, we see that the composite $(n+1)$ -cell r is in S . $\square$

We record the closure properties of the class of flat equivalences we have established.

Corollary 4.2.1.3. We have $\text{pst}(\nu\Psi) \subseteq \nu\Psi$ and $\text{rinv}(\nu\Psi) \subseteq \nu\Psi$. $\lrcorner$

Proof. By Corollary 1.4.7 and Proposition 4.2.1.2. $\square$

4.2.2. Flat equivalences are spherical equivalences. We are now ready to show that the two definitions of equivalence cells agree.

Proposition 4.2.2.1. A cell in a weak ω -category is a flat equivalence if and only if it is a spherical equivalence. $\lrcorner$

Proof. It suffices to show that all flat equivalences in a weak ω -category X are spherical equivalences. For this, it suffices to show $\nu\Psi \subseteq \Phi(\nu\Psi)$. Suppose that $u: x \rightarrow y$ is a flat equivalence n -cell in X . Thus we have n -cells $v, w: y \rightarrow x$ and flat equivalence $(n+1)$ -cells $p: u*_{n-1}v \rightarrow \text{id}_n(x)$ and $q: w*_{n-1}u \rightarrow \text{id}_n(y)$ in X . We now construct a flat equivalence $(n+1)$ -cell $v*_{n-1}u \rightarrow \text{id}_n(y)$. The $(n+1)$ -cell $r: v \rightarrow w$ constructed as in the proof of Proposition 4.2.1.2 is a flat equivalence by Corollary 4.2.1.3. Hence so is $(r*_{n-1}u)*_n q: v*_{n-1}u \rightarrow \text{id}_n(y)$, again by Corollary 4.2.1.3. Therefore we have $u \in \Phi(\nu\Psi)$ as desired. $\square$

Having now shown that spherical equivalences and flat equivalences coincide, we may call them simply *equivalences* in what follows.

Finally we redefine the notion of *inverse* of an equivalence cell.

Proposition 4.2.2.2. Let X be a weak ω -category, $n \geq 1$, and $u: x \rightarrow y$ be an equivalence n -cell in X . Then, for any n -cell $v: y \rightarrow x$ in X , the following conditions are equivalent.

- (i) We have $u*_{n-1}v \sim \text{id}_n(x)$ in X .
- (ii) We have $v*_{n-1}u \sim \text{id}_n(y)$ in X .

Moreover, these conditions imply that v is an equivalence in X . $\lrcorner$

Proof. By Definition 4.2.0.2, there exists an n -cell $v': y \rightarrow x$ with $u*_{n-1}v' \sim \text{id}_n(x)$ and $v'*_{n-1}u \sim \text{id}_n(y)$. Given $u*_{n-1}v \sim \text{id}_n(x)$, we can derive $v*_{n-1}u \sim \text{id}_n(y)$ as follows.

$$\begin{aligned}
v*_{n-1}u &\sim \text{id}_n(y)*_{n-1}(v*_{n-1}u) && \text{(by Corollary 2.4.4)} \\
&\sim (v'*_{n-1}u)*_{n-1}(v*_{n-1}u) && \text{(by Corollary 2.4.14)} \\
&\sim v'*_{n-1}((u*_{n-1}v)*_{n-1}u) && \text{(by Corollaries 2.3.8, 2.4.13 and 2.4.14)} \\
&\sim v'*_{n-1}(\text{id}_n(x)*_{n-1}u) && \text{(by Corollary 2.4.14)} \\
&\sim v'*_{n-1}u && \text{(by Corollaries 2.4.4 and 2.4.14)} \\
&\sim \text{id}_n(y).
\end{aligned}$$

The converse direction is similar.

The second statement is immediate from Definition 4.2.0.2. $\square$

Definition 4.2.2.3 (Definition 2.2.1 revisited). Let X be a weak ω -category and $u: x \rightarrow y$ be an equivalence n -cell in X , with $n \geq 1$. An n -cell $v: y \rightarrow x$ in X is called an *inverse* of u if the equivalent conditions of Proposition 4.2.2.2 are satisfied. $\lrcorner$

Recall from Corollary 2.4.15 that inverses are unique up to $\sim$.

4.3. ω -equifibrations

In this section, we define the class of ω -equifibrations between weak ω -categories and show that it can be characterised by the right lifting property with respect to a set $\mathcal{J}$ of morphisms in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. Whereas formally we will deal with weak ω -categories, the corresponding result for strict ω -categories can be obtained by replacing ‘weak’ by ‘strict’ (and $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ by $\mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s$) throughout. (Alternatively, one can apply the strict ω -categorical reflection functor $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s$ to $\mathcal{J}$; see Lemma 4.5.2.1.)

4.3.1. The definition of ω -equifibrations. The following definition generalises the notion of isofibration between categories to the context of weak ω -categories. Analogous classes of fibrations between 2-categories [Lac02], bicategories [Lac04], and Gray-categories [Lac11] have been investigated by Lack.

Definition 4.3.1.1. We say that a strict ω -functor $f: X \rightarrow Y$ between weak ω -categories is an ω -*equifibration* if for each $n \geq 1$, $(n-1)$ -cell x in X , and equivalence n -cell $u: fx \rightarrow y$ in Y , there exists an equivalence n -cell $\bar{u}: x \rightarrow \bar{y}$ in X such that $f\bar{u} = u$. $\lrcorner$

Although it may seem arbitrary that we consider equivalences u with domain (rather than codomain) fx in the above definition, this asymmetry is only superficial.

Proposition 4.3.1.2. Let $f: X \rightarrow Y$ be an ω -equifibration between weak ω -categories. Then for each $n \geq 1$, $(n-1)$ -cell x in X , and equivalence n -cell $v: y \rightarrow fx$ in Y , there exists an equivalence n -cell $\bar{v}: \bar{y} \rightarrow x$ in X such that $f\bar{v} = v$. $\lrcorner$

Proof. Let $u: fx \rightarrow y$ be an inverse of v . Since u is also an equivalence, there exists an equivalence n -cell $\bar{u}: x \rightarrow \bar{y}$ in X such that $f\bar{u} = u$. Take an inverse $\bar{v}'$ of $\bar{u}$ in X . Since f preserves equivalence $(n+1)$ -cells by Proposition 2.3.1, it sends inverses of $\bar{u}$ to inverses of $f\bar{u} = u$. Therefore, $f\bar{v}'$ and v are both inverses of u . By Corollary 2.4.15, there exists an equivalence $(n+1)$ -cell $p: f\bar{v}' \rightarrow v$ in Y , which we can lift to an equivalence $(n+1)$ -cell $\bar{p}: \bar{v}' \rightarrow \bar{v}$ in X . In particular, we have $f\bar{v} = v$. Thanks to Corollary 2.4.16, $\bar{v}$ is an equivalence in X since $\bar{v}'$ is. $\square$

As a sanity check, we show the expected identity

$$\{\text{trivial fibrations}\} = \{\text{weak equivalences}\} \cap \{\text{fibrations}\}.$$

Recall from Example 1.3.3 that a *trivial fibration* $f: X \rightarrow Y$ between weak ω -categories is just an $\mathcal{I}$ -injective morphism, i.e., a strict ω -functor that has the right lifting property against $\iota^n: \partial\mathcal{C}^n \rightarrow \mathcal{C}^n$ for each $n \geq 0$. Equivalently, f is a trivial fibration if

- for each 0-cell y in Y , there exists a 0-cell x in X such that $fx = y$, and
- for each $n \geq 1$, parallel pair of $(n-1)$ -cells x, x' in X , and n -cell $u: fx \rightarrow fx'$ in Y , there exists an n -cell $\bar{u}: x \rightarrow x'$ such that $f\bar{u} = u$.

Also recall from Definitions 3.2.1.1 and 3.2.1.3 that a strict ω -functor $f: X \rightarrow Y$ between weak ω -categories is an ω -*weak equivalence* if

- for each 0-cell y in Y , there exists a 0-cell x in X such that $fx \sim y$, and
- for each $n \geq 1$, parallel pair of $(n-1)$ -cells x, x' in X , and n -cell $u: fx \rightarrow fx'$ in Y , there exists an n -cell $\bar{u}: x \rightarrow x'$ such that $f\bar{u} \sim u$.

Proposition 4.3.1.3. For a strict ω -functor $f: X \rightarrow Y$ between weak ω -categories, the following conditions are equivalent.

- f is a trivial fibration.
- f is both an ω -weak equivalence and an ω -equifibration. $\lrcorner$

Proof. If f is a trivial fibration, then it is clearly an ω -weak equivalence, as is observed in Lemma 3.3.2.2. In particular, by Proposition 2.3.3 or Proposition 3.2.1.8, f reflects equivalence cells. Using this, we can easily see that f is an ω -equifibration.

Conversely, suppose that f is both an ω -weak equivalence and an ω -equifibration. We fix $n \geq 1$ and prove that f has the right lifting property with respect to $\iota^n: \partial\mathcal{C}^n \rightarrow \mathcal{C}^n$; the case $n = 0$ can be proven similarly. Let x, x' be a parallel pair of $(n-1)$ -cells in X and let $u: fx \rightarrow fx'$ be an n -cell in Y . Then, since f is an ω -weak equivalence, there exist an n -cell $\bar{u}': x \rightarrow x'$ in X and an equivalence $(n+1)$ -cell $v: f\bar{u}' \rightarrow u$ in Y . Moreover, since f is an ω -equifibration, there exists an equivalence $(n+1)$ -cell $\bar{v}: \bar{u}' \rightarrow \bar{u}$ in X with $f\bar{v} = v$. In particular, we have $f\bar{u} = u$. This completes the proof. $\square$

4.3.2. Marked weak ω -categories. In the introduction, we argued that the ω -equifibrations are difficult to characterise via the right lifting property because their definition refers to the *property*, rather than a *structure*, of certain cells being equivalences. It turns out that such a characterisation is easy to obtain if one can encode properties of cells as extra structure. This is our motivation for introducing the following notion.

Definition 4.3.2.1. A *marked weak ω -category* (X, tX) is a weak ω -category X equipped with distinguished subsets $tX_n \subseteq X_n$ of *marked* cells for each $n \geq 1$. A strict ω -functor $f: X \rightarrow Y$ between the underlying weak ω -categories of marked weak ω -categories (X, tX) and (Y, tY) is *marking-preserving* if for each $n \geq 1$ and each $x \in tX_n$, we have $fx \in tY_n$. $\lrcorner$

Let $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ be the category of marked weak ω -categories and marking-preserving strict ω -functors between them. The forgetful functor $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ mapping (X, tX) to X has both left and right adjoints. In particular, the left adjoint $(-)^b: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ maps each weak ω -category X to $X^b = (X, \emptyset)$. (The right adjoint $(-)^{\sharp}: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ maps X to $X^{\sharp} = (X, \coprod_{n \geq 1} X_n)$, but we shall not use this.) Thanks to Proposition 2.3.1, we have another functor $(-)^{\natural}: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ mapping each weak ω -category X to the marked weak ω -category $X^{\natural} = (X, eX)$, where eX is the set of all equivalence cells in X .

Remark 4.3.2.2. In the above definition of marked weak ω -category (X, tX) , we do not require the set tX to contain the identity cells nor be closed under compositions. Thus, even if the underlying weak ω -category X happens to be a strict ω -category, the pair (X, tX) may not be an ∞ -marked ∞ -category in the sense of [HL25, Definition 2.15]. $\lrcorner$

Recall from Notation 1.2.2.11 that $\mathcal{C}^n \in \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is the weak ω -category freely generated by the representable globular set G^n for each $n \geq 0$.

Definition 4.3.2.3. For each $n \geq 1$, we write $\mathbf{m}\mathcal{C}^n \in \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ for the marked weak ω -category $(\mathcal{C}^n, \{c_n\})$ where c_n is the image of the unique n -cell in G^n under the inclusion $G^n \rightarrow \mathcal{C}^n$. We refer to c_n as the *fundamental n -cell* of $\mathcal{C}^n$. $\lrcorner$

For each $n \geq 1$, the strict ω -functor $\sigma^{n-1}: \mathcal{C}^{n-1} \rightarrow \mathcal{C}^n$ as in (1.2.2.11.1) gives rise to the marking-preserving strict ω -functor $\sigma^{n-1}: (\mathcal{C}^{n-1})^b \rightarrow \mathbf{m}\mathcal{C}^n$. The following is the easy characterisation of ω -equifibrations alluded to at the beginning of this subsection.

Proposition 4.3.2.4. A strict ω -functor $f: X \rightarrow Y$ between weak ω -categories is an ω -equifibration if and only if the morphism $f^{\natural}: X^{\natural} \rightarrow Y^{\natural}$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ has the right lifting property² with respect to the morphism $\sigma^{n-1}: (\mathcal{C}^{n-1})^b \rightarrow \mathbf{m}\mathcal{C}^n$ for each $n \geq 1$. $\lrcorner$

We conclude this subsection by establishing the following result, which will be used later when we perform the small object argument in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$.

Proposition 4.3.2.5. The category $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is locally finitely presentable. $\lrcorner$

Proof. Observe that $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ fits into the following pullback square:

$$\begin{array}{ccc} \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s & \xrightarrow{\quad} & \mathbf{Sub}(\mathbf{Set}^{\mathbb{N}}) \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s & \xrightarrow{\quad} & \mathbf{GSet} \xrightarrow{X \mapsto (n \mapsto X_{n+1})} \mathbf{Set}^{\mathbb{N}} \end{array}$$

(The functor from $\mathbf{GSet}$ to $\mathbf{Set}^{\mathbb{N}}$ forgets X_0 ; recall that, by definition, marked cells in a marked weak ω -category have positive dimension.) Here, the right vertical functor is the subobject fibration of $\mathbf{Set}^{\mathbb{N}}$, which is in particular an isofibration. Therefore the pullback square is a bipullback square [JS93]. By Corollary 1.2.3.5, all functors in the cospan defining the pullback are finitely accessible right adjoint functors between locally finitely presentable categories. This concludes that $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is locally finitely presentable by [Bir84, Theorem 2.17]. $\square$

²In terms of Section 1.3, this means that $f^{\natural}$ is injective with respect to the set of morphisms $\{\sigma^{n-1}: (\mathcal{C}^{n-1})^b \rightarrow \mathbf{m}\mathcal{C}^n \mid n \geq 1\}$, although we will not use this viewpoint here since we do not need algebraic version for this generating set.

4.3.3. Characterising ω -equifibrations via the right lifting property in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. In this subsection, we turn Proposition 4.3.2.4 into a characterisation of ω -equifibrations via the right lifting property in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ (rather than $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$) assuming the existence of a (small) set $\mathcal{K}$ of morphisms in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ satisfying conditions (K1)–(K3) stated below. We will present an example of such a set $\mathcal{K}$ in Section 4.3.4, and another example in Section A.6.

Let $\mathcal{K}$ be a class of morphisms, seen as an indexed family of morphisms (see Remark 1.3.1), in a category $\mathbf{C}$ with a terminal object 1 . Recall from Definition 1.3.2 that a morphism $f: X \rightarrow Y$ in $\mathbf{C}$ with right lifting property with respect to each morphism in $\mathcal{K}$ is called a $\mathcal{K}$ -*injective morphism*, and $\mathcal{K}$ -cofibrations are those having the left lifting property with respect to all $\mathcal{K}$ -injective morphisms. An object X is $\mathcal{K}$ -injective if and only if the unique morphism $X \rightarrow 1$ is $\mathcal{K}$ -injective.

Now let $\mathcal{K}$ be a set of morphisms in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. We consider the following conditions on $\mathcal{K}$.

- (K1) For any ω -equifibration $f: X \rightarrow Y$ between weak ω -categories, the morphism $f^\natural: X^\natural \rightarrow Y^\natural$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is $\mathcal{K}$ -injective.

Note that the unique strict ω -functor $X \rightarrow 1$ to the terminal weak ω -category 1 is an ω -equifibration for any weak ω -category X , and $1^\natural = 1^\sharp$ is the terminal object in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. Thus, if (K1) holds, then the functor $(-)^{\natural}: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ factors through the full subcategory $\mathcal{K}\text{-}\mathbf{Inj}$ of $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. The following condition (K2) is stated under this assumption.

- (K2) The functor $(-)^{\natural}: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathcal{K}\text{-}\mathbf{Inj}$ is right adjoint to the forgetful functor $\mathcal{K}\text{-}\mathbf{Inj} \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$.
 (K3) For each $n \geq 0$, the object $(\mathcal{C}^n)^{\flat}$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is in $\mathcal{K}\text{-}\mathbf{Inj}$.

For the rest of this subsection, we fix a (small) set $\mathcal{K}$ of morphisms in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ satisfying the conditions (K1)–(K3). For each $n \geq 1$, we further fix an object $\mathcal{E}^n \in \mathcal{K}\text{-}\mathbf{Inj}$ and a morphism $i^n: \mathbf{m}\mathcal{C}^n \rightarrow \mathcal{E}^n$ in $\mathcal{K}\text{-}\mathbf{cof}$. Such data can be obtained, for example, by applying the small object argument with respect to $\mathcal{K}$ to the unique morphism $\mathbf{m}\mathcal{C}^n \rightarrow 1$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, where 1 is the terminal object of $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. (Proposition 4.3.2.5 ensures that we may indeed apply the small object argument in this category.)

Remark 4.3.3.1. Under the assumptions (K1) and (K2), the underlying weak ω -category of the $\mathcal{K}$ -injective object $\mathcal{E}^n$ can be thought of as a *walking equivalence n -cell*, in the following sense. Let X be a weak ω -category, and let x be an n -cell in X seen as a morphism $\mathcal{C}^n \rightarrow X$ in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. Then $\mathcal{E}^n$ represents equivalence n -cells in the sense that x is an equivalence n -cell in X if and only if the morphism $\mathcal{C}^n \rightarrow X$ factors through the underlying weak ω -functor $i^n: \mathcal{C}^n \rightarrow \mathcal{E}^n$ of the $\mathcal{K}$ -cofibration. The ‘if’ direction follows since i^n is a $\mathcal{K}$ -cofibration and $X^\natural$ is $\mathcal{K}$ -injective by (K1). The ‘only if’ direction follows since any morphism $\mathcal{E}^n \rightarrow X$ in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ defines a morphism $\mathcal{E}^n \rightarrow X^\natural$ in $\mathcal{K}\text{-}\mathbf{Inj}$ by (K2). Here we use that the unit for the adjunction in (K2) is identity on underlying weak ω -categories by one of the triangle identities. $\lrcorner$

Define a set $\mathcal{J}$ of morphisms in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ by

$$\mathcal{J} = \left\{ \mathcal{C}^{n-1} \xrightarrow{\sigma^{n-1}} \mathcal{C}^n \xrightarrow{i^n} \mathcal{E}^n \mid n \geq 1 \right\}. \quad (4.3.3.1.1)$$

Theorem 4.3.3.2. Let $f: X \rightarrow Y$ be a strict ω -functor between weak ω -categories. Then f is an ω -equifibration if and only if f has the right lifting property with respect to $\mathcal{J}$. $\lrcorner$

Proof. We show that the following three conditions on f are equivalent.

- (a) The morphism $f^\natural: X^\natural \rightarrow Y^\natural$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ has the right lifting property (in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$) with respect to $\sigma^{n-1}: (\mathcal{C}^{n-1})^\flat \rightarrow \mathbf{m}\mathcal{C}^n$ for each $n \geq 1$.
- (b) The morphism $f^\natural: X^\natural \rightarrow Y^\natural$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ has the right lifting property (in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$) with respect to $i^n \sigma^{n-1}: (\mathcal{C}^{n-1})^\flat \rightarrow \mathcal{E}^n$ for each $n \geq 1$.
- (c) The morphism $f: X \rightarrow Y$ in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ has the right lifting property (in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$) with respect to $i^n \sigma^{n-1}: \mathcal{C}^{n-1} \rightarrow \mathcal{E}^n$ for each $n \geq 1$.

Notice that this suffices because f is an ω -equifibration if and only if f satisfies (a) by Proposition 4.3.2.4. The equivalence of (b) and (c) follows from (K2), because the object $(\mathcal{C}^{n-1})^\flat$ is in $\mathcal{K}\text{-}\mathbf{Inj}$ by (K3), whereas the object $\mathcal{E}^n$ is in $\mathcal{K}\text{-}\mathbf{Inj}$ by construction. Therefore it remains to show the equivalence of (a) and (b).

First suppose that (a) holds. Take any $n \geq 1$ and any commutative diagram

$$\begin{array}{ccc} (\mathcal{C}^{n-1})^b & \xrightarrow{x} & X^b \\ \sigma^{n-1} \downarrow & & \downarrow f^b \\ m\mathcal{C}^n & & Y^b \\ i^n \downarrow & & \downarrow \\ \mathcal{E}^n & \xrightarrow{u} & Y^b \end{array}$$

in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$. By (a), we obtain a morphism $v: m\mathcal{C}^n \rightarrow X^b$ making the diagram

$$\begin{array}{ccc} (\mathcal{C}^{n-1})^b & \xrightarrow{x} & X^b \\ \sigma^{n-1} \downarrow & \nearrow v & \downarrow f^b \\ m\mathcal{C}^n & & Y^b \\ i^n \downarrow & & \downarrow \\ \mathcal{E}^n & \xrightarrow{u} & Y^b \end{array}$$

commute. Since $i^n \in \mathcal{K}\text{-cof}$ and f^b is $\mathcal{K}$ -injective, ((K1)), we obtain a morphism $w: \mathcal{E}^n \rightarrow X^b$ making the diagram

$$\begin{array}{ccc} (\mathcal{C}^{n-1})^b & \xrightarrow{x} & X^b \\ \sigma^{n-1} \downarrow & \nearrow v & \downarrow f^b \\ m\mathcal{C}^n & \nearrow w & Y^b \\ i^n \downarrow & & \downarrow \\ \mathcal{E}^n & \xrightarrow{u} & Y^b \end{array}$$

commute. Therefore (b) holds.

Now suppose that (b) holds and take any $n \geq 1$ and any commutative square

$$\begin{array}{ccc} (\mathcal{C}^{n-1})^b & \xrightarrow{x} & X^b \\ \sigma^{n-1} \downarrow & & \downarrow f^b \\ m\mathcal{C}^n & \xrightarrow{u} & Y^b \end{array} \quad (4.3.3.2.1)$$

in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$. By $i^n \in \mathcal{K}\text{-cof}$ and $Y^b \in \mathcal{K}\text{-Inj}$, there exists a morphism v in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$ making the following triangle commute.

$$\begin{array}{ccc} m\mathcal{C}^n & \xrightarrow{u} & Y^b \\ i^n \downarrow & \nearrow v & \\ \mathcal{E}^n & & \end{array}$$

Thus we obtain the commutative square

$$\begin{array}{ccc} (\mathcal{C}^{n-1})^b & \xrightarrow{x} & X^b \\ i^n \sigma^{n-1} \downarrow & & \downarrow f^b \\ \mathcal{E}^n & \xrightarrow{v} & Y^b \end{array}$$

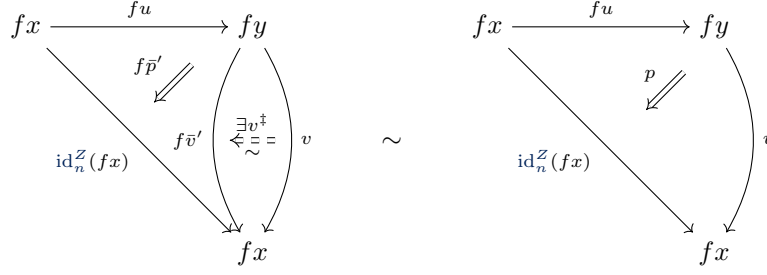
in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$, which admits a diagonal filler $w: \mathcal{E}^n \rightarrow X^b$ by (b). The morphism $w i^n: m\mathcal{C}^n \rightarrow X^b$ provides a diagonal filler for (4.3.3.2.1), and hence (a) holds. $\square$

4.3.4. An example of $\mathcal{K}$. In this subsection, we construct a set $\mathcal{K}$ of morphisms in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$ satisfying conditions (K1)–(K3).

Definition 4.3.4.1. For each $n \geq 1$, we define a marked weak ω -category $(\mathcal{F}^n, t\mathcal{F}^n)$. (The letter F stands for ‘flat (equivalence)’.) The underlying weak ω -category $\mathcal{F}^n$ is constructed as follows:

- start with $\mathcal{C}^n$, and call its fundamental n -cell $u_{\mathcal{F}}: x_{\mathcal{F}} \rightarrow y_{\mathcal{F}}$,
- freely adjoin n -cells $v_{\mathcal{F}}: y_{\mathcal{F}} \rightarrow x_{\mathcal{F}}$ and $w_{\mathcal{F}}: y_{\mathcal{F}} \rightarrow x_{\mathcal{F}}$, that is, take the colimit of the solid part of the following diagram in $\mathbf{Wk-}\omega\text{-Cat}_s$:

$$\begin{array}{ccccc} \partial\mathcal{C}^n & \xrightarrow{\langle y_{\mathcal{F}}, x_{\mathcal{F}} \rangle} & \mathcal{C}^n & \xleftarrow{\langle y_{\mathcal{F}}, x_{\mathcal{F}} \rangle} & \partial\mathcal{C}^n \\ \downarrow & & \downarrow u_{\mathcal{F}} & & \downarrow \\ \mathcal{C}^n & \xrightarrow{v_{\mathcal{F}}} & \mathcal{D}^n & \xleftarrow{w_{\mathcal{F}}} & \mathcal{C}^n \end{array}$$

FIGURE 1. Lifting v

and

- freely adjoin $(n+1)$ -cells $p_{\mathcal{F}}: u_{\mathcal{F}} *_{n-1} v_{\mathcal{F}} \rightarrow \text{id}_n(x_{\mathcal{F}})$ and $q_{\mathcal{F}}: w_{\mathcal{F}} *_{n-1} u_{\mathcal{F}} \rightarrow \text{id}_n(y_{\mathcal{F}})$.

The marked cells are $u_{\mathcal{F}}$, $p_{\mathcal{F}}$, and $q_{\mathcal{F}}$. Notice that there exists a unique marking-preserving strict ω -functor $k_{\mathcal{F}}^n: \mathbf{mC}^n \rightarrow \mathcal{F}^n$, which maps the fundamental n -cell c_n of $\mathbf{mC}^n$ to $u_{\mathcal{F}}$. $\lrcorner$

We claim that the set $\mathcal{K}_{\mathcal{F}} = \{k_{\mathcal{F}}^n: \mathbf{mC}^n \rightarrow \mathcal{F}^n \mid n \geq 1\}$ satisfies conditions (K1)–(K3). We first verify (K1).

Proposition 4.3.4.2. For any ω -equifibration $f: X \rightarrow Z$ between weak ω -categories, the marking-preserving strict ω -functor $f^\natural: X^\natural \rightarrow Z^\natural$ is $\mathcal{K}_{\mathcal{F}}$ -injective. $\lrcorner$

Proof. Consider a commutative square in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$, with $n \geq 1$:

$$\begin{array}{ccc} \mathbf{mC}^n & \xrightarrow{u} & X^\natural \\ k_{\mathcal{F}}^n \downarrow & & \downarrow f^\natural \\ \mathcal{F}^n & \longrightarrow & Z^\natural \end{array}$$

Such data are equivalent to:

- an equivalence n -cell $u: x \rightarrow y$ in X ,
- n -cells $v: fy \rightarrow fx$ and $w: fy \rightarrow fx$ in Z , and
- equivalence $(n+1)$ -cells $p: fu *_{n-1}^Z v \rightarrow \text{id}_n^Z(fx)$ and $q: w *_{n-1}^Z fu \rightarrow \text{id}_n^Z(fy)$ in Z .

We wish to extend u to a quintuple $(u, \bar{v}, \bar{w}, \bar{p}, \bar{q})$ of cells of suitable types in X (so that it corresponds to a marking-preserving map $\mathcal{F}^n \rightarrow X^\natural$) such that f sends it to (fu, v, w, p, q) . Observe that, since $u: x \rightarrow y$ is an equivalence in X , we can choose

- n -cells $\bar{v}': y \rightarrow x$ and $\bar{w}': y \rightarrow x$ in X , and
- equivalence $(n+1)$ -cells $\bar{p}': u *_{n-1}^X \bar{v}' \rightarrow \text{id}_n^X(x)$ and $\bar{q}': \bar{w}' *_{n-1}^X u \rightarrow \text{id}_n^X(y)$ in X .

Of course there is no guarantee that $f\bar{v}' = v$ (or any of the analogous equations for the other cells) holds, which is why we are calling it $\bar{v}'$ rather than $\bar{v}$. Nevertheless, we can rectify this defect and turn it into a proper lift $\bar{v}$ as follows; part of the following argument is visualised in fig. 1.

Since $f\bar{p}': fu *_{n-1}^Z f\bar{v}' \rightarrow \text{id}_n^Z(fx)$ (which indeed has this type by Lemma 1.2.3.3) is an equivalence in Z by Proposition 2.3.1, (the dual of) Proposition 4.1.1.1.(i) implies that there exists an equivalence $(n+1)$ -cell

$$v^\dagger: fu *_{n-1}^Z v \rightarrow fu *_{n-1}^Z f\bar{v}'$$

in Z satisfying

$$v^\dagger *_{n-1}^Z f\bar{p}' \sim p. \quad (4.3.4.2.1)$$

Similarly, since $fu: fx \rightarrow fy$ is an equivalence in Z , Proposition 4.1.1.1.(ii) implies that there exists an equivalence $(n+1)$ -cell

$$v^\ddagger: v \rightarrow f\bar{v}'$$

in Z satisfying

$$fu *_{n-1}^Z v^\ddagger \sim v^\dagger. \quad (4.3.4.2.2)$$

Since f is an ω -equifibration, Proposition 4.3.1.2 allows us to lift $v^\ddagger$ to an equivalence $(n+1)$ -cell in X , which we denote as $\bar{v}^\ddagger: \bar{v} \rightarrow \bar{v}'$. This completes the lifting of v .

Next, we rectify the $(n+1)$ -cell $\bar{p}'$. Observe that we have an equivalence $(n+1)$ -cell

$$(u *_{n-1}^X \bar{v}^\ddagger) *_{n-1}^X \bar{p}': u *_{n-1}^X \bar{v} \rightarrow \text{id}_n^X(x)$$

in X , and moreover (4.3.4.2.2) and (4.3.4.2.1) provide equivalence $(n+2)$ -cells

$$f((u*_{n-1}^X \bar{v}^\dagger)*_n^X \bar{p}') = (fu*_{n-1}^Z v^\dagger)*_n^Z f\bar{p}' \sim v^\dagger*_n^Z f\bar{p}' \sim p$$

in Z . Since f is an ω -equifibration, we can lift the composite of the latter to an equivalence $(n+2)$ -cell

$$(u*_{n-1}^X \bar{v}^\dagger)*_n^X \bar{p}' \sim \bar{p}$$

in X . The resulting $(n+1)$ -cell $\bar{p}$ is an equivalence in X by Corollary 2.4.16, and this completes the lifting of p .

In a similar manner, one can lift w and q . $\square$

Next we verify (K2).

Proposition 4.3.4.3.

- (i) If a marked weak ω -category (X, tX) is $\mathcal{K}_{\mathcal{F}}$ -injective, then all cells in tX are equivalences in the underlying weak ω -category X .
- (ii) For any weak ω -category X , the marked weak ω -category $X^\flat$ is $\mathcal{K}_{\mathcal{F}}$ -injective. $\lrcorner$

Proof. For any marked weak ω -category (X, tX) , the containment $tX \subseteq \Psi(tX)$ holds if and only if (X, tX) is $\mathcal{K}_{\mathcal{F}}$ -injective, which implies both statements. $\square$

Finally, (K3) is obvious because, for each $n \geq 1$, $\mathbf{mC}^n$ has a marked cell. In fact, for any weak ω -category X , the object $X^\flat$ in $\mathbf{m}\mathbf{Wk}\mathbf{-}\omega\mathbf{-Cat}_s$ is in $\mathcal{K}_{\mathcal{F}}\mathbf{-Inj}$ simply because there is no morphism $\mathbf{mC}^n \rightarrow X^\flat$ in $\mathbf{m}\mathbf{Wk}\mathbf{-}\omega\mathbf{-Cat}_s$.

4.4. Presentations of $\mathcal{E}^n$

Let $\mathcal{K}_{\mathcal{F}} = \{k_{\mathcal{F}}^n: \mathbf{mC}^n \rightarrow \mathcal{F}^n \mid n \geq 1\}$ be the set of morphisms in $\mathbf{m}\mathbf{Wk}\mathbf{-}\omega\mathbf{-Cat}_s$ constructed in Section 4.3.4. Then Theorem 4.3.3.2 provides a characterisation of ω -equifibrations in terms of a family of $\mathcal{K}_{\mathcal{F}}$ -injective marked weak ω -categories $\mathcal{E}^n$, but we said almost nothing about the structure of $\mathcal{E}^n$ in Section 4.3.3 aside from pointing out that they can be obtained using the small object argument. The aim of this section is to provide an explicit presentation of a certain universal model of $\mathcal{E}^n$, which we denote by $\mathcal{E}_{\mathcal{F}}^n$.

In fact, we provide *two* presentations of $\mathcal{E}_{\mathcal{F}}^n$. The first presentation (given in Definitions 4.4.1.1 and 4.4.1.5 and justified in Proposition 4.4.1.7) essentially traces the steps of the algebraic small object argument outlined in [Gar09b, Section 6] (although we do not explicitly prove that the individual steps of our construction actually coincide with those in [Gar09b]) whereas the second (given in Definitions 4.4.3.3 and 4.4.3.4 and justified in Proposition 4.4.3.5) constructs everything in one go. We will also deduce from the latter presentation that the strict ω -categorical reflection of $\mathcal{E}_{\mathcal{F}}^1$ is isomorphic to the *coherent walking ω -equivalence* $\widehat{\omega\mathcal{E}}$ constructed in [OR24] (which was later shown in [Had+25] to be indeed weakly equivalent to the terminal strict ω -category in the folk model structure [LMW10]).

Let us first make precise the sense in which our $\mathcal{E}_{\mathcal{F}}^n$ is universal; precisely speaking, we define $\mathcal{E}_{\mathcal{F}}^n$ as the *algebraically $\mathcal{K}_{\mathcal{F}}$ -injective replacement* (Definition 1.3.14) of $\mathbf{mC}^n$.

Notation 4.4.0.1 (cf. [FHM25b, Definition 5.0.1]). Following Definition 1.3.7, we write $\mathcal{K}_{\mathcal{F}}\mathbf{-AInj}$ for the category of algebraically $\mathcal{K}_{\mathcal{F}}$ -injective objects in $\mathbf{m}\mathbf{Wk}\mathbf{-}\omega\mathbf{-Cat}_s$. Given an algebraically $\mathcal{K}_{\mathcal{F}}$ -injective object (X, κ^X) , $n \geq 1$, and a marked n -cell $x \in tX_n$, we proceed as follows. Writing $k_{\mathcal{F}}^n: \mathbf{mC}^n \rightarrow \mathcal{F}^n \in \mathcal{K}_{\mathcal{F}}$ simply as n , and identifying morphisms from $\mathbf{mC}^n$ with marked n -cells, we define $\kappa^X(n; x)$ to be $\kappa^X(k_{\mathcal{F}}^n; [x]): \mathcal{F}^n \rightarrow X$, where $[x] = (c_n \mapsto x): \mathbf{mC}^n \rightarrow X$ is the morphism corresponding to x . Diagrammatically, this means that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{mC}^n & \xrightarrow{c_n \mapsto x} & X \\ k_{\mathcal{F}}^n \downarrow & \nearrow \kappa^X(n; x) & \\ \mathcal{F}^n & & \end{array}$$

By Proposition 4.3.2.5, we may apply Proposition 1.3.15 (which is essentially the *algebraic small object argument*, see Remark 1.3.10)) and we have the following definition.

Definition 4.4.0.2. For each $n \geq 1$, we write $(\mathcal{E}_{\mathcal{F}}^n, \kappa_{\mathcal{F}}^n)$ for the algebraically $\mathcal{K}_{\mathcal{F}}$ -injective replacement of $\mathbf{mC}^n$, and $i_{\mathcal{F}}^n: \mathbf{mC}^n \rightarrow \mathcal{E}_{\mathcal{F}}^n$ for the unit map. $\lrcorner$

Remark 1.3.16 implies that this indeed gives an example of $\mathcal{E}^n$ as in Section 4.3.3.

Lemma 4.4.0.3. For each $n \geq 1$, the unit map $i_{\mathcal{F}}^n: \mathbf{mC}^n \rightarrow \mathcal{E}_{\mathcal{F}}^n$ is in $\mathcal{K}_{\mathcal{F}}\mathbf{-cof}$. $\lrcorner$

(An alternative proof for this lemma can be obtained at the end of Section 4.4.1: Proposition 4.4.1.7 shows that $i_{\mathcal{F}}^n = \lambda^0$, the latter being constructed from members of $\mathcal{K}_{\mathcal{F}}$ by pushouts and transfinite composition.)

Combining Theorem 4.3.3.2 and the content of Section 4.3.4, we obtain the following.

Proposition 4.4.0.4. Let $f: X \rightarrow Y$ be a strict ω -functor between weak ω -categories. Then f is an ω -equifibration if and only if f has the right lifting property with respect to

$$\mathcal{J}_{\mathcal{F}} = \{ \mathcal{C}^{n-1} \xrightarrow{\sigma^{n-1}} \mathcal{C}^n \xrightarrow{i_{\mathcal{F}}^n} \mathcal{E}_{\mathcal{F}}^n \mid n \geq 1 \}. \quad \lrcorner$$

Since we will be dealing with various colimits in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, let us record the following observation.

Remark 4.4.0.5. The forgetful functor $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ admits a right adjoint (see Section 4.3.2), and thus preserves colimits. In other words, for any small diagram $X: I \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, the underlying weak ω -category of $\text{colim } X$ may be computed in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. The marking $t(\text{colim } X)$ is given by $\bigcup_{i \in I} \lambda^i(tX(i))$ where $\lambda^i: X(i) \rightarrow \text{colim } X$ is the leg of the colimiting cocone at $i \in I$. $\lrcorner$

4.4.1. Presenting $\mathcal{E}_{\mathcal{F}}^n$ by unravelling the algebraic small object argument. Fix $n \geq 1$. In this subsection, we present the marked weak ω -category $\mathcal{E}_{\mathcal{F}}^n$ as the colimit of an infinite sequence in the manner outlined in [Gar09b, Section 6]. This sequence comprises of the following family of marked weak ω -categories.

Definition 4.4.1.1. For each $m \in \mathbb{N}$, we define

- a marked weak ω -category $\mathcal{E}_{\mathcal{F}}^{n,m} = (\mathcal{E}_{\mathcal{F}}^{n,m}, t\mathcal{E}_{\mathcal{F}}^{n,m})$, whose set of marked $(n+m)$ -cells is denoted by $T^{n,m} = (t\mathcal{E}_{\mathcal{F}}^{n,m})_{n+m}$,
- a marking-preserving strict ω -functor $\mathbf{e}^{m+1}: \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,m+1}$, and
- a marking-preserving strict ω -functor $\mathbf{f}^{m+1}: \coprod_{u \in T^{n,m}} \mathcal{F}^{n+m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,m+1}$

inductively as follows.

- Set $\mathcal{E}_{\mathcal{F}}^{n,0} = \mathbf{m}\mathcal{C}^n$ (so that $T^{n,0} = \{c_n\}$).
- For $m \geq 0$, the marked weak ω -category $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ is defined by the following pushout in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$:

$$\begin{array}{ccc} \coprod_{u \in T^{n,m}} \mathbf{m}\mathcal{C}^{n+m} & \xrightarrow{(u, c_{n+m}) \mapsto u} & \mathcal{E}_{\mathcal{F}}^{n,m} \\ \downarrow & \lrcorner & \downarrow \mathbf{e}^{m+1} \\ \coprod_{u \in T^{n,m}} \mathcal{F}^{n+m} & \xrightarrow{\mathbf{f}^{m+1}} & \mathcal{E}_{\mathcal{F}}^{n,m+1} \end{array} \quad (4.4.1.1.1)$$

Thus the restriction of $\mathbf{f}^{m+1}$ to the u -th summand, which we denote by $\mathbf{f}_u^{m+1}$, sends $u_{\mathcal{F}}$ to $\mathbf{e}^{m+1}(u)$. $\lrcorner$

Remark 4.4.1.2. By Remark 4.4.0.5, the underlying weak ω -category of $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ for $m \geq 0$ can be obtained by computing the pushout (4.4.1.1.1) in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, and a cell in $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ is marked if and only if it is the image of a marked cell under either $\mathbf{f}^{m+1}$ or $\mathbf{e}^{m+1}$. In fact, one can easily check by induction on m that

- $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ has no marked k -cells for $k < n$,
- a k -cell in $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ with $n \leq k < n+m$ is marked if and only if it is the image of a marked k -cell under $\mathbf{e}^{m+1}$ if and only if it is the image of a marked k -cell under $\mathbf{e}^{m+1} \circ \dots \circ \mathbf{e}^{k-n+1}$,
- an $(n+m)$ -cell in $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ is marked if and only if it is the image of a marked $(n+m)$ -cell under $\mathbf{f}^{m+1}$ if and only if it is the image of a marked $(n+m)$ -cell under $\mathbf{e}^{m+1}$,
- an $(n+m+1)$ -cell in $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ is marked if and only if it is the image of a marked $(n+m+1)$ -cell under $\mathbf{f}^{m+1}$, and
- $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ has no marked k -cells for $k > n+m+1$. $\lrcorner$

The following lemma provides an even more explicit description of the marked cells in $\mathcal{E}_{\mathcal{F}}^{n,m+1}$.

Lemma 4.4.1.3. For any $m \geq 0$ and $n \leq k \leq n+m$, the functions

$$(t\mathbf{e}^{m+1})_k: (t\mathcal{E}_{\mathcal{F}}^{n,m})_k \rightarrow (t\mathcal{E}_{\mathcal{F}}^{n,m+1})_k \quad \text{and} \quad (t\mathbf{f}^{m+1})_{n+m+1}: \coprod_{u \in T^{n,m}} (t\mathcal{F}^{n+m})_{n+m+1} \rightarrow (t\mathcal{E}_{\mathcal{F}}^{n,m+1})_{n+m+1} \quad (4.4.1.3.1)$$

(between sets of marked k - and $(n+m+1)$ -cells) are bijective. $\lrcorner$

Proof. First observe that the pushout square (4.4.1.1.1) induces the commutative square

$$\begin{array}{ccc} \coprod_{u \in T^{n,m}} (t(\mathbf{m}^{n+m}))_k & \longrightarrow & (t\mathcal{E}_{\mathcal{F}}^{n,m})_k \\ \downarrow & & \downarrow (te^{m+1})_k \\ \coprod_{u \in T^{n,m}} (t\mathcal{F}^{n+m})_k & \xrightarrow{(tf^{m+1})_k} & (t\mathcal{E}_{\mathcal{F}}^{n,m+1})_k \end{array} \quad (4.4.1.3.2)$$

of functions between sets of marked k -cells for each $k \geq 1$, in which the functions $(te^{m+1})_k$ and $(tf^{m+1})_k$ are jointly surjective by Remark 4.4.0.5 (or by Remark 4.4.1.2). We begin with the proof of the surjectivity of the functions (4.4.1.3.1). When $k < n + m$, the function $(te^{m+1})_k$ is surjective because the set $\coprod_{u \in T^{n,m}} (t\mathcal{F}^{n+m})_k$ is empty. Similarly, the function $(tf^{m+1})_{n+m+1}$ is surjective because the set $(t\mathcal{E}_{\mathcal{F}}^{n,m})_{n+m+1}$ is empty. Finally, when $k = n + m$, both the top horizontal arrow and the left vertical arrow in (4.4.1.3.2) are bijections, and hence both $(te^{m+1})_{n+m}$ and $(tf^{m+1})_{n+m}$ are surjective.

To show the injectivity of the functions (4.4.1.3.1), we introduce the following strict ω -categories.

Definition 4.4.1.4. For $k \geq 1$, we write W^k for the following strict ω -category. Its cells are given by

$$W_l^k = \begin{cases} \{\star\} & l < k, \\ \{w_k, \star\} & l = k, \\ \{w_k, \star\}^2 & l > k, \end{cases}$$

and the source and target operations are given by the identity except for

$$\begin{aligned} s_{k-1}(w_k) &= s_{k-1}(\star) = t_{k-1}(w_k) = t_{k-1}(\star) = \star, \\ s_k &= \pi_1 \quad \text{and} \quad t_k = \pi_2. \end{aligned}$$

In other words, the underlying globular set of W^k is the one that classifies ‘sets of k -cells’, in the sense that for any globular set X , giving a globular map $g: X \rightarrow W^k$ is equivalent to giving a subset of X_k corresponding to $g^{-1}(w_k)$. This follows from the fact that the underlying globular set of W^k is obtained by applying the right adjoint of the ‘evaluation at k ’ functor $\mathbf{GSet} \rightarrow \mathbf{Set}$, mapping each $X \in \mathbf{GSet}$ to $X_k \in \mathbf{Set}$, to the two-element set.

The strict ω -category structure on W^k is given as follows. Each l -cell shares its name with the identity $(l+1)$ -cell on it except for

$$\text{id}_{k+1}(w_k) = (w_k, w_k) \quad \text{and} \quad \text{id}_{k+1}(\star) = (\star, \star).$$

Note that we have only identity cells in all positive dimensions but k and $k+1$. The compositions of k -cells are given by

$$x *_l y = \begin{cases} \star & x = y = \star, \\ w_k & \text{otherwise} \end{cases}$$

for all $0 \leq l < k$. The compositions of $(k+1)$ -cells are given by

$$(x_1, x_2) *_l (y_1, y_2) = \begin{cases} (x_1 *_l y_1, x_2 *_l y_2) & 0 \leq l < k, \\ (x_1, y_2) & l = k \text{ (and hence } x_2 = y_1). \end{cases}$$

The compositions of l -cells for $l > k$ are determined by these because all l -cells are identity cells. $\square$

Observe that the unique strict ω -functor $W^k \rightarrow 1$ from W^k to the terminal weak ω -category 1 is a trivial fibration in the sense of Example 1.3.3. This can be seen by transposing lifting problems along suitable adjunctions as follows. Suppose that we are given the lifting problem in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ as in the left below. Using the adjunction $F_L \dashv U_L$ and the fact that $\iota_n: \partial \mathcal{C}^n \rightarrow \mathcal{C}^n$ is in the image of F_L , we see that this lifting problem is equivalent to the lifting problem in $\mathbf{GSet}$ as in the right below.

$$\begin{array}{ccc} \partial \mathcal{C}^n & \longrightarrow & W^k \\ \iota^n \downarrow & & \downarrow \\ \mathcal{C}^n & \longrightarrow & 1 \end{array} \quad \begin{array}{ccc} \partial G^n & \longrightarrow & U_L W^k \\ \iota^n \downarrow & & \downarrow \\ G^n & \longrightarrow & 1 \end{array}$$

Using the adjunction consisting of the ‘evaluation at k ’ functor $\mathbf{GSet} \rightarrow \mathbf{Set}$ and its right adjoint, and the fact that $U_L W^k \rightarrow 1$ is in the image of this right adjoint, we see that the latter lifting problem is equivalent to a trivial lifting problem in $\mathbf{Set}$, and thus can be solved.

Claim 1. Let $m \geq 0$, and let a, b be distinct marked $(n + m + 1)$ -cells in $\coprod_{u \in T^{n,m}} \mathcal{F}^{n+m}$. Then there exists a strict ω -functor $g: \mathcal{E}_{\mathcal{F}}^{n,m+1} \rightarrow \mathbf{W}^{n+m+1}$ such that $gf^{m+1}(a) = w_{n+m+1}$ and $gf^{m+1}(b) = \star$.

$\therefore$ (Proof of Claim 1) Observe that (the underlying weak ω -category of) $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ can be written as the iterated pushout

$$\begin{array}{ccccc} \coprod_{u \in T^{n,m}} \mathcal{C}^{n+m} & \xrightarrow{(u, c_{n+m}) \mapsto u} & \mathcal{E}_{\mathcal{F}}^{n,m} & & \\ \downarrow & & \downarrow & \lrcorner & \\ X & \xrightarrow{\quad} & Y & \xleftarrow{\quad} & \partial \mathcal{C}^{n+m+1} \\ & & \downarrow & \lrcorner & \downarrow \\ & & \mathcal{E}_{\mathcal{F}}^{n,m+1} & \xleftarrow{a} & \mathcal{C}^{n+m+1} \end{array}$$

where X is ' $\coprod_{u \in T^{n,m}} \mathcal{F}^{n+m}$ with a removed'. The desired g can be thus obtained as the unique strict ω -functor induced by $Y \xrightarrow{!} \mathcal{C}^0 \xrightarrow{\star} \mathbf{W}^{n+m+1}$ and $\mathcal{C}^{n+m+1} \xrightarrow{w_{n+m+1}} \mathbf{W}^{n+m+1}$. This completes the proof of Claim 1. $\diamond$

Claim 2. For any $m \geq 0$ and any distinct marked k -cells a, b in $\mathcal{E}_{\mathcal{F}}^{n,m}$ with $k \geq 1$, there exist

- (1) a strict ω -functor $g: \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathbf{W}^k$ such that $g(a) = w_k$ and $g(b) = \star$, and
- (2) a strict ω -functor $h: \mathcal{E}_{\mathcal{F}}^{n,m+1} \rightarrow \mathbf{W}^k$ such that $he^{m+1}(a) = w_k$ and $he^{m+1}(b) = \star$.

$\therefore$ (Proof of Claim 2) We proceed by induction on m . The base case $m = 0$ is vacuous since $\mathcal{E}_{\mathcal{F}}^{n,0} = \mathcal{M}\mathcal{C}^n$ has only one marked cell. For the inductive step, let $m \geq 0$ and let a, b be distinct marked k -cells in $\mathcal{E}_{\mathcal{F}}^{n,m+1}$. Note that we have $n \leq k \leq n + m + 1$.

We first prove (1) for $m+1$, that is, the existence of a suitable strict ω -functor $g: \mathcal{E}_{\mathcal{F}}^{n,m+1} \rightarrow \mathbf{W}^k$ (note the index). In the case $k \leq n + m$, there exist (necessarily distinct) marked k -cells a', b' in $\mathcal{E}_{\mathcal{F}}^{n,m}$ with $e^{m+1}(a') = a$ and $e^{m+1}(b') = b$. By the inductive hypothesis (2), we have a strict ω -functor $h': \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathbf{W}^k$ such that $h'(a') = w_k$ and $h'(b') = \star$, and we can simply take $g = h'$. In the case $k = n + m + 1$, there exist (necessarily distinct) marked k -cells a', b' in $\coprod_{u \in T^{n,m}} \mathcal{F}^{n+m}$ such that $f^{m+1}(a') = a$ and $f^{m+1}(b') = b$. The desired g can be thus obtained by applying Claim 1 to the pair a', b' .

Next we prove (2) for $m+1$, that is, the existence of a suitable strict ω -functor $h: \mathcal{E}_{\mathcal{F}}^{n,m+2} \rightarrow \mathbf{W}^k$. Since we have already constructed $g: \mathcal{E}_{\mathcal{F}}^{n,m+1} \rightarrow \mathbf{W}^k$, it suffices to construct a strict ω -functor $h_u: \mathcal{F}^{n+m+1} \rightarrow \mathbf{W}^k$ satisfying $h_u(u_{\mathcal{F}}) = g(u)$ for each $u \in T^{n,m+1}$. This can be achieved by solving the following lifting problem:

$$\begin{array}{ccc} \mathcal{C}^{n+m+1} & \xrightarrow{g(u)} & \mathbf{W}^k \\ u_{\mathcal{F}} \downarrow & & \downarrow \\ \mathcal{F}^{n+m+1} & \longrightarrow & 1 \end{array}$$

in which the strict ω -functor $u_{\mathcal{F}}: \mathcal{C}^{n+m+1} \rightarrow \mathcal{F}^{n+m+1}$ is a *cofibration* (i.e., is an $\mathcal{I}$ -cofibration) and $\mathbf{W}^k \rightarrow 1$ is a trivial fibration (i.e., is an $\mathcal{I}$ -injective morphism). This completes the proof of Claim 2. $\diamond$

The injectivity of the functions (4.4.1.3.1) follows from (2) of Claim 2 and Claim 1. This completes the proof of Lemma 4.4.1.3. $\square$

Now we consider the colimit of the marked weak ω -categories $\mathcal{E}_{\mathcal{F}}^{n,m}$, corresponding to the final result of the algebraic small object argument.

Definition 4.4.1.5. We write $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ for the colimit of the sequence

$$\mathcal{E}_{\mathcal{F}}^{n,0} \xrightarrow{e^1} \mathcal{E}_{\mathcal{F}}^{n,1} \xrightarrow{e^2} \mathcal{E}_{\mathcal{F}}^{n,2} \xrightarrow{e^3} \dots \quad (4.4.1.5.1)$$

in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, and write $(\lambda^m: \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,\omega})_{m \geq 0}$ for the colimiting cocone. $\lrcorner$

Again by Remark 4.4.0.5, the underlying weak ω -category of $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ can be computed as the colimit of the sequence in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$, and the marking $t\mathcal{E}_{\mathcal{F}}^{n,\omega}$ is given by the union $\bigcup_{m \geq 0} \lambda^m(t\mathcal{E}_{\mathcal{F}}^{n,m})$. In particular, all marked cells in $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ have dimension at least n , and $\lambda^0(c_n)$ is the only marked n -cell in $\mathcal{E}_{\mathcal{F}}^{n,\omega}$. In fact, we can say the following about the marked cells in $\mathcal{E}_{\mathcal{F}}^{n,\omega}$.

Lemma 4.4.1.6. The (graded) function $t\lambda^m: t\mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow t\mathcal{E}_{\mathcal{F}}^{n,\omega}$ is a monomorphism for each $m \geq 0$. Consequently, $\lambda^m: \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,\omega}$ induces a bijection $T^{n,m} \cong (t\mathcal{E}_{\mathcal{F}}^{n,\omega})_{n+m}$ for each $m \geq 0$. $\lrcorner$

Proof. By Corollary 1.2.3.5, the (filtered) colimit defining $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ is created by the forgetful functor to **GSet**. Therefore we have $(\mathcal{E}_{\mathcal{F}}^{n,\omega})_k = \coprod_{m \in \mathbb{N}} (\mathcal{E}_{\mathcal{F}}^{n,m})_k / \sim$, where $(m, x) \sim (m', x')$ if and only if there exists $\bar{m} \geq \max(m, m')$ such that $e^{\bar{m}/m}(x) = e^{\bar{m}/m'}(x')$ where $e^{\bar{m}/m}: \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,\bar{m}}$ is the composite $e^{\bar{m}} \circ \dots \circ e^{m+1}$. Hence, given $x, x' \in t\mathcal{E}_{\mathcal{F}}^{n,m}$ with $\lambda^m(x) = \lambda^m(x')$, we can pick $\bar{m} \geq m$ such that $e^{\bar{m}/m}(x) = e^{\bar{m}/m}(x')$. Since $te^{\bar{m}/m}$ is a monomorphism by Lemma 4.4.1.3, we can conclude $x = x'$. This completes the proof of the first assertion, and the second assertion follows from Remark 4.4.1.2. $\square$

Finally we are ready to prove the main result of this subsection.

Proposition 4.4.1.7. There is a structure κ of algebraically $\mathcal{K}_{\mathcal{F}}$ -injective object on $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ such that

$$\mathbf{mC}^n = \mathcal{E}_{\mathcal{F}}^{n,0} \xrightarrow{\lambda^0} \mathcal{E}_{\mathcal{F}}^{n,\omega}$$

exhibits $(\mathcal{E}_{\mathcal{F}}^{n,\omega}, \kappa)$ as the algebraically $\mathcal{K}_{\mathcal{F}}$ -injective replacement of $\mathbf{mC}^n$. In particular, we have $(\mathcal{E}_{\mathcal{F}}^{n,\omega}, \kappa) \cong (\mathcal{E}_{\mathcal{F}}^n, \kappa_{\mathcal{F}})$. $\dashv$

Proof. We must specify, for each marked $(n+m)$ -cell u_{ω} in $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ with $m \geq 0$, a strict ω -functor

$$\kappa(n+m; u_{\omega}): \mathcal{F}^{n+m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,\omega}$$

that sends $u_{\mathcal{F}}$ to u_{ω} . Given such u_{ω} , Lemma 4.4.1.6 provides unique $u \in T^{n,m}$ such that $\lambda^m(u) = u_{\omega}$. We set

$$\kappa(n+m; u_{\omega}) = \lambda^{m+1} \circ f_u^{m+1},$$

and this defines an algebraically $\mathcal{K}_{\mathcal{F}}$ -injective object $(\mathcal{E}_{\mathcal{F}}^{n,\omega}, \kappa)$.

We now prove that $(\mathcal{E}_{\mathcal{F}}^{n,\omega}, \kappa)$ enjoys the desired universal property. Let (Z, κ^Z) be an algebraically $\mathcal{K}_{\mathcal{F}}$ -injective object and z be a marked n -cell in Z . We wish to show that there exists a unique morphism

$$g: (\mathcal{E}_{\mathcal{F}}^{n,\omega}, \kappa) \rightarrow (Z, \kappa^Z)$$

in $\mathcal{K}_{\mathcal{F}}\text{-}\mathbf{AInj}$ such that $g \circ \lambda^0(c_n) = z$. Define a cocone $(g^m)_{m \in \mathbb{N}}$ under the sequence (4.4.1.5.1) in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ inductively as follows.

- $g^0: \mathbf{mC}^n \rightarrow Z$ is the unique morphism that sends c_n to z .
- Define $\bar{g}^{m+1}: \coprod_{u \in T^{n,m}} \mathcal{F}^{n+m} \rightarrow Z$ so that its restriction to the u -th summand is

$$\kappa^Z(n+m; g^m(u)): \mathcal{F}^{n+m} \rightarrow Z,$$

and define g^{m+1} as the morphism induced by g^m and $\bar{g}^{m+1}$ via the universality of the pushout square (4.4.1.1.1). In other words, the strict ω -functor g^{m+1} is the unique one satisfying $g^{m+1} \circ e^{m+1} = g^m$ and $g^{m+1} \circ f_u^{m+1} = \kappa^Z(n+m; g^m(u))$ for all $u \in T^{n,m}$.

This cocone induces a marking-preserving strict ω -functor $g: \mathcal{E}_{\mathcal{F}}^{n,\omega} \rightarrow Z$, and we claim that this is the desired morphism in $\mathcal{K}_{\mathcal{F}}\text{-}\mathbf{AInj}$.

For g to be a morphism in $\mathcal{K}_{\mathcal{F}}\text{-}\mathbf{AInj}$, we must have

$$g \circ \kappa(n+m; u_{\omega}) = \kappa^Z(n+m; g(u_{\omega}))$$

for each marked $(n+m)$ -cell u_{ω} in $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ with $m \geq 0$. Let $u \in T^{n,m}$ be the unique $(n+m)$ -cell provided by Remark 4.4.1.2 such that $\lambda^m(u) = u_{\omega}$. The desired equality can be verified as

$$\begin{aligned} g \circ \kappa(n+m; u_{\omega}) &= g \circ \lambda^{m+1} \circ f_u^{m+1} && \text{(definition of } \kappa) \\ &= g^{m+1} \circ f_u^{m+1} && \text{(definition of } g) \\ &= \kappa^Z(n+m; g^m(u)) && \text{(definition of } g^{m+1}) \\ &= \kappa^Z(n+m; g \circ \lambda^m(u)) && \text{(definition of } g) \\ &= \kappa^Z(n+m; g(u_{\omega})). \end{aligned}$$

It remains to prove the uniqueness of g . To this end, let $g': (\mathcal{E}_{\mathcal{F}}^{n,\omega}, \kappa) \rightarrow (Z, \kappa^Z)$ be a morphism in $\mathcal{K}_{\mathcal{F}}\text{-}\mathbf{AInj}$ such that $g' \circ \lambda^0(c_n) = z$. We will prove by induction on $m \geq 0$ that we necessarily have $g' \circ \lambda^m = g^m$. The base case $g' \circ \lambda^0 = g^0$ is precisely the assumption $g \circ \lambda^0(c_n) = z$. For the inductive step, let $m \geq 0$ and suppose that $g' \circ \lambda^m = g^m$ holds. Since $\mathcal{E}_{\mathcal{F}}^{n,m+1}$ is defined as the pushout (4.4.1.1.1), in order to prove $g' \circ \lambda^{m+1} = g^{m+1}$, it suffices to show that

- $g' \circ \lambda^{m+1} \circ f_u^{m+1} = g^{m+1} \circ f_u^{m+1}$ holds for each $u \in T^{n,m}$, and
- $g' \circ \lambda^{m+1} \circ e^{m+1} = g^{m+1} \circ e^{m+1}$ holds.

These conditions can be verified as

$$\begin{aligned}
 g' \circ \lambda^{m+1} \circ f_u^{m+1} &= g' \circ \kappa(n+m; \lambda^m(u)) && \text{(definition of } \kappa) \\
 &= \kappa^Z(n+m; g' \circ \lambda^m(u)) && (g' \text{ is a morphism in } \mathcal{K}_{\mathcal{F}}\text{-}\mathbf{AInj}) \\
 &= \kappa^Z(n+m; g^m(u)) && \text{(inductive hypothesis)} \\
 &= g^{m+1} \circ f_u^{m+1} && \text{(definition of } g^{m+1})
 \end{aligned}$$

and

$$\begin{aligned}
 g' \circ \lambda^{m+1} \circ e^{m+1} &= g' \circ \lambda^m && \text{(cocone condition)} \\
 &= g^m && \text{(inductive hypothesis)} \\
 &= g^{m+1} \circ e^{m+1}. && \text{(cocone condition)}
 \end{aligned}$$

Thus we can conclude $g' = g$, and this completes the proof. $\square$

4.4.2. Suspension construction. Before giving another presentation of $\mathcal{E}_{\mathcal{F}}^n$, we need to introduce and analyse a *suspension* construction for (marked) weak ω -categories. In order to reassure that it is compatible with the familiar suspension construction for strict ω -categories, we first recall from [CF22] the following fact about the *hom weak ω -category* (see Section 1.2.4).

Remark 4.4.2.1. Given a weak ω -category X and a pair of 0-cells x, y in X , we defined in Fact 1.2.4.1 the hom weak ω -category $X(x, y)$. Varying x and y , we obtain a $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$ -enriched graph, and this extends to a functor which fits into middle row of the following diagram:

$$\begin{array}{ccc}
 \mathbf{Str}\text{-}\omega\text{-Cat}_s & \longrightarrow & (\mathbf{Str}\text{-}\omega\text{-Cat}_s)\text{-Gph} \\
 \downarrow & \cong & \downarrow \\
 \mathbf{Wk}\text{-}\omega\text{-Cat}_s & \longrightarrow & (\mathbf{Wk}\text{-}\omega\text{-Cat}_s)\text{-Gph} \\
 \downarrow & \cong & \downarrow \\
 \mathbf{GSet} & \xrightarrow{\sim} & \mathbf{GSet}\text{-Gph},
 \end{array} \tag{4.4.2.1.1}$$

where the other two horizontal maps are the more familiar hom functors. This is essentially the main result of [CF22, Section 3], but a clarification is in order.

The authors of [CF22] use the category $\omega\text{-Gph}$ of ω -graphs (informally speaking, ‘ ω -fold’ enriched graphs) instead of $\mathbf{GSet}$ as the underlying category of the monads $\mathbf{T}$ and $\mathbf{L}$ (so that, in particular, the bottom horizontal map in (4.4.2.1.1) becomes an isomorphism). We can obtain (4.4.2.1.1) by transferring the corresponding (strictly commutative) diagram for $\omega\text{-Gph}$ along an equivalence $\mathbf{GSet} \simeq \omega\text{-Gph}$ (cf. [Lei04, Appendix F] and Remark B.3.4). Such a diagram can be extrapolated from [CF22] as follows. The perimeter of [CF22, (19)] is a commutative square

$$\begin{array}{ccc}
 [\mathbf{L}] & \longrightarrow & \mathbf{L} \\
 \downarrow & & \downarrow \\
 (\mathbf{T}^{(\omega)}\text{-}\mathbf{Gph})1 & \longrightarrow & \mathbf{T}^{(\omega)}1
 \end{array}$$

of *globular operads* [Bat98; Lei04] and Section A.2.1 (or $\mathbf{T}^{(\omega)}$ -operads in [CF22]), which can be thought of as representing certain monads on $\mathbf{GSet}$ or $\omega\text{-Gph}$ (see [CF22, Subsection 2.1] and Section A.2.1 for a summary or [Lei04] for full details). Unravelling their notations [CF22, Subsections 2.2, 2.4, and Section 3], one can check that

- $\mathbf{T}^{(\omega)}1$ corresponds to the monad for strict ω -categories,
- $(\mathbf{T}^{(\omega)}\text{-}\mathbf{Gph})1$ corresponds to the monad for $\mathbf{Str}\text{-}\omega\text{-Cat}_s$ -enriched graphs,
- $\mathbf{L}$ corresponds to the monad for weak ω -categories, and
- $[\mathbf{L}]$ corresponds to the monad for $\mathbf{Wk}\text{-}\omega\text{-Cat}_s$ -enriched graphs.

Taking the categories of algebras thus yields the upper half of (4.4.2.1.1), and the lower half simply witnesses that these functors commute with the forgetful functors to the base category of the monads. $\lrcorner$

Proposition 4.4.2.2. Taking each 2-pointed weak ω -category $(x, y): \mathcal{C}^0 \sqcup \mathcal{C}^0 \rightarrow X$ to the hom weak ω -category $X(x, y)$ extends to a functor $\mathcal{C}^0 \sqcup \mathcal{C}^0 / \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ which fits into the following diagram:

$$\begin{array}{ccc}
 \mathcal{C}^0 \sqcup \mathcal{C}^0 / \mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s & \xrightarrow{\text{hom}} & \mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s \\
 \downarrow & \cong & \downarrow \\
 \mathcal{C}^0 \sqcup \mathcal{C}^0 / \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s & \xrightarrow{\text{hom}} & \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \\
 \downarrow & \cong & \downarrow \\
 (\mathbf{GSet}_{**})^{G^0 \sqcup G^0} / \mathbf{GSet} & \xrightarrow{\text{hom}} & \mathbf{GSet}
 \end{array} \tag{4.4.2.2.1}$$

Moreover, all functors in this diagram are right adjoints. $\lrcorner$

Proof. The first assertion follows from Remark 4.4.2.1, so we only need to prove the second assertion. We know that all categories in (4.4.2.2.1) are cocomplete and moreover the functors

$$\mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{GSet}, \quad \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{GSet}$$

and their sliced variants are all monadic. Hence, by [BW85, Theorem 3.7.3.(b)], it suffices to show that the bottom horizontal functor is a right adjoint. As observed in Remark 1.1.1.9. it indeed admits a left adjoint. $\square$

Remark 4.4.2.3. We do not know whether the hom-suspension adjunction for weak ω -categories in Proposition 4.4.2.2 coincides with the version constructed by Benjamin and Markakis [BM24]; see the ‘related work’ section of that paper. $\lrcorner$

Definition 4.4.2.4. For each weak ω -category X , we write $(\Sigma X, (\star, \star'))$ for its image under the left adjoint of $\text{hom}: \mathcal{C}^0 \sqcup \mathcal{C}^0 / \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. The components of the unit of this adjunction will be denoted by $\Sigma: X \rightarrow \Sigma X(\star, \star')$. The *suspension* functor $\Sigma: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is the composite of this left adjoint with the forgetful functor $\mathcal{C}^0 \sqcup \mathcal{C}^0 / \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. $\lrcorner$

Note that the functor Σ preserves all connected colimits.

Proposition 4.4.2.5. For each weak ω -category X , the family of functions $(\Sigma: X_n \rightarrow \Sigma X_{n+1})_{n \in \mathbb{N}}$ preserves binary compositions and identities. More precisely, we have

$$\Sigma \left(\text{id}_{n+1}^X(z) \right) = \text{id}_{n+2}^{\Sigma X}(\Sigma z) \quad \text{and} \quad \Sigma(u *_{n+1}^X v) = \Sigma u *_{n+1}^{\Sigma X} \Sigma v$$

for any $n \geq 0$, n -cell z , and composable pair of $n+1$ -cells u, v in X . $\lrcorner$

Proof. This follows from Corollary 1.2.4.3 and the fact that $\Sigma: X \rightarrow \Sigma X(\star, \star')$ is a strict ω -functor using Lemma 1.2.3.3. $\square$

For each $n \geq 0$, the left adjoint of $\text{hom}: G^0 \sqcup G^0 / \mathbf{GSet} \rightarrow \mathbf{GSet}$ sends G^n and ∂G^n to G^{n+1} and ∂G^{n+1} (with the obvious distinguished points) respectively. Therefore, Proposition 4.4.2.2 implies that we have canonical isomorphisms $\Sigma \mathcal{C}^n \cong \mathcal{C}^{n+1}$ and $\Sigma \partial \mathcal{C}^n \cong \partial \mathcal{C}^{n+1}$. For each weak ω -category X and an n -cell x in X , the strict ω -functor $\mathcal{C}^{n+1} \rightarrow \Sigma X$ picking out Σx is the composite

$$\mathcal{C}^{n+1} \cong \Sigma \mathcal{C}^n \rightarrow \Sigma X$$

where the first isomorphism is this canonical one and the second factor is obtained by applying Σ to the the strict ω -functor $\mathcal{C}^n \rightarrow X$ picking out x .

Definition 4.4.2.6. We define the *suspension* of a marked weak ω -category (X, tX) to be the weak ω -category ΣX equipped with the marking $t\Sigma X = \{\Sigma x \mid x \in tX\}$. This assignation extends to a functor $\Sigma: \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ in the obvious manner. $\lrcorner$

It easily follows from Remark 4.4.0.5 that the suspension functor $\Sigma: \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ preserves all connected colimits. In particular, it preserves the marked weak ω -categories $\mathcal{F}^n$ in the following sense.

Proposition 4.4.2.7. For each $n \geq 1$, there is an isomorphism $\mathcal{F}^{n+1} \cong \Sigma \mathcal{F}^n$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ that sends the generating cells $u_{\mathcal{F}}, v_{\mathcal{F}}, w_{\mathcal{F}}, p_{\mathcal{F}}$, and $q_{\mathcal{F}}$ of $\mathcal{F}^{n+1}$ in Definition 4.3.4.1 to $\Sigma u_{\mathcal{F}}, \Sigma v_{\mathcal{F}}, \Sigma w_{\mathcal{F}}, \Sigma p_{\mathcal{F}}$, and $\Sigma q_{\mathcal{F}}$ respectively. $\lrcorner$

Proof. Recall that $\mathcal{F}^n$ is constructed in the following two steps. Firstly we take the colimit of the solid part of

$$\begin{array}{ccccc} (\partial\mathcal{C}^n)^b & \xrightarrow{\langle\tau^{n-1}, \sigma^{n-1}\rangle} & \mathbf{m}\mathcal{C}^n & \xleftarrow{\langle\tau^{n-1}, \sigma^{n-1}\rangle} & (\partial\mathcal{C}^n)^b \\ \langle\sigma^{n-1}, \tau^{n-1}\rangle \downarrow & \dashrightarrow \langle y_{\mathcal{F}}, x_{\mathcal{F}} \rangle & \downarrow u_{\mathcal{F}} & & \downarrow \langle\sigma^{n-1}, \tau^{n-1}\rangle \\ (\mathcal{C}^n)^b & \dashrightarrow v_{\mathcal{F}} & \mathcal{D}^n & \dashleftarrow w_{\mathcal{F}} & (\mathcal{C}^n)^b \end{array}$$

which yields a marked weak ω -category $\mathcal{D}^n$. Then we take the colimit of the solid part of

$$\begin{array}{ccccc} (\partial\mathcal{C}^{n+1})^b & \xrightarrow{\langle u_{\mathcal{F}} *_{n-1} v_{\mathcal{F}}, \text{id}_n(x_{\mathcal{F}}) \rangle} & \mathcal{D}^n & \xleftarrow{\langle w_{\mathcal{F}} *_{n-1} u_{\mathcal{F}}, \text{id}_n(y_{\mathcal{F}}) \rangle} & (\partial\mathcal{C}^{n+1})^b \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{m}\mathcal{C}^{n+1} & \dashrightarrow p_{\mathcal{F}} & \mathcal{F}^n & \dashleftarrow q_{\mathcal{F}} & \mathbf{m}\mathcal{C}^{n+1} \end{array}$$

which yields $\mathcal{F}^n$. The suspension functor $\Sigma: \mathbf{m}\text{-Wk-}\omega\text{-Cat}_s \rightarrow \mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$ preserves connected colimits, and in particular the above defining colimit diagram of $\mathcal{D}^n$. Combining the resulting diagram with the canonical isomorphisms $\Sigma\partial\mathcal{C}^n \cong \partial\mathcal{C}^{n+1}$ and $\Sigma\mathcal{C}^n \cong \mathcal{C}^{n+1}$ yields the following colimit diagram.

$$\begin{array}{ccccc} (\partial\mathcal{C}^{n+1})^b & \xrightarrow{\langle\tau^n, \sigma^n\rangle} & \mathbf{m}\mathcal{C}^{n+1} & \xleftarrow{\langle\tau^n, \sigma^n\rangle} & (\partial\mathcal{C}^{n+1})^b \\ \langle\sigma^n, \tau^n\rangle \downarrow & & \downarrow & & \downarrow \langle\sigma^n, \tau^n\rangle \\ (\mathcal{C}^{n+1})^b & \dashrightarrow & \Sigma\mathcal{D}^n & \dashleftarrow & (\mathcal{C}^{n+1})^b \end{array}$$

Comparing it with the defining colimit diagram of $\mathcal{D}^{n+1}$, we obtain an isomorphism $\Sigma\mathcal{D}^n \cong \mathcal{D}^{n+1}$ that sends $\Sigma u, \Sigma v, \Sigma w$ to u, v, w in $\mathcal{D}^{n+1}$.

The desired isomorphism $\Sigma\mathcal{F}^n \cong \mathcal{F}^{n+1}$ can be obtained by applying a similar argument to the second diagram. The compatibility of the upper horizontal maps and the canonical isomorphisms $\Sigma\partial\mathcal{C}^{n+1} \cong \partial\mathcal{C}^{n+2}$ follows from Proposition 4.4.2.5. $\square$

4.4.3. Presenting $\mathcal{E}_{\mathcal{F}}^n$ as the free system of witnesses. Fix $n \geq 1$. In this subsection, we provide another presentation of $\mathcal{E}_{\mathcal{F}}^n$ which uses the suspension construction. The starting point is the following observation.

Lemma 4.4.3.1. For each $m \geq 0$, there is a bijection

$$T^{n,m} \times \{p, q\} \cong T^{n,m+1}$$

which sends (u, p) to $\mathbf{f}_u^{m+1}(p_{\mathcal{F}})$ and (u, q) to $\mathbf{f}_u^{m+1}(q_{\mathcal{F}})$ where $p_{\mathcal{F}}$ and $q_{\mathcal{F}}$ are the (generating) marked $(n+m+1)$ -cells of $\mathcal{F}^{n+m}$ in Definition 4.3.4.1 and $\mathbf{f}_u^{m+1}: \mathcal{F}^{n+m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n+m+1}$ is the strict ω -functor in Definition 4.4.1.1. $\square$

Proof. This follows from Lemma 4.4.1.3, as we have $(t\mathcal{F}^{n+m})_{n+m+1} = \{p_{\mathcal{F}}, q_{\mathcal{F}}\}$ and $(t\mathcal{E}_{\mathcal{F}}^{n,m+1})_{n+m+1} = T^{n,m+1}$. $\square$

Applying this lemma iteratively, we obtain an isomorphism $T^{n,m} \cong \{p, q\}^m$ and so the pushout square (4.4.1.1.1) can be rewritten as

$$\begin{array}{ccc} \coprod_{\phi \in \{p, q\}^m} \mathbf{m}\mathcal{C}^{n+m} & \xrightarrow{g^m} & \mathcal{E}_{\mathcal{F}}^{n,m} \\ \downarrow & \lrcorner & \downarrow \mathbf{e}^{m+1} \\ \coprod_{\phi \in \{p, q\}^m} \mathcal{F}^{n+m} & \xrightarrow{\mathbf{f}^{m+1}} & \mathcal{E}_{\mathcal{F}}^{n,m+1} \end{array}$$

where $g^0: \mathbf{m}\mathcal{C}^n \rightarrow \mathcal{E}_{\mathcal{F}}^{n,0} = \mathbf{m}\mathcal{C}^n$ is the identity, and g^{m+1} is characterised as the unique strict ω -functor such that, for each $\phi \in \{p, q\}^m$, its restrictions

$$g_{\phi p}^{m+1}, g_{\phi q}^{m+1}: \mathbf{m}\mathcal{C}^{n+m+1} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,m+1}$$

onto the ϕp -th and the ϕq -th factors send c_{n+m+1} to $\mathbf{f}_{\phi}^m(p_{\mathcal{F}})$ and $\mathbf{f}_{\phi}^m(q_{\mathcal{F}})$ respectively; here the expression ϕp means the sequence obtained from ϕ by appending p , and similarly for ϕq .

It follows that unpacking the data of a cocone under the sequence (4.4.1.5.1) with nadir X (we borrowed this terminology from [Rie16, Definition 3.1.2]) yields precisely the following data.

Definition 4.4.3.2. Let X be a marked weak ω -category and let $n \geq 1$. A *system of witnesses* of dimension n in X consists of

- a marked $(n+m)$ -cell x_ϕ in X , and
- $(n+m)$ -cells $x_{\phi v}, x_{\phi w}$ in X

for each sequence $\phi \in \{p, q\}^m$ of length $m \geq 0$, whose source and target are given by

$$\begin{aligned} x_{\phi v}, x_{\phi w} &: t_{n+m-1}(x_\phi) \rightarrow s_{n+m-1}(x_\phi), \\ x_{\phi p} &: x_\phi *_{n+m-1} x_{\phi v} \rightarrow \text{id}_{n+m}(s_{n+m-1}(x_\phi)), \text{ and} \\ x_{\phi q} &: x_{\phi w} *_{n+m-1} x_\phi \rightarrow \text{id}_{n+m}(t_{n+m-1}(x_\phi)). \end{aligned}$$

Equivalently, the quintuple $(x_\phi, x_{\phi v}, x_{\phi w}, x_{\phi p}, x_{\phi q})$ must specify a (marking-preserving) strict ω -functor $\mathcal{F}^{n+m} \rightarrow X$ for each $\phi \in \{p, q\}^m$. $\lrcorner$

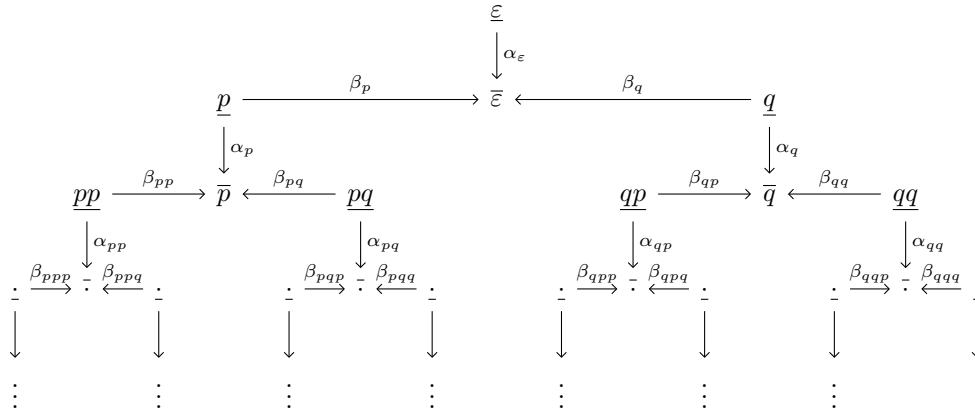
We now describe a more elementary diagram in **m-Wk- ω -Cat_s** such that a cocone under that diagram with nadir X is precisely a system of witnesses in X .

Definition 4.4.3.3. We write Ξ for the free category generated by the following graph.

- The set of vertices is $\{\underline{\phi} \mid \phi \in \{p, q\}^*\} \cup \{\bar{\phi} \mid \phi \in \{p, q\}^*\}$.
- For each $\phi \in \{p, q\}^*$, there are edges

$$\alpha_\phi: \underline{\phi} \rightarrow \bar{\phi}, \quad \beta_{\phi p}: \underline{\phi p} \rightarrow \bar{\phi}, \quad \text{and} \quad \beta_{\phi q}: \underline{\phi q} \rightarrow \bar{\phi}. \quad \lrcorner$$

The generating graph of the category Ξ can be visualised as follows:



Note that specifying a diagram $D: \Xi \rightarrow \mathbf{C}$ in a category $\mathbf{C}$ is equivalent to specifying the image of the full subcategory

$$\begin{array}{c} \underline{\phi} \\ \downarrow \alpha_\phi \\ \underline{\phi p} \xrightarrow{\beta_{\phi p}} \bar{\phi} \xleftarrow{\beta_{\phi q}} \underline{\phi q} \end{array} \quad (4.4.3.3.1)$$

for each $\phi \in \{p, q\}^m$ in a coherent manner. More precisely, for each $\phi \in \{p, q\}^m$ with $m \geq 1$, the object $\underline{\phi}$ appears in exactly two of such full subcategories (once at the top and once in the bottom row), and the value $D(\underline{\phi})$ must be consistent between these two appearances.

Definition 4.4.3.4. For each $n \geq 1$, define a functor $\frac{\mathcal{C}}{\mathcal{F}}^n: \Xi \rightarrow \mathbf{m-Wk-}\omega\text{-Cat}_s$ by sending (4.4.3.3.1) to

$$\begin{array}{c} m\mathcal{C}^{n+m} \\ \downarrow u_{\mathcal{F}} \\ m\mathcal{C}^{n+m+1} \xrightarrow{p_{\mathcal{F}}} \mathcal{F}^{n+m} \xleftarrow{q_{\mathcal{F}}} m\mathcal{C}^{n+m+1}, \end{array} \quad (4.4.3.4.1)$$

for each $\phi \in \{p, q\}^m$ where $u_{\mathcal{F}}, p_{\mathcal{F}}, q_{\mathcal{F}}$ are the generating marked cells of $\mathcal{F}^{n+m}$ in Definition 4.3.4.1. $\lrcorner$

The admittedly strange notation $\frac{\mathcal{C}}{\mathcal{F}}^n$ is motivated by the following mnemonics.

- The relative position $\frac{\mathcal{C}}{\mathcal{F}}$ suggests that the vertical map in (4.4.3.4.1) has domain $m\mathcal{C}^{n+m}$ and codomain $\mathcal{F}^{n+m}$.

- The upper half $\underline{\mathcal{C}}$ suggests that underlined objects $\underline{\phi}$ are sent to $\mathbf{m}\mathcal{C}^{n+m}$.
- The lower half $\overline{\mathcal{F}}$ suggests that overlined objects $\overline{\phi}$ are sent to $\mathcal{F}^{n+m}$.

We have defined the functor $\underline{\mathcal{C}}_{\overline{\mathcal{F}}}^n$ so that the following holds.

Proposition 4.4.3.5. The category of cocones under $\underline{\mathcal{C}}_{\overline{\mathcal{F}}}^n$ is isomorphic to the category of cocones under the sequence (4.4.1.5.1), and this isomorphism commutes with the evident forgetful functors to $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. In particular, $\mathcal{E}_{\mathcal{F}}^n$ is a colimit of $\underline{\mathcal{C}}_{\overline{\mathcal{F}}}^n$. $\lrcorner$

Corollary 4.4.3.6. For each $n \geq 1$, the isomorphisms $(\mathcal{F}^{n+1+m} \cong \Sigma \mathcal{F}^{n+m})_{m \in \mathbb{N}}$ and $(\mathbf{m}\mathcal{C}^{n+1+m} \cong \Sigma \mathbf{m}\mathcal{C}^{n+m})_{m \in \mathbb{N}}$ in Proposition 4.4.2.7 induce an isomorphism $\mathcal{E}_{\mathcal{F}}^{n+1} \cong \Sigma \mathcal{E}_{\mathcal{F}}^n$. $\lrcorner$

Proof. Since the category Ξ is connected, we have

$$\Sigma \mathcal{E}_{\mathcal{F}}^n \cong \Sigma(\operatorname{colim} \underline{\mathcal{C}}_{\overline{\mathcal{F}}}^n) \cong \operatorname{colim}(\Sigma \circ \underline{\mathcal{C}}_{\overline{\mathcal{F}}}^n)$$

by the comment just below Definition 4.4.2.6. By Proposition 4.4.2.7, the diagram $\Sigma \circ \underline{\mathcal{C}}_{\overline{\mathcal{F}}}^n$ is isomorphic to $\underline{\mathcal{C}}_{\overline{\mathcal{F}}}^{n+1}$. $\square$

4.4.4. Comparison to the Ozornova–Rovelli model. In this subsection, we prove that reflecting (the underlying weak ω -category of) $\mathcal{E}_{\mathcal{F}}^1$ to $\mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s$ yields the strict ω -category $\widehat{\omega\mathcal{E}}$, which was first introduced in [OR24] and later shown to be weakly contractible (in the folk model structure) in [Had+25]. We first define a weak ω -category $\mathcal{E}_{\text{OR}}^1$ by imitating the original construction of $\widehat{\omega\mathcal{E}}$.

Definition 4.4.4.1. Let $\mathcal{E}_{\text{OR}}^{1,0} = \mathcal{C}^0 \amalg \mathcal{C}^0$, whose 0-cells we call x_{OR} and y_{OR} . Let $\mathcal{E}_{\text{OR}}^{1,1}$ be the weak ω -category obtained from $\mathcal{E}_{\text{OR}}^{1,0}$ by freely adjoining 1-cells

$$u_{\text{OR}}: x_{\text{OR}} \rightarrow y_{\text{OR}} \quad \text{and} \quad v_{\text{OR}}, w_{\text{OR}}: y_{\text{OR}} \rightarrow x_{\text{OR}}.$$

Write $e^1: \mathcal{E}_{\text{OR}}^{1,0} \rightarrow \mathcal{E}_{\text{OR}}^{1,1}$ for the inclusion, and write $f_p^1, f_q^1: \Sigma \mathcal{E}_{\text{OR}}^{1,0} \rightarrow \mathcal{E}_{\text{OR}}^{1,1}$ for the strict ω -functors corresponding via the adjunction $\Sigma \dashv \operatorname{hom}$ to

$$\langle u_{\text{OR}} *_0 v_{\text{OR}}, \operatorname{id}_1(x_{\text{OR}}) \rangle: \mathcal{E}_{\text{OR}}^{1,0} \rightarrow \mathcal{E}_{\text{OR}}^{1,1}(x_{\text{OR}}, x_{\text{OR}}) \quad \text{and} \quad \langle w_{\text{OR}} *_0 u_{\text{OR}}, \operatorname{id}_1(y_{\text{OR}}) \rangle: \mathcal{E}_{\text{OR}}^{1,0} \rightarrow \mathcal{E}_{\text{OR}}^{1,1}(y_{\text{OR}}, y_{\text{OR}}),$$

respectively, where we identify morphisms from $\mathcal{E}_{\text{OR}}^{1,0}$ with pairs of 0-cells. For $m \geq 1$, define the weak ω -category $\mathcal{E}_{\text{OR}}^{1,m+1}$ and strict ω -functors

$$e^{m+1}: \mathcal{E}_{\text{OR}}^{1,m} \rightarrow \mathcal{E}_{\text{OR}}^{1,m+1} \quad \text{and} \quad f_p^{m+1}, f_q^{m+1}: \Sigma \mathcal{E}_{\text{OR}}^{1,m} \rightarrow \mathcal{E}_{\text{OR}}^{1,m+1}$$

to be the legs of the colimiting cocone

$$\begin{array}{ccccc} \Sigma \mathcal{E}_{\text{OR}}^{1,m-1} & \xrightarrow{f_p^m} & \mathcal{E}_{\text{OR}}^{1,m} & \xleftarrow{f_q^m} & \Sigma \mathcal{E}_{\text{OR}}^{1,m-1} \\ \Sigma e^m \downarrow & & \downarrow e^{m+1} & & \downarrow \Sigma e^m \\ \Sigma \mathcal{E}_{\text{OR}}^{1,m} & \xrightarrow{\quad f_p^{m+1} \quad} & \mathcal{E}_{\text{OR}}^{1,m+1} & \xleftarrow{\quad f_q^{m+1} \quad} & \Sigma \mathcal{E}_{\text{OR}}^{1,m} \end{array} \quad (4.4.4.1.1)$$

in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. We write $\mathcal{E}_{\text{OR}}^1$ for the colimit of the sequence

$$\mathcal{E}_{\text{OR}}^{1,0} \xrightarrow{e^1} \mathcal{E}_{\text{OR}}^{1,1} \xrightarrow{e^2} \mathcal{E}_{\text{OR}}^{1,2} \xrightarrow{e^3} \dots \quad (4.4.4.1.2)$$

in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. $\lrcorner$

Remark 4.4.4.2. Although we have intentionally chosen the notations

$$e^{m+1}: \mathcal{E}_{\text{OR}}^{1,m} \rightarrow \mathcal{E}_{\text{OR}}^{1,m+1} \quad \text{and} \quad f_p^{m+1}, f_q^{m+1}: \Sigma \mathcal{E}_{\text{OR}}^{1,m} \rightarrow \mathcal{E}_{\text{OR}}^{1,m+1}$$

which resemble

$$e^{m+1}: \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,m+1} \quad \text{and} \quad f_u^{m+1}: \mathcal{F}^{n+m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,m+1}$$

from Section 4.4.1, we are not claiming any rigorous relationship between them beyond their playing similar roles. In particular, we are giving the same name e^{m+1} to two different strict ω -functors, but this should cause no serious confusion because $e^{m+1}: \mathcal{E}_{\mathcal{F}}^{n,m} \rightarrow \mathcal{E}_{\mathcal{F}}^{n,m+1}$ never appears in this subsection outside of this remark. $\lrcorner$

Proposition 4.4.4.3. The strict ω -categorical reflection of $\mathcal{E}_{\text{OR}}^1$ is isomorphic to the coherent walking ω -equivalence $\widehat{\omega\mathcal{E}}$ constructed in [OR24]. $\lrcorner$

Proof. By virtue of Proposition 4.4.2.2, the strict ω -categorical reflection functor $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s$ sends the defining colimits of each $\mathcal{E}_{\text{OR}}^{1,m}$ and $\mathcal{E}_{\text{OR}}^1$ to those of $\widehat{\omega\mathcal{E}}^{(m)}$ and $\widehat{\omega\mathcal{E}}$ in [OR24, Constructions 1.5.12 and 1.5.13] respectively. $\square$

We now prove that $\mathcal{E}_{\text{OR}}^1$ is in fact isomorphic to $\mathcal{E}_{\mathcal{F}}^1$. To this end, let $\frac{E}{E}: \Xi \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ be the unique diagram that sends (4.4.3.1) to

$$\begin{array}{ccccc} & & \Sigma^m \mathcal{E}_{\text{OR}}^{1,0} & & \\ & & \downarrow \Sigma^m e^1 & & \\ \Sigma^{m+1} \mathcal{E}_{\text{OR}}^{1,0} & \xrightarrow{\Sigma^m f_p^1} & \Sigma^m \mathcal{E}_{\text{OR}}^{1,1} & \xleftarrow{\Sigma^m f_q^1} & \Sigma^{m+1} \mathcal{E}_{\text{OR}}^{1,0} \end{array} \quad (4.4.4.3.1)$$

for each $\phi \in \{p, q\}^m$.

Lemma 4.4.4.4. The category of cocones under $\frac{E}{E}$ is isomorphic to the category of cocones under the sequence (4.4.4.1.2), and this isomorphism commutes with the evident forgetful functors to $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. In particular, $\mathcal{E}_{\text{OR}}^1$ is a colimit of $\frac{E}{E}$. $\dashv$

Proof. For each $m \geq 0$, we write Ξ^m for the full subcategory of Ξ spanned by the subset

$$\{\phi \mid \phi \in \{p, q\}^{\leq m}\} \sqcup \{\bar{\phi} \mid \phi \in \{p, q\}^{< m}\},$$

and $\frac{E}{E}^m$ for the restriction of $\frac{E}{E}$ to Ξ^m . The inclusion $\Xi^m \hookrightarrow \Xi^{m+1}$ induces a strict ω -functor

$$g^{m+1}: \text{colim}(\frac{E}{E}^m) \rightarrow \text{colim}(\frac{E}{E}^{m+1})$$

on their colimits. This defines a sequence

$$\text{colim}(\frac{E}{E}^0) \xrightarrow{g^1} \text{colim}(\frac{E}{E}^1) \xrightarrow{g^2} \text{colim}(\frac{E}{E}^2) \xrightarrow{g^3} \dots$$

and clearly the category of cocones under this sequence is isomorphic to that of cocones under $\frac{E}{E}$. Therefore it suffices to show that this sequence is naturally isomorphic to the sequence (4.4.4.1.2). We will do so by constructing suitable strict ω -functors $h_p^{m+1}, h_q^{m+1}: \Sigma \text{colim}(\frac{E}{E}^m) \rightarrow \text{colim}(\frac{E}{E}^{m+1})$ and comparing the diagram

$$\begin{array}{ccccc} \Sigma(\text{colim} \frac{E}{E}^{m-1}) & \xrightarrow{h_p^m} & \text{colim} \frac{E}{E}^m & \xleftarrow{h_q^m} & \Sigma(\text{colim} \frac{E}{E}^{m-1}) \\ \Sigma g^m \downarrow & & \downarrow g^{m+1} & & \downarrow \Sigma g^m \\ \Sigma(\text{colim} \frac{E}{E}^m) & \xrightarrow{h_p^{m+1}} & \text{colim} \frac{E}{E}^{m+1} & \xleftarrow{h_q^{m+1}} & \Sigma(\text{colim} \frac{E}{E}^m) \end{array} \quad (4.4.4.4.1)$$

to (4.4.4.1.1).

For each $r \in \{p, q\}$, the function $r \dashv \dashv: \{p, q\}^* \rightarrow \{p, q\}^*$ given by $r \dashv \dashv \phi = r\phi$ defines a fully faithful functor $\Xi^m \rightarrow \Xi^{m+1}$. We write $r\Xi^m$ for its image, which is precisely the full subcategory of Ξ^{m+1} spanned by those objects of the form $r\phi$ or $r\bar{\phi}$. Moreover, we have a commutative square

$$\begin{array}{ccc} \Xi^m & \xrightarrow{\frac{E}{E}^m} & \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \\ r \dashv \dashv \downarrow & & \downarrow \Sigma \\ \Xi^{m+1} & \xrightarrow{\frac{E}{E}^{m+1}} & \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \end{array}$$

which induces a map between colimits $\text{colim}(\Sigma \circ \frac{E}{E}^m) \rightarrow \text{colim}(\frac{E}{E}^{m+1})$. Since Ξ^m is connected, the functor Σ preserves Ξ^m -shaped colimits, and we obtain a morphism

$$h_r^{m+1}: \Sigma \text{colim}(\frac{E}{E}^m) \rightarrow \text{colim}(\frac{E}{E}^{m+1}).$$

Clearly these strict ω -functors fit into the commutative diagram (4.4.4.4.1).

Observe that the category Ξ^{m+1} can be written as the union of three full subcategories

$$\Xi^{m+1} = p\Xi^m \cup q\Xi^m \cup \Xi^m$$

and the intersection $r\Xi^m \cap \Xi^m$ is precisely $r\Xi^{m-1}$ for each $r \in \{p, q\}$ whereas $p\Xi^m \cap q\Xi^m$ is empty. It follows that (4.4.4.4.1) presents $\text{colim}(\frac{E}{E}^{m+1})$ as the colimit of the solid part. Since h_r^1 and g^1 are isomorphic to f_r^1 and

i.e., it comes equipped with a distinguished n -cell $i_\star^n$.) Then the composite $\partial C^n \xrightarrow{\iota^n} C^n \xrightarrow{i_\star^n} E_\star^n$ is clearly a folk cofibration, so

$$\{C^{n-1} \xrightarrow{\sigma^{n-1}} C^n \xrightarrow{i_\star^n} E_\star^n \mid n \geq 1\}$$

is a generating set of folk trivial cofibrations by Proposition 4.5.0.1. Note that, since the trivial fibration $r_\star^n: E_\star^n \rightarrow C^{n-1}$ reflects equivalences by [LMW10, Lemma 4.9], the n -cell $i_\star^n(c_n)$ (and similarly every k -cell in $E_\star^n$ with $k \geq n$) is an equivalence.

The statement of the following lemma appears in [HL25, Proposition 4.22]. Although the authors of [HL25] state that it is merely a reformulation of [LMW10, Lemma 4.36], we believe that this is not quite true, so we provide a separate proof.

Lemma 4.5.1.1. Let $u: x \rightarrow y$ be an n -cell in a strict ω -category X with $n \geq 1$. Then u is an equivalence if and only if the map $C^n \rightarrow X$ picking out u can be extended along $i_\star^n: C^n \rightarrow E_\star^n$. $\lrcorner$

Proof. The ‘if’ direction follows from Proposition 2.3.1. The ‘only if’ direction is deferred to Section A.4. $\square$

Proposition 4.5.1.2. Any folk fibration is an ω -equifibration. $\lrcorner$

Proof. Let $f: X \rightarrow Y$ be a folk fibration between strict ω -categories. Let $n \geq 1$ and suppose that we are given an equivalence n -cell of the form $u: fx \rightarrow y$ in Y . Then by Lemma 4.5.1.1, the map $C^n \rightarrow Y$ picking out u can be extended to $E_\star^n$, which fits into the commutative square

$$\begin{array}{ccc} C^{n-1} & \xrightarrow{x} & X \\ i_\star^n \sigma^{n-1} \downarrow & & \downarrow f \\ E_\star^n & \longrightarrow & Y. \end{array}$$

Since the left vertical map is a folk trivial cofibration, this square admits a lift $E_\star^n \rightarrow X$. In particular, it sends the n -cell picked out by $i_\star^n: C^n \rightarrow E_\star^n$ to an equivalence n -cell of the form $\bar{u}: x \rightarrow \bar{y}$ with $f\bar{u} = u$. $\square$

4.5.2. ω -equifibrations are folk fibrations. Recall from Section 4.4 that, for each $n \geq 1$, $i_\mathcal{F}^n: C^n \rightarrow \mathcal{E}_\mathcal{F}^n$ is (the underlying strict ω -functor of) the unit map associated to the algebraically $\mathcal{K}_\mathcal{F}$ -injective replacement (of $\mathbf{m}C^n$). We will write $i_\mathcal{F}^n: C^n \rightarrow E_\mathcal{F}^n$ for its image under the strict ω -categorical reflection functor $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s$ defined in Notation 1.2.2.10; in particular, we have $E_\mathcal{F}^1 \cong \widehat{\omega\mathcal{E}}$ by Proposition 4.4.4.3 and Theorem 4.4.4.5. The following is an immediate consequence of Proposition 4.4.0.4.

Lemma 4.5.2.1. A strict ω -functor between strict ω -categories is an ω -equifibration if and only if it has the right lifting property with respect to

$$\mathcal{J}_\mathcal{F} = \{C^{n-1} \xrightarrow{\sigma^{n-1}} C^n \xrightarrow{i_\mathcal{F}^n} E_\mathcal{F}^n \mid n \geq 1\}. \quad \lrcorner$$

In order to prove that ω -equifibrations are folk fibrations, it thus suffices to show that $\mathcal{J}_\mathcal{F}$ is a generating set of folk trivial cofibrations. The first step is straightforward.

Lemma 4.5.2.2 (See [Had+25, Theorem 1.33] for the second assertion). The map $i_\mathcal{F}^n \sigma^{n-1}: C^{n-1} \rightarrow E_\mathcal{F}^n$ is a folk trivial cofibration for each $n \geq 1$. In particular, the coherent walking ω -equivalence $\widehat{\omega\mathcal{E}}$ is weakly equivalent to the terminal strict ω -category in the folk model structure. $\lrcorner$

Proof. Proposition 4.5.1.2 and Lemma 4.5.2.1 imply that each $i_\mathcal{F}^n \sigma^{n-1}$ has the left lifting property with respect to all folk fibrations, which proves the first assertion. The second assertion follows from the case $n = 1$. $\square$

Our next step is to modify Proposition 4.5.0.1 using a seemingly trivial observation regarding ω -equifibrations.

Definition 4.5.2.3. We call a strict ω -category X *gaunt* if every equivalence in X is an identity. $\lrcorner$

Lemma 4.5.2.4. Let $f: X \rightarrow Y$ be a morphism in $\mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat}_s$ with Y gaunt. Then f is an ω -equifibration. $\lrcorner$

Proof. Let $x \in X_{n-1}$ and let $u: fx \rightarrow y$ be an equivalence n -cell in Y . Then u is the identity on fx by assumption, so we may simply lift it to the identity on x . $\square$

Lemma 4.5.2.5. Suppose that, for each $n \geq 1$, we are given a factorisation

$$\partial C^n \xrightarrow{i_\partial^n} E^n \xrightarrow{r^n} C^{n-1}$$

of the collapse map such that

- i_∂^n is a folk cofibration, and
- the composite $\mathcal{C}^{n-1} \xrightarrow{\sigma_\partial^{n-1}} \partial\mathcal{C}^n \xrightarrow{i_\partial^n} \mathcal{E}^n$ is a folk trivial cofibration.

Then

$$\left\{ \mathcal{C}^{n-1} \xrightarrow{\sigma_\partial^{n-1}} \partial\mathcal{C}^n \xrightarrow{i_\partial^n} \mathcal{E}^n \mid n \geq 1 \right\}$$

is a generating set of folk trivial cofibrations. $\lrcorner$

Proof. Since $\mathcal{C}^{n-1}$ is gaunt, the map r^n is automatically an ω -equifibration. Moreover, since r^n is a retraction of $i_\partial^n \sigma_\partial^{n-1}$, the 2-out-of-3 property implies that r^n is an ω -weak equivalence. It follows that r^n is a folk trivial fibration. Hence the claim follows by Proposition 4.5.0.1. $\square$

Now we are ready to prove the desired statement.

Theorem 4.5.2.6. The set $\mathcal{J}_F$ is a generating set of folk trivial cofibrations. Consequently, the folk fibrations are precisely the ω -equifibrations between strict ω -categories. $\lrcorner$

Proof. We apply Lemma 4.5.2.5 to the strict ω -functors i_∂^n and r^n defined as follows. The former is the strict ω -categorical reflection of $\partial\mathcal{C}^n \xrightarrow{\ell^n} \mathcal{C}^n \xrightarrow{i_{\mathcal{F}}^n} \mathcal{E}_{\mathcal{F}}^n$. Note that this composite in $\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ can be written as a transfinite composite of pushouts of maps of the form $\partial\mathcal{C}^m \hookrightarrow \mathcal{C}^m$, so its image i_∂^n is a transfinite composite of pushouts of $\partial\mathcal{C}^m \hookrightarrow \mathcal{C}^m$, and in particular it is a folk cofibration. Moreover, the composite

$$\left(\mathcal{C}^{n-1} \xrightarrow{\sigma_\partial^{n-1}} \partial\mathcal{C}^n \xrightarrow{i_\partial^n} \mathcal{E}_F^n \right) = \left(\mathcal{C}^{n-1} \xrightarrow{\sigma^{n-1}} \mathcal{C}^n \xrightarrow{i_F^n} \mathcal{E}_F^n \right)$$

is a folk trivial cofibration by Lemma 4.5.2.2.

Recall that the map $i_{\mathcal{F}}^n: m\mathcal{C}^n \rightarrow \mathcal{E}_{\mathcal{F}}^n$ in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ is in $\mathcal{K}_{\mathcal{F}}\text{-cof}$. It follows that the map $m\mathcal{C}^n \rightarrow (\mathcal{C}^{n-1})^\natural$ picking out $\text{id}_n(c_{n-1})$ can be extended along $i_{\mathcal{F}}^n: m\mathcal{C}^n \rightarrow \mathcal{E}_{\mathcal{F}}^n$. The strict ω -functor $r^n: \mathcal{E}_F^n \rightarrow \mathcal{C}^{n-1}$ is the strict ω -categorical reflection of the underlying map of this extension. Clearly the composite $r^n i_\partial^n$ is the collapse map by construction, and this completes the proof. $\square$

Deferred proofs and additional remarks

In this chapter we collect several technical results and deferred proofs whose statements were used in the main body of the thesis. Our aim is to separate these more involved arguments from the main narrative while keeping the exposition essentially self-contained.

In Section A.1 we give a detailed proof that realisation embeds the discrete category of pasting schemes into the core groupoid of **GSet**, via an analysis of the auxiliary functor **Br**.

In the following section we recall the description of **L** as a globular operad and show that the associated globular operad is *normalised*, a fact used later in Section A.3. Section A.3 is then devoted to the proof that ε_X is bijective on objects, obtained by an adjointness argument involving the coskeleton construction.

In Section A.4 we complete the proof of Lemma 4.5.1.1 by recalling and adapting the cylinder machinery of [LMW10].

Section A.5 constructs the quotient bicategory $\tau_2(X)$ by identifying 2-cells up to the equivalence relation generated by equivalence cells, and relates equivalences in X to equivalences in $\tau_2(X)$. Using this construction, in Section A.6 we present an alternative choice of the set $\mathcal{K}$ based on half-adjoint equivalences and verify that it satisfies conditions (K1)–(K3), thereby yielding a second characterisation of ω -equifibrations.

A.1. Realisation embeds pasting schemes into **GSet**

We show the following proposition, which rephrases Proposition 1.1.2.17.

Proposition A.1.1. The realisation defines a fully faithful functor $\widehat{(-)}: \mathbf{PS} \rightarrow \mathbf{k}(\mathbf{GSet})$ from the discrete category **PS** of pasting schemes to the core groupoid of the category **GSet**. $\lrcorner$

To this end, we first check the following lemma.

Lemma A.1.2. Consider the functor $\mathbf{Br}: \mathbf{Set}^* \rightarrow \mathbf{Gph}$ defined in Construction B.3.21 for $\mathbf{C} = 1$ that sends a list of sets $(X_1, \dots, X_k)$ to the colimit

$$\begin{array}{ccccccc} & \underline{\Sigma} X_1 & & \underline{\Sigma} X_2 & & \dots & & \underline{\Sigma} X_k \\ & \nearrow \star & \nwarrow \star' & \nearrow \star & \nwarrow \star' & \nearrow \star & \nwarrow \star' & \nearrow \star \\ G^0 & & G^0 & & G^0 & & G^0 & & G^0 \end{array}$$

Here $\underline{\Sigma} S$ is the graph that has exactly two vertices $\star$ and $\star'$ and whose only edges are those in $\underline{\Sigma} S(\star, \star') = S$.

Then, given two lists of sets $(X_1, \dots, X_k)$ and $(Y_1, \dots, Y_l)$ such that each X_i and Y_j is non-empty, and an isomorphism $\phi: \mathbf{Br}(X_1, \dots, X_k) \xrightarrow{\sim} \mathbf{Br}(Y_1, \dots, Y_l)$ in **Gph**, we have $k = l$ and there exists a unique list of bijections $\phi_i: X_i \rightarrow Y_i$ for $i = 1, \dots, k$ such that $\mathbf{Br}(\phi_1, \dots, \phi_k) = \phi$. $\lrcorner$

Proof. $\mathbf{Br}(X_1, \dots, X_k)$ is described as follows:

- $\mathbf{Br}(X_1, \dots, X_k)_0 = \{\star_0, \star_1, \star_2, \dots, \star_k\}$ is a $k + 1$ -element set.
- $\mathbf{Br}(X_1, \dots, X_k)(\star_i, \star_{i+1}) = X_{i+1}$ for each $i < k$, and those are the only non-empty hom-sets.

Now the statement is clear from this description. $\square$

Now we are ready to prove the proposition.

Proof of Proposition A.1.1. Suppose we are given two pasting schemes $\mathbf{k} = \mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_l]$ and $\mathbf{k}' = \mathbf{br}[\mathbf{k}'_1, \dots, \mathbf{k}'_{l'}]$ and an isomorphism $\varphi: \widehat{\mathbf{k}} \xrightarrow{\sim} \widehat{\mathbf{k}}'$ in **GSet**. When $\mathbf{k} = [^0] = \mathbf{br}[]$, we have $\widehat{\mathbf{k}} = G^0 \cong \widehat{\mathbf{k}}'$, which implies $\mathbf{k}' = [^0]$

since $\mathbf{k}'$ must be of dimension 0, and $[^0]$ is the only pasting scheme of dimension 0. By (1.1.2.13.2), any isomorphism φ in **GSet** induces an isomorphism

$$\varphi_0: \text{Br}((\widehat{\mathbf{k}}_1)_0, \dots, (\widehat{\mathbf{k}}_l)_0) \cong \text{Br}((\widehat{\mathbf{k}}'_1)_0, \dots, (\widehat{\mathbf{k}}'_{l'})_0) \text{ in } \mathbf{Gph},$$

which, by Lemma A.1.2, implies $l = l'$ and that it decomposes as $\text{Br}(\varphi_0^1, \dots, \varphi_0^l)$. Now this shows that φ sends cells coming from $\widehat{\mathbf{k}}_i$ to those coming from $\widehat{\mathbf{k}}'_i$ since their source and target 0-cells are sent correspondingly by φ_0 . Thus φ decomposes as the colimit of isomorphisms $\underline{\Sigma}(\varphi^i): \underline{\Sigma}\widehat{\mathbf{k}}_i \xrightarrow{\sim} \underline{\Sigma}\widehat{\mathbf{k}}'_i$, and by induction on the level of $\mathbf{k}$, we have $\mathbf{k}_i = \mathbf{k}'_i$ and $\varphi^i = \text{id}_{\widehat{\mathbf{k}}_i}$ for each i , which concludes $\mathbf{k} = \mathbf{k}'$ and $\varphi = \text{id}_{\widehat{\mathbf{k}}}$. $\square$

A.2. **L** as a normalised globular operad

Using the fact that **T** is a cartesian monad, one can see that the entire structure of a cartesian monad over **T** with contraction $(P, \eta^P, \mu^P, \text{ar}^P, \bar{\kappa}^P)$ is determined by the tuple $(P1, \eta_1^P, \mu_1^P, \text{ar}_1^P, \bar{\kappa}^P)$ of its components at the terminal globular set 1. This latter description leads to the notion of *globular operad with contraction*, which is the actual structure used by Leinster to define weak ω -categories in [Lei04]. From this point of view, we will prove that the globular operad corresponding to the initial cartesian monad **L** over **T** with contraction is *normalised*, which has been mentioned in literature but the proof has not been explicitly given. Most of the contents in this section are splitted out from [FHM25a, Appendix A].

A.2.1. Globular operad. We first recall the notion of globular operads and observe that this is equivalent to cartesian monads over **T**, and hence **L** can be seen as a globular operad (see also [CF22, Section 2]).

Definition A.2.1. Let **E** be a finitely complete category. We define $[\mathbf{E}, \mathbf{E}]_{\text{cart}}$ as the category of all pullback-preserving endofunctors on **E** and cartesian natural transformations between them. Since pullback-preserving functors and cartesian natural transformations are closed under composition of functors and horizontal composition of natural transformations respectively, this category $[\mathbf{E}, \mathbf{E}]_{\text{cart}}$ is equipped with a structure of a sub-monoidal category of the (strict) monoidal category $[\mathbf{E}, \mathbf{E}]$. $\lrcorner$

Remark A.2.2. For a finitely complete category **E**, a cartesian monad on **E** is precisely a monoid object in the monoidal category $[\mathbf{E}, \mathbf{E}]_{\text{cart}}$. Moreover, a cartesian monad equipped with a cartesian monad morphism to a fixed cartesian monad S on **E** is precisely a monoid object in the slice monoidal category $[\mathbf{E}, \mathbf{E}]_{\text{cart}}/S$. $\lrcorner$

The following is the core of the equivalence between cartesian monads over **T** and globular operads:

Lemma A.2.3 ([Kel92, 3.1 and 3.2]). Let **E** be a finitely complete category, and S be a cartesian monad on **E**. Then the evaluation functor $\text{ev}_1: [\mathbf{E}, \mathbf{E}]_{\text{cart}}/S \rightarrow \mathbf{E}/S1$ at the terminal object $1 \in \mathbf{E}$ is an equivalence of categories. $\lrcorner$

Definition A.2.4. Let **E** be a finitely complete category, and S be a cartesian monad on **E**. Along the equivalence ev_1 in Lemma A.2.3, we can transport the (strict) monoidal structure on the slice monoidal category $[\mathbf{E}, \mathbf{E}]_{\text{cart}}/S$ to induce a (not necessarily strict) monoidal structure on $\mathbf{E}/S1$.

An *S-operad* is a monoid object in $\mathbf{E}/S1$ with respect to this monoidal structure. $\lrcorner$

From the monoidal equivalence ev_1 , it is clear that an *S-operad* corresponds to a cartesian monad on **E** equipped with a cartesian monad morphism to S .

Remark A.2.5. More explicitly, the monoidal structure on $\mathbf{E}/S1$ is described as follows. The monoidal unit in $\mathbf{E}/S1$ is the component $\eta_1^S: 1 \rightarrow S1$ of the unit η^S of the monad S at the terminal object $1 \in \mathbf{E}$. Given two objects $f: X \rightarrow S1$ and $g: Y \rightarrow S1$ in $\mathbf{E}/S1$, their monoidal product $f \otimes g$ in $\mathbf{E}/S1$ is the composite of the top horizontal row in the following diagram.

$$\begin{array}{ccccc} SX \times_{S1} Y & \longrightarrow & SX & \xrightarrow{Sf} & S^2 1 & \xrightarrow{\mu_1^S} & S1 \\ \downarrow & \lrcorner & \downarrow S! & & & & \\ Y & \xrightarrow{g} & S1 & & & & \end{array}$$

The most important fact about *S-operads* for us is the following, which is essential in the proof of the existence of **L** (Proposition 1.2.2.7).

Fact A.2.6 ([Lei04, Proposition 6.5.4 and 6.5.6]). If $\mathbf{E}$ is a presheaf category and S is finitary and preserves wide pullbacks, then the forgetful functor $\mathbf{Mon}(\mathbf{E}/S1) \rightarrow \mathbf{E}/S1$ from the category of S -operads is a finitary monadic functor. $\lrcorner$

Remark A.2.7. This fact can be checked in a more general setting as follows. In general, given a monoidal category $\mathbf{V}$ that is locally presentable as a category and whose monoidal product $\otimes$ preserves filtered colimits in each variable, the category $\mathbf{Mon}(\mathbf{V})$ of monoid objects in $\mathbf{V}$ is also locally presentable, and the forgetful functor $\mathbf{Mon}(\mathbf{V}) \rightarrow \mathbf{V}$ is finitary monadic. This is observed in [Por08] for symmetric case, and the same argument works for non-symmetric case as well, which is observed in [AV25, Proposition 2.1.7]. Since $\mathbf{E}/S1$ is locally presentable if so is $\mathbf{E}$, it suffices to show that the monoidal product on $\mathbf{E}/S1$, equivalently, the monoidal product on $[\mathbf{E}, \mathbf{E}]_{\text{cart}}/S$, preserves filtered colimits in each variable, provided that $\mathbf{E}$ is locally presentable and S preserves filtered colimits.

To see this, first observe that filtered colimits in $[\mathbf{E}, \mathbf{E}]_{\text{cart}}/S$ are computed pointwise if $\mathbf{E}$ is locally presentable, because filtered colimits commutes with pullbacks in $\mathbf{E}$. This immediately implies that the monoidal product on $[\mathbf{E}, \mathbf{E}]_{\text{cart}}/S$ preserves filtered colimits in one variable. For the other variable, it suffices to show that if S preserves filtered colimits and there is an endofunctor F over S in $[\mathbf{E}, \mathbf{E}]_{\text{cart}}/S$, then F also preserves filtered colimits. This can be shown in a similar way as Proposition 1.2.3.4. $\lrcorner$

Definition A.2.8. A *globular operad* is a $\mathbf{T}$ -operad, where $\mathbf{T}$ is the free strict ω -category monad on the category $\mathbf{GSet}$ of globular sets. $\lrcorner$

Remark A.2.9. Explicitly, a globular operad consists of an object $a: O \rightarrow \mathbf{T}1$ and morphisms $e: \eta_1^{\mathbf{T}} \rightarrow a$ and $m: a \otimes a \rightarrow a$ in $\mathbf{GSet}/\mathbf{T}1$ satisfying the monoid axioms. In particular, e is a globular map $e: 1 \rightarrow O$, and hence it gives rise to an n -cell $e_n \in O_n$ of the globular set O for each natural number n . A contraction on the monad over $\mathbf{T}$ corresponding to a globular operad (O, a, e, m) is precisely an $\mathcal{I}_O^\times$ -lifting structure on $a: O \rightarrow \mathbf{T}1$, where $\mathcal{I}_O^\times = \{\iota_n: \partial \mathcal{C}^n \rightarrow \mathcal{C}^n \mid n \geq 1\}$ is the set of boundary inclusions defined in Example 1.3.4. See also Example 1.3.22 for this perspective. $\lrcorner$

A.2.2. L is normalised. A globular operad O is said to be *normalised* if the set O_0 of its 0-cells is a singleton. The objective of this subsection is to show that $\mathbf{L}$ is normalised as a globular operad; i.e., the set $(\mathbf{L}1)_0$ of 0-cells of the globular set $\mathbf{L}1$ is a singleton. This result is in particular used in the next section Section A.3. (Although it should be possible, as mentioned in [Gar09a, Proof of Theorem 10], to deduce this statement from results in [Che10], we found it rather non-trivial to actually do so, which is why we give a different proof here.)

Following Remark A.2.9, let $O = (O, a, e, m)$ be a globular operad, and write $e_n \in O_n$ for the n -cell of O corresponding to the unique element of 1_n via the globular map $e: 1 \rightarrow O$.

Definition A.2.10. We define the *normal part* O^{norm} of the $\mathbf{T}$ -operad $O = (a: O \rightarrow \mathbf{T}1, e, m)$ as the globular subset $O^{\text{norm}} \subseteq O$ defined as follows.

- There is only one 0-cell in O^{norm} , namely e_0 .
- For $n > 0$, the set O_n^{norm} of all n -cells of O^{norm} consists of all n -cells $x \in O_n$ of O satisfying $s_0^O(x) = e_0 = t_0^O(x)$.

Note that to give a globular map $X \rightarrow O^{\text{norm}}$ from $X \in \mathbf{GSet}$ is equivalent to giving a globular map $X \rightarrow O$ which maps every 0-cell of X to e_0 . $\lrcorner$

Lemma A.2.11. O^{norm} (more precisely, the subobject $O^{\text{norm}} \hookrightarrow O \xrightarrow{a} \mathbf{T}1$ of a in $\mathbf{GSet}/\mathbf{T}1$) is closed under the monoid structure of O , and hence defines a globular suboperad of O . $\lrcorner$

Proof. It is clear that the globular map $e: 1 \rightarrow O$ factors through O^{norm} . To see that O^{norm} is closed under m , observe that the monad $\mathbf{T}$ is trivial at the dimension 0, i.e., the unit component $\eta_X^{\mathbf{T}}: X \rightarrow \mathbf{T}X$ induces a bijection $X_0 \cong (\mathbf{T}X)_0$ for any globular set X . This implies that the functor $(-)_0: \mathbf{GSet}/\mathbf{T}1 \rightarrow \mathbf{Set}$, mapping each object $f: X \rightarrow \mathbf{T}1$ of $\mathbf{GSet}/\mathbf{T}1$ to the set X_0 , is strong monoidal with respect to the cartesian monoidal structure on $\mathbf{Set}$. Hence the set O_0 of 0-cells of the globular operad O is an ordinary monoid, with $e_0 \in O_0$ as the unit element. Now it is easy to see that O^{norm} is closed under m , using the fact that m commutes with the source and target operations. $\square$

If the underlying object $a: O \rightarrow \mathbf{T}1$ of a globular operad $O = (a: O \rightarrow \mathbf{T}1, e, m)$ has a contraction $\bar{\kappa}^O$, then $\bar{\kappa}^O$ restricts to the normal part O^{norm} of O because the globular map $\iota_n: \partial G^n \rightarrow G^n$ is bijective on 0-cells for all $n > 0$. Now, recall that the globular operad $O_{\mathbf{L}}$ for weak ω -categories (which corresponds to the monad $\mathbf{L}$ for weak ω -categories) is defined as the initial object in the category of globular operads with contraction; see [Lei04, Section 9.2] for details.

Proposition A.2.12. O_L is a normalised globular operad. $\lrcorner$

In terms of the monad L , this means that the set $(L1)_0$ is a singleton.

Proof. The above observations imply that the normal part O_L^{norm} of O_L is also a globular operad with contraction, and the inclusion map $O_L^{\text{norm}} \rightarrow O_L$, being a monomorphism into an initial object, must then be an isomorphism. Therefore we have $O_L^{\text{norm}} \cong O_L$, i.e., the set $(O_L)_0$ is a singleton. $\square$

Moreover, we have the following corollary by considering the cartesianness of the unit η^L :

Corollary A.2.13. The component $\eta_X^L: X \rightarrow LX$ of the unit induces a bijection $X_0 \cong (LX)_0$ for any globular set X . $\lrcorner$

A.3. ε_X is bijective on objects

This section is devoted to the proof of Proposition 3.3.2.7, which is split off from [FHM25a, Appendix A].

Proposition A.3.1. The functor $(-)_0: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Set}$, mapping each weak ω -category X to its set X_0 of objects, has a right adjoint $\text{Cosk}: \mathbf{Set} \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. $\lrcorner$

Proof. Observe that we have a lax monad morphism (i.e., a *monad functor* in the sense of [Str72])

$$((-)_0, (-)_0 \circ \eta^L) : (\mathbf{GSet}, L) \rightarrow (\mathbf{Set}, \text{Id}_{\mathbf{Set}}),$$

where $(-)_0: \mathbf{GSet} \rightarrow \mathbf{Set}$ is the functor mapping each globular set X to its set X_0 of 0-cells. The functor $(-)_0: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Set}$ is induced by this lax monad morphism. Now, Corollary A.2.13 implies that $((-)_0, (-)_0 \circ \eta^L)$ is in fact a *strong* monad morphism.

The underlying functor $(-)_0: \mathbf{GSet} \rightarrow \mathbf{Set}$ of this strong monad morphism has both left and right adjoints, given by left and right Kan extensions along the (fully faithful) inclusion $\lceil 0 \rceil: 1 \rightarrow \mathbf{G}$. In particular, the right adjoint $\text{Ran}_{\lceil 0 \rceil}: \mathbf{Set} \rightarrow \mathbf{GSet}$ of $(-)_0: \mathbf{GSet} \rightarrow \mathbf{Set}$ induces (by *doctrinal adjunction* [Kel74]) a right adjoint of $((-)_0, (-)_0 \circ \eta^L) : (\mathbf{GSet}, L) \rightarrow (\mathbf{Set}, \text{Id}_{\mathbf{Set}})$ in the 2-category of monads and lax monad morphisms, which induces the desired right adjoint $\text{Cosk}: \mathbf{Set} \rightarrow \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ of $(-)_0: \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s \rightarrow \mathbf{Set}$. $\square$

Proof of Proposition 3.3.2.7. Let X be a weak ω -category. For any object $x \in X_0$ of X , let $\bar{x} \in (\mathbf{Q}X)_0$ be the object of $\mathbf{Q}X$ corresponding to the diagonal strict ω -functor $\bar{\kappa}_X(0; !^{\mathbf{Q}X}, x)$ below:

$$\begin{array}{ccc} \emptyset & \xrightarrow{!^{\mathbf{Q}X}} & \mathbf{Q}X \\ \downarrow \iota_0 & \nearrow \bar{\kappa}_X^{\mathbf{Q}}(0; !^{\mathbf{Q}X}, x) & \downarrow \varepsilon_X \\ \mathcal{C}^0 & \xrightarrow{x} & X \end{array}$$

We claim that the function $(-): X_0 \rightarrow (\mathbf{Q}X)_0$, mapping $x \in X_0$ to $\bar{x} \in (\mathbf{Q}X)_0$, is an inverse of the function $(\varepsilon_X)_0: (\mathbf{Q}X)_0 \rightarrow X_0$, mapping $y \in (\mathbf{Q}X)_0$ to $\varepsilon_X(y) \in X_0$. For any $x \in X_0$, we clearly have $\varepsilon_X(\bar{x}) = x$ by the commutativity of the above diagram.

It suffices to show that the function $(-): X_0 \rightarrow (\mathbf{Q}X)_0$ is surjective. To this end, consider a two-element set $S = \{a, b\}$, and let $E = \text{Cosk } S \in \mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. The adjointness $(-)_0 \dashv \text{Cosk}$ implies that E is right orthogonal to $\mathcal{I}^\times = \{\iota_n: \partial \mathcal{C}^n \rightarrow \mathcal{C}^n \mid n \geq 1\}$. Hence, in order to equip the unique strict ω -functor $!_E: E \rightarrow 1$ to the terminal weak ω -category 1 with an $\mathcal{I}$ -lifting structure λ , we only need to specify $\lambda(0; !^E, *): \mathcal{C}^0 \rightarrow E$. We set it to the transpose of the function $(\mathcal{C}^0)_0 \rightarrow S$ mapping the unique element of $(\mathcal{C}^0)_0 = (LG^0)_0 \cong (G^0)_0$ to a . The last bijection follows from Corollary A.2.13.

By the universality of $(\varepsilon_X: \mathbf{Q}X \rightarrow X, \kappa_X^{\mathbf{Q}})$, the unique strict ω -functor $!_X: X \rightarrow 1$ uniquely extends to a morphism $(h, !_X): (\varepsilon_X, \kappa_X^{\mathbf{Q}}) \rightarrow (!_E, \lambda)$ in $\mathcal{I}\text{-}\mathbf{AIMor}$. The strict ω -functor $h: \mathbf{Q}X \rightarrow E$ is necessarily the transpose of some function $\hat{h}: (\mathbf{Q}X)_0 \rightarrow S$, and one can check that the pair $(h, !_X)$ is a morphism $(\varepsilon_X, \kappa_X^{\mathbf{Q}}) \rightarrow (!_E, \lambda)$ in $\mathcal{I}\text{-}\mathbf{AIMor}$ if and only if we have $\hat{h}(y) = a$ for all y in the image of the function $(-): X_0 \rightarrow (\mathbf{Q}X)_0$. Therefore the uniqueness of such h , or equivalently that of such $\hat{h}$, implies that $(-): X_0 \rightarrow (\mathbf{Q}X)_0$ is surjective. $\square$

A.4. Proof of Lemma 4.5.1.1

Based on [FHM25b, Appendix A], in this section, we complete the proof of Lemma 4.5.1.1. Our proof below is essentially a ‘proof-relevant’ version of that of [LMW10, Lemma 4.36], and in particular, we will use their notion of *cylinder*. (In the following definition of cylinder, we adopt (and slightly modify) the notations $(-)^b$, $(-)^{\sharp}$, and $(-)^{\natural}$ from [LMW10], but note that they are unrelated to the functors $\mathbf{Wk}\text{-}\omega\text{-Cat}_s \rightarrow \mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-Cat}_s$ introduced in Section 4.3.2.)

Definition A.4.1. Let X be a strict ω -category and let x be a cell of dimension ≥ 1 in X . We write $x^b = s_0(x)$ and $x^{\sharp} = t_0(x)$. $\lrcorner$

Definition A.4.2 ([LMW10, Definition 4.14]). The notion of n -cylinder $U: x \curvearrowright y$ in a strict ω -category X , where x and y are n -cells in X , is defined inductively as follows.

- A 0-cylinder $U: x \curvearrowright y$ is an equivalence 1-cell $U: x \rightarrow y$.
- For $n \geq 1$, an n -cylinder $U: x \curvearrowright y$ is a triple $U = (U^b, U^{\sharp}, U^{\natural})$ where
 - $U^b: x^b \rightarrow y^b$ is an equivalence 1-cell,
 - $U^{\sharp}: x^{\sharp} \rightarrow y^{\sharp}$ is an equivalence 1-cell, and
 - $U^{\natural}: (x *_0^X U^{\sharp}) \curvearrowright (U^b *_0^X y)$ is an $(n-1)$ -cylinder in the hom strict ω -category $X(x^b, y^{\sharp})$.

For $n \geq 1$, the *source* of such U is the $(n-1)$ -cylinder $s_{n-1}(U): s_{n-1}(x) \curvearrowright s_{n-1}(y)$ defined inductively as follows.

- If $n = 1$, then $s_0(U) = U^b$.
- If $n \geq 2$, then $s_{n-1}(U)^b = U^b$, $s_{n-1}(U)^{\sharp} = U^{\sharp}$, and $s_{n-1}(U)^{\natural} = s_{n-2}(U^{\natural})$.

The *target* $(n-1)$ -cylinder $t_{n-1}(U): t_{n-1}(x) \curvearrowright t_{n-1}(y)$ is defined similarly. These operations make the collection of all cylinders in X into a globular set, which we denote by $\Gamma(X)$. $\lrcorner$

Theorem A.4.3 ([LMW10, Theorem 4.21 and Corollary 4.23]). The globular set $\Gamma(X)$ can be made into a strict ω -category such that the projections

$$\pi_1^X, \pi_2^X: \Gamma(X) \longrightarrow X$$

given by $\pi_1^X(U: x \curvearrowright y) = x$ and $\pi_2^X(U: x \curvearrowright y) = y$ are folk trivial fibrations (and in particular they are strict ω -functors). $\lrcorner$

Lemma A.4.4. Let X be a strict ω -category and let

$$x \xrightarrow{u} y \xrightarrow{v} x' \xrightarrow{u'} y'$$

be a chain of equivalence n -cells in X with $n \geq 1$. Then there exists an n -cylinder

$$U_n^X(u, v, u'): u \curvearrowright u'.$$

Proof. We proceed by induction on n . In the case $n = 1$, we must provide a 1-cylinder $u \curvearrowright u'$, which amounts to a commutative-up-to-equivalence-2-cell square such that

- the top edge is u ,
- the bottom edge is u' , and
- both vertical edges are equivalences.

We indeed have such a square, namely:

$$\begin{array}{ccc} x & \xrightarrow{u} & y \\ \downarrow u *_1^X v & \swarrow \text{id}_3^X(u *_1^X v *_1^X u') & \downarrow v *_1^X u' \\ x' & \xrightarrow{u'} & y' \end{array}$$

For $n \geq 2$, we define the n -cylinder $U_n^X(u, v, u')$ by

- $U_n^X(u, v, u')^b = \text{id}_1^X(x^b)$,
- $U_n^X(u, v, u')^\sharp = \text{id}_1^X(x^\sharp)$, and
- $U_n^X(u, v, u')^\natural = U_{n-1}^{X(x^b, x^\sharp)}(u, v, u')$.

□

Proof of Lemma 4.5.1.1 continued. Suppose that $u: x \rightarrow y$ is an equivalence n -cell in X . Observe that the n -cylinder $U_n^X(\text{id}_n^X(x), \text{id}_n^X(x), u)$ has type $\text{id}_n^X(x) \curvearrowright u$, so the strict ω -functor $\mathbf{C}^n \rightarrow \Gamma(X)$ picking out this cylinder fits into the following commutative diagram:

$$\begin{array}{ccccc}
 & & u & & \\
 & \nearrow & & \searrow & \\
 \mathbf{C}^n & \xrightarrow{U_n^X(\text{id}_n^X(x), \text{id}_n^X(x), u)} & \Gamma(X) & \xrightarrow{\pi_2^X} & X \\
 \downarrow i_*^n & \nearrow & \downarrow \pi_1^X & & \\
 \mathbf{E}_*^n & \xrightarrow{r_*^n} & \mathbf{C}^{n-1} & \xrightarrow{x} & X
 \end{array}$$

Since i_*^n is a folk cofibration by construction, and π_1^X is a folk trivial fibration by Theorem A.4.3, we obtain a lift as indicated. Composing this lift with π_2^X yields the desired extension. □

A.5. Quotient bicategory of a weak ω -category

This section is based on [FHM25b, Appendix B].

Recall from Corollary 2.4.13 that the binary relation $\sim$ defined by the existence of equivalence cells is an equivalence relation on the set of cells in a given weak ω -category X . The purpose of this appendix is to show that quotienting the 2-cells by this relation $\sim$ yields a bicategory $\tau_2(X)$. This construction will be used in Section A.6 in which we provide another example of a set $\mathcal{K}$ of maps in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$ satisfying (K1)–(K3) based on coinductive half-adjoint equivalences.

We will define the quotient *bicategory* construction by applying the quotient *category* construction to the hom weak ω -categories, which is in turn defined by applying the quotient *set* construction to the hom. The seemingly strange choice of letters in the definitions and propositions below is intended to accord with this context.

Definition A.5.1. Let Z be a weak ω -category. We write $\tau_0(Z)$ for the quotient set $Z_0/\sim$, and $\llbracket p \rrbracket \in \tau_0(Z)$ for the equivalence class containing $p \in Z_0$. ▮

Definition A.5.2. Let Y be a weak ω -category. We write $\tau_1(Y)$ for the graph whose vertex set is Y_0 and whose edges are given by

$$\tau_1(Y)(u, v) = \tau_0(Y(u, v)).$$

▮

Proposition A.5.3. Let Y be a weak ω -category. Then setting

- $\text{id}_1^{\tau_1(Y)}(u) = \llbracket \text{id}_1^Y(u) \rrbracket$ for each 0-cell u in Y , and
- $\llbracket p \rrbracket *_0 \tau_1(Y) \llbracket q \rrbracket = \llbracket p *_0^Y q \rrbracket$ for each composable pair of 1-cells p, q in Y

makes $\tau_1(Y)$ into a category. ▮

Proof. The composition is well defined by Corollary 2.4.14, and the category axioms follow from Corollaries 2.3.8 and 2.4.4. □

The following straightforward observation will be useful later.

Proposition A.5.4. Let Y be a weak ω -category and let $p: u \rightarrow v$ be a 1-cell in Y . Then p is an equivalence in Y if and only if $\llbracket p \rrbracket$ is an isomorphism in $\tau_1(Y)$. ▮

Proof. By definition of $\tau_1(Y)$, the morphism $\llbracket p \rrbracket$ is an isomorphism if and only if there exists a 1-cell $q: v \rightarrow u$ such that $p *_0^Y q \sim \text{id}_1^Y(u)$ and $q *_0^Y p \sim \text{id}_1^Y(v)$. □

Now we can define the underlying category-enriched graph of the desired quotient bicategory.

Definition A.5.5. Let X be a weak ω -category. We write $\tau_2(X)$ for the category-enriched graph whose vertex set is X_0 and whose hom categories are given by

$$\tau_2(X)(a, b) = \tau_1(X(a, b)).$$

▮

Before describing the entire bicategory structure, we need to give names to certain instances of *coherence* (Proposition 2.3.7).

Definition A.5.6. Given an n -cell $u: a \rightarrow b$ in a weak ω -category X , we write:

- $\lambda_{n+1}^X(u): \text{id}_n^X(a) *_{n-1}^X u \rightarrow u$
- $\check{\lambda}_{n+1}^X(u): u \rightarrow \text{id}_n^X(a) *_{n-1}^X u$
- $\rho_{n+1}^X(u): u *_{n-1}^X \text{id}_n^X(b) \rightarrow u$
- $\check{\rho}_{n+1}^X(u): u \rightarrow u *_{n-1}^X \text{id}_n^X(b)$

for equivalence $(n+1)$ -cells in X witnessing the unit law (Corollary 2.4.4).

Given a composable triple of n -cells $u_i: a_{i-1} \rightarrow a_i$ for $1 \leq i \leq 3$, we write

- $\alpha_{n+1}^X(u_1, u_2, u_3): (u_1 *_{n-1}^X u_2) *_{n-1}^X u_3 \rightarrow u_1 *_{n-1}^X (u_2 *_{n-1}^X u_3)$
- $\check{\alpha}_{n+1}^X(u_1, u_2, u_3): u_1 *_{n-1}^X (u_2 *_{n-1}^X u_3) \rightarrow (u_1 *_{n-1}^X u_2) *_{n-1}^X u_3$

for equivalence $(n+1)$ -cells in X witnessing the associative law (Corollary 2.3.8). $\lrcorner$

Proposition A.5.7. Let $(X, \xi: LX \rightarrow X)$ be a weak ω -category. For each 0-cell a in X , define $\text{id}_1^{\tau_2(X)}(a) = \text{id}_1^X(a)$. For each triple of 0-cells a, b, c in X , define a graph morphism

$$-*_0^{\tau_2(X)} -: \tau_1(X(a, b)) \times \tau_1(X(b, c)) \rightarrow \tau_1(X(a, c))$$

as follows.

- For 1-cells $u: a \rightarrow b$ and $v: b \rightarrow c$ in X , the 1-cell $u*_0^{\tau_2(X)}v: a \rightarrow c$ is $u*_0^Xv$.
- For 2-cells $p: u \rightarrow u': a \rightarrow b$ and $q: v \rightarrow v': b \rightarrow c$ in X , the equivalence class

$$[p] *_0^{\tau_2(X)} [q] \in \tau_1(X(a, c)) (u*_0^Xv, u'*_0^Xv')$$

is $[p*_0^Xq]$.

This is well defined and moreover defines a functor $-*_0^{\tau_2(X)} -: \tau_1(X(a, b)) \times \tau_1(X(b, c)) \rightarrow \tau_1(X(a, c))$.

Furthermore, the morphisms

$$\alpha_2^{\tau_2(X)}(u, v, w) = [\alpha_2^X(u, v, w)] : (u*_0^{\tau_2(X)}v) *_0^{\tau_2(X)}w \rightarrow u*_0^{\tau_2(X)}(v*_0^{\tau_2(X)}w),$$

$$\lambda_2^{\tau_2(X)}(u) = [\lambda_2^X(u)] : \text{id}_1^{\tau_2(X)}(x) *_0^{\tau_2(X)}u \rightarrow u, \text{ and}$$

$$\rho_2^{\tau_2(X)}(u) = [\rho_2^X(u)] : u*_0^{\tau_2(X)}\text{id}_1^{\tau_2(X)}(y) \rightarrow u$$

in the hom-categories of $\tau_2(X)$ form suitable natural isomorphisms, making $\tau_2(X)$ into a bicategory. $\lrcorner$

Proof. We first verify that $-*_0^{\tau_2(X)} -$ is well defined. Suppose we are given

- 1-cells $u, u': a \rightarrow b$ and $v, v': b \rightarrow c$,
- 2-cells $p_1, p_2: u \rightarrow u'$ and $q_1, q_2: v \rightarrow v'$, and
- equivalence 3-cells $h: p_1 \rightarrow p_2$ and $k: q_1 \rightarrow q_2$

in X . We must provide an equivalence 3-cell $p_1*_0^Xq_1 \rightarrow p_2*_0^Xq_2$. A 3-cell of correct type can be constructed as $h*_0^Xk$, and it is moreover an equivalence in X by Remark 2.4.8.

To see that $-*_0^{\tau_2(X)} -$ is a functor, suppose we are given

- 1-cells $u, u', u'': a \rightarrow b$ and $v, v', v'': b \rightarrow c$, and
- 2-cells $p: u \rightarrow u', p': u' \rightarrow u'', q: v \rightarrow v',$ and $q': v' \rightarrow v''$

in X . We wish to prove that

$$\text{id}_1^{X(a, b)}(u) *_0^X \text{id}_1^{X(b, c)}(v) \sim \text{id}_1^{X(a, c)}(u*_0^Xv)$$

and

$$(p*_0^Xp') *_0^X (q*_0^Xq') \sim (p*_0^Xq) *_0^X (p'*_0^Xq')$$

hold. By Corollary 1.2.4.3, we can rewrite these expressions as

$$\text{id}_2^X(u) *_0^X \text{id}_2^X(v) \sim \text{id}_2^X(u*_0^Xv) \tag{A.5.7.1}$$

and

$$(p*_1^Xp') *_0^X (q*_1^Xq') \sim (p*_0^Xq) *_1^X (p'*_0^Xq').$$

Because their proofs are similar, we will only spell out the former.

The left-hand side of (A.5.7.1) can be rewritten in terms of the corresponding operations in $L1$ (as opposed to X) as follows.

$$\begin{aligned}
\text{id}_2^X(u) *_{\tilde{e}_0}^X \text{id}_2^X(v) &= \xi \left(\tilde{e}_2 *_{\tilde{e}_0}^{L1} \tilde{e}_2, \begin{bmatrix} \text{id}_2^X(u) & \text{id}_2^X(v) \\ & b \end{bmatrix} \right) \quad (\text{Remark 2.4.8}) \\
&= \xi \left(\tilde{e}_2 *_{\tilde{e}_0}^{L1} \tilde{e}_2, \begin{bmatrix} \xi \left(\text{id}_2^{L1}(\tilde{e}_1), [u] \right) & \xi \left(\text{id}_2^{L1}(\tilde{e}_1), [v] \right) \\ \xi(\tilde{e}_0, [b]) & \end{bmatrix} \right) \\
&\quad (\text{Proposition 1.2.3.13.(i) and Remark 1.2.3.2.(iv)}) \\
&= \xi \left(\mu_1^L \left(\tilde{e}_2 *_{\tilde{e}_0}^{L1} \tilde{e}_2, \begin{bmatrix} \text{id}_2^{L1}(\tilde{e}_1) & \text{id}_2^{L1}(\tilde{e}_1) \\ & \tilde{e}_0 \end{bmatrix} \right), \mu_X^T \begin{bmatrix} [u] & [v] \\ & [b] \end{bmatrix} \right) \\
&\quad (\text{Remarks 1.2.3.2.(iii) and 1.2.3.2.(v)}) \\
&= \xi \left(\text{id}_2^{L1}(\tilde{e}_1) *_{\tilde{e}_0}^{L1} \text{id}_2^{L1}(\tilde{e}_1), \begin{bmatrix} u & v \\ & b \end{bmatrix} \right) \\
&\quad (\text{Proposition 1.2.3.13.(i) and action of } \mu_X^T \text{ (see Remark 1.1.3.7 or Remark 1.2.3.11)})
\end{aligned}$$

Similarly, the right-hand side of (A.5.7.1) can be rewritten as

$$\text{id}_2^X(u *_{\tilde{e}_0}^X v) = \xi \left(\text{id}_2^{L1}(\tilde{e}_1 *_{\tilde{e}_0}^{L1} \tilde{e}_1), \begin{bmatrix} u & v \\ & b \end{bmatrix} \right).$$

Since both $\text{id}_2^{L1}(\tilde{e}_1) *_{\tilde{e}_0}^{L1} \text{id}_2^{L1}(\tilde{e}_1)$ and $\text{id}_2^{L1}(\tilde{e}_1 *_{\tilde{e}_0}^{L1} \tilde{e}_1)$ are 2-cells in $L1$ of type $\tilde{e}_1 *_{\tilde{e}_0}^{L1} \tilde{e}_1 \rightarrow \tilde{e}_1 *_{\tilde{e}_0}^{L1} \tilde{e}_1$ and arity $\begin{bmatrix} 1 & 1 \\ 0 & \end{bmatrix}^{(2)}$, the existence of the desired invertible 3-cell can be deduced from Proposition 2.3.7.

For the second part of the proposition, that they are invertible follows from Proposition A.5.4. Their naturality, the pentagon identity, and the triangle identity can all be proved using the same strategy as above. To give a little more detail, by Corollary 1.2.4.3, we can reduce the desired commutativity conditions to the following statements: given

- 0-cells a, b, c, d, e ,
- 1-cells $u, u': a \rightarrow b$, $v, v': b \rightarrow c$, $w, w': c \rightarrow d$, $x: d \rightarrow e$, and
- 2-cells $p: u \rightarrow u'$, $q: v \rightarrow v'$, $r: w \rightarrow w'$

in X , there exist equivalence 3-cells

$$\begin{aligned}
\alpha_2^X(u, v, w) *_{\tilde{e}_1}^X (p *_{\tilde{e}_0}^X (q *_{\tilde{e}_0}^X r)) &\sim ((p *_{\tilde{e}_0}^X q) *_{\tilde{e}_0}^X r) *_{\tilde{e}_1}^X \alpha_2^X(u', v', w'), \\
\lambda_2^X(u) *_{\tilde{e}_1}^X p &\sim (\text{id}_1^X(a) *_{\tilde{e}_0}^X p) *_{\tilde{e}_1}^X \lambda_2^X(u'), \\
\rho_2^X(u) *_{\tilde{e}_1}^X p &\sim (p *_{\tilde{e}_0}^X \text{id}_1^X(b)) *_{\tilde{e}_1}^X \rho_2^X(u') \\
\alpha_2^X(u *_{\tilde{e}_0}^X v, w, x) *_{\tilde{e}_1}^X \alpha_2^X(u, v, w *_{\tilde{e}_0}^X x) &\sim \left((\alpha_2^X(u, v, w) *_{\tilde{e}_0}^X \text{id}_2^X(x)) *_{\tilde{e}_1}^X \alpha_2^X(u, v *_{\tilde{e}_0}^X w, x) \right) *_{\tilde{e}_1}^X \left(\text{id}_2^X(u) *_{\tilde{e}_0}^X \alpha_2^X(v, w, x) \right), \\
\alpha_2^X(u, \text{id}_1^X(b), v) *_{\tilde{e}_1}^X (\text{id}_2^X(u) *_{\tilde{e}_0}^X \lambda_2^X(v)) &\sim \rho_2^X(u) *_{\tilde{e}_0}^X v.
\end{aligned}$$

Each of these can be proved by rewriting both sides in terms of the corresponding operations in $L1$ using Proposition 1.2.3.13 and then applying Proposition 2.3.7. $\square$

Proposition A.5.8. Let X be a weak ω -category, and let $u: x \rightarrow y$ be a 1-cell in X . Then u is an equivalence in X if and only if u is an equivalence when regarded as a 1-cell in $\tau_2(X)$. $\dashv$

Proof. By definition of $\tau_2(X)$, the 1-cell u is an equivalence in $\tau_2(X)$ if and only if there exists a 1-cell $v: y \rightarrow x$ such that there are isomorphisms $u *_{\tilde{e}_0}^X v \cong \text{id}_1^X(x)$ and $v *_{\tilde{e}_0}^X u \cong \text{id}_1^X(y)$ in $\tau_1(X(x, x))$ and $\tau_1(X(y, y))$ respectively, which, by Proposition A.5.4, is equivalent to $u *_{\tilde{e}_0}^X v \sim \text{id}_1^X(x)$ and $v *_{\tilde{e}_0}^X u \sim \text{id}_1^X(y)$. $\square$

A.6. An example of $\mathcal{K}$ based on half-adjoint equivalences

This section is based on [FHM25b, Appendix C].

Here we give another example of a set $\mathcal{K}$ of morphisms in $\mathbf{m}\text{-Wk-}\omega\text{-Cat}_s$ satisfying conditions (K1)–(K3) stated in Section 4.3.2. By the results of Section 4.3.3, it gives rise to a set of morphisms in $\mathbf{Wk-}\omega\text{-Cat}_s$ characterising ω -equifibrations via the right lifting property. Whereas the example in Section 4.3.4 is based on the notion of flat equivalence, the example given here is based on half-adjoint equivalences. See Section 4.1 for the homotopical intuition.

Remark A.6.1. We declare each $*_n^X$ to be notationally left associative. That is, given enough cells u_i of dimension $> n$ in a weak ω -category X with each consecutive pair being composable along the n -dimensional boundary, the expression $u_1 *_n^X u_2 *_n^X u_3$ is understood as $(u_1 *_n^X u_2) *_n^X u_3$, and $u_1 *_n^X u_2 *_n^X u_3 *_n^X u_4$ is understood as $((u_1 *_n^X u_2) *_n^X u_3) *_n^X u_4$ etc. $\lrcorner$

Definition A.6.2. For each $n \geq 1$, we define a marked weak ω -category $(\mathcal{H}^n, t\mathcal{H}^n)$. (Here $\mathcal{H}$ stands for ‘half-adjoint’.) The underlying weak ω -category $\mathcal{H}^n$ is constructed as follows:

- start with $\mathcal{C}^n$, and call its fundamental n -cell $u_{\mathcal{H}}: x_{\mathcal{H}} \rightarrow y_{\mathcal{H}}$,
- freely adjoin an n -cell $v_{\mathcal{H}}: y_{\mathcal{H}} \rightarrow x_{\mathcal{H}}$, that is, take the pushout

$$\begin{array}{ccc} \partial\mathcal{C}^n \xrightarrow{(y_{\mathcal{H}}, x_{\mathcal{H}})} \mathcal{C}^n & & \\ \downarrow & & \downarrow u_{\mathcal{H}} \\ \mathcal{C}^n & \xrightarrow{v_{\mathcal{H}}} & \bullet \end{array}$$

- freely adjoin $(n+1)$ -cells $p_{\mathcal{H}}: \text{id}_n(x_{\mathcal{H}}) \rightarrow u_{\mathcal{H}} *_n v_{\mathcal{H}}$ and $q_{\mathcal{H}}: v_{\mathcal{H}} *_n u_{\mathcal{H}} \rightarrow \text{id}_n(y_{\mathcal{H}})$, and
- freely adjoin an $(n+2)$ -cell $r_{\mathcal{H}}$ from $\text{id}_{n+1}(u_{\mathcal{H}})$ to

$$\check{\lambda}_{n+1}(u_{\mathcal{H}}) *_n (p_{\mathcal{H}} *_n v_{\mathcal{H}}) *_n \alpha_{n+1}(u_{\mathcal{H}}, v_{\mathcal{H}}, u_{\mathcal{H}}) *_n (u_{\mathcal{H}} *_n q_{\mathcal{H}}) *_n \rho_{n+1}(u_{\mathcal{H}}).$$

The marked cells are $u_{\mathcal{H}}$, $p_{\mathcal{H}}$, $q_{\mathcal{H}}$, and $r_{\mathcal{H}}$.

We define the marking-preserving strict ω -functor $k_{\mathcal{H}}^n: \mathcal{M}\mathcal{C}^n \rightarrow \mathcal{H}^n$ as the one picking out $u_{\mathcal{H}}$. $\lrcorner$

We claim that the set

$$\mathcal{K}_{\mathcal{H}} = \{k_{\mathcal{H}}^n: \mathcal{M}\mathcal{C}^n \rightarrow \mathcal{H}^n \mid n \geq 1\}$$

of morphisms in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$ satisfies conditions (K1)–(K3).

We first show that $\mathcal{K}_{\mathcal{H}}$ satisfies (K1). The proof of the following proposition (especially the use of Proposition 4.1.1.1 in it) is inspired by [Lac04, Proof of Proposition 6].

Proposition A.6.3. Let $f: X \rightarrow Z$ be an ω -equifibration between weak ω -categories, and let $n \geq 1$. Suppose that we are given a commutative square

$$\begin{array}{ccc} \mathcal{M}\mathcal{C}^n & \xrightarrow{u} & X^{\natural} \\ k_{\mathcal{H}}^n \downarrow & & \downarrow f^{\natural} \\ \mathcal{H}^n & \longrightarrow & Z^{\natural} \end{array}$$

in $\mathbf{m}\text{-}\mathbf{Wk}\text{-}\omega\text{-}\mathbf{Cat}_s$. Then this square admits a lift. $\lrcorner$

Proof. The data of the given commutative square are equivalent to:

- an equivalence n -cell $u: x \rightarrow y$ in X ,
- an n -cell $v: fy \rightarrow fx$ in Z ,
- equivalence $(n+1)$ -cells $p: \text{id}_n^Z(fx) \rightarrow fu *_n^Z v$ and $q: v *_n^Z fu \rightarrow \text{id}_n^Z(fy)$ in Z , and
- an equivalence $(n+2)$ -cell r from $\text{id}_{n+1}^Z(fu)$ to

$$\check{\lambda}_{n+1}^Z(fu) *_n^Z (p *_n^Z v) *_n^Z \alpha_{n+1}^Z(fu, v, fu) *_n^Z (fu *_n^Z q) *_n^Z \rho_{n+1}^Z(fu)$$

in Z .

We wish to extend u to a quintet $(u, \bar{v}, \bar{p}, \bar{q}, \bar{r})$ of suitable type in X (so that it corresponds to a marking-preserving map $\mathcal{H}^n \rightarrow X^{\natural}$) such that f sends it to (fu, v, p, q, r) .

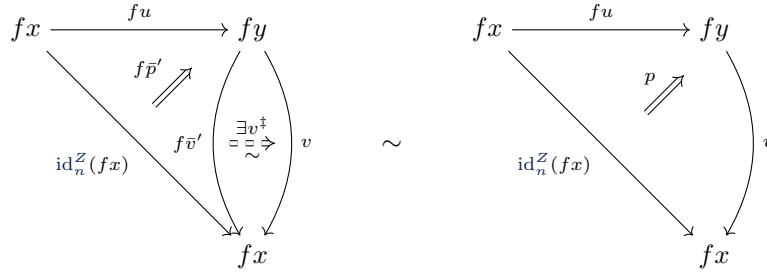
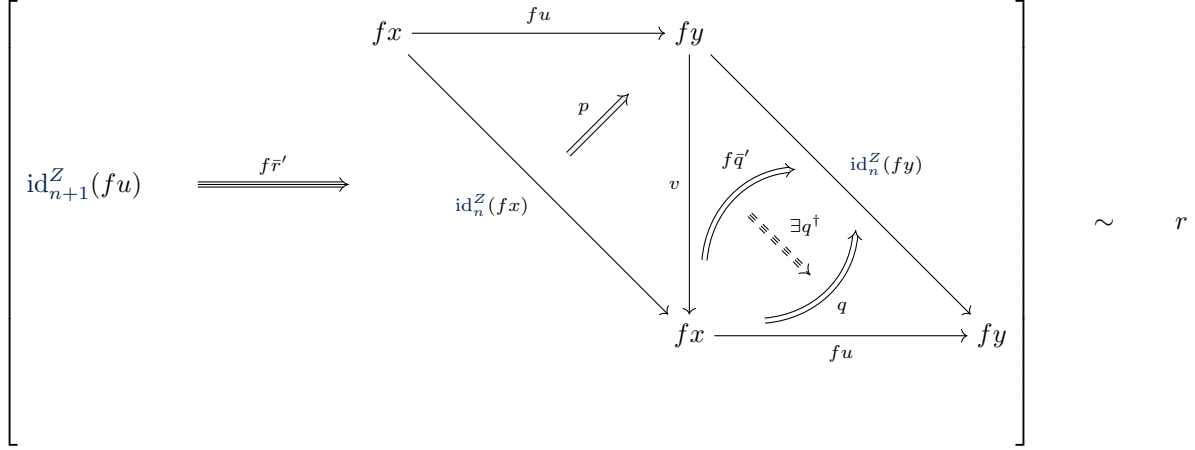
We lift v and p using an argument similar to that in the proof of Proposition 4.3.4.2; the details are as follows.

Since u is an equivalence, we can choose an n -cell $\bar{v}': y \rightarrow x$ and an equivalence $(n+1)$ -cell $\bar{p}': \text{id}_n^X(x) \rightarrow u *_n^X \bar{v}'$ in X . Since $f\bar{p}': \text{id}_n^Z(fx) \rightarrow fu *_n^Z f\bar{v}'$ is an equivalence in Z by Proposition 2.3.1, there exists an equivalence $(n+1)$ -cell

$$v^{\dagger}: fu *_n^Z f\bar{v}' \rightarrow fu *_n^Z v$$

in Z such that $f\bar{p}' *_n^Z v^{\dagger} \sim p$ by (1) of Proposition 4.1.1.1. Similarly, since fu is an equivalence in Z , there exists an equivalence $(n+1)$ -cell

$$v^{\ddagger}: f\bar{v}' \rightarrow v$$

FIGURE 1. Lifting v FIGURE 2. Lifting q (with $\check{\lambda}$, α and ρ omitted)

such that $fu *_{n-1} Zv^\dagger \sim v^\dagger$ by (2) of Proposition 4.1.1.1. Since f is an ω -equifibration, we can lift $v^\dagger$ to an equivalence $(n+1)$ -cell in X , which we denote as $\bar{v}^\dagger: \bar{v}' \rightarrow \bar{v}$. This completes the lifting of v .

Next, we lift the $(n+1)$ -cell p . Observe that we have an equivalence $(n+1)$ -cell

$$\bar{p}' *_{n-1}^X (u *_{n-1}^X \bar{v}^\dagger): \text{id}_n^X(x) \rightarrow u *_{n-1}^X \bar{v}$$

in X , and equivalence $(n+2)$ -cells witnessing

$$f(\bar{p}' *_{n-1}^X (u *_{n-1}^X \bar{v}^\dagger)) = f\bar{p}' *_{n-1}^Z (fu *_{n-1}^Z v^\dagger) \sim f\bar{p}' *_{n-1}^Z v^\dagger \sim p$$

in Z . Since f is an ω -equifibration, we can lift the composite of the latter to an equivalence $(n+2)$ -cell

$$\bar{p}' *_{n-1}^X (u *_{n-1}^X \bar{v}^\dagger) \sim \bar{p}$$

in X . Note that $\bar{p}$ is an equivalence in X by Corollary 2.4.16.

The way we will lift q and r will be similar to how we lifted v and p , and fig. 2 provides its summary. Consider the quotient bicategory $\tau_2(X(s_{n-2}(x), t_{n-2}(x)))$, where $X(s_{n-2}(x), t_{n-2}(x))$ is the suitable (iterated) hom weak ω -category of X . (When $n=1$, we interpret $X(s_{n-2}(x), t_{n-2}(x))$ as X .) In this bicategory, u and $\bar{v}'$ are pseudo-inverse equivalence 1-cells, and moreover we have $\bar{v}' \cong \bar{v}$. It follows that $\bar{v}$ is also a pseudo-inverse of u , and so the triple $(u, \bar{v}, [\bar{p}])$ can be completed to an adjoint equivalence. In other words, we can pick an equivalence $(n+1)$ -cell $\bar{q}': \bar{v} *_{n-1}^X u \rightarrow \text{id}_n^X(y)$ in X and an equivalence $(n+2)$ -cell $\bar{r}'$ from $\text{id}_{n+1}^X(u)$ to

$$\check{\lambda}_{n+1}^X(u) *_{n-1}^X (\bar{p} *_{n-1}^X u) *_{n-1}^X \alpha_{n+1}^X(u, \bar{v}, u) *_{n-1}^X (u *_{n-1}^X \bar{q}') *_{n-1}^X \rho_{n+1}^X(u)$$

in X . Making use of Proposition 4.1.1.1 with various equivalence cells, we can obtain an equivalence $(n+2)$ -cell $q^\dagger: f\bar{q}' \rightarrow q$ such that there is an equivalence $(n+3)$ -cell from

$$f\bar{r}' *_{n+2}^Z (\check{\lambda}_{n+1}^Z(fu) *_{n-1}^Z (p *_{n-1}^Z fu) *_{n-1}^Z \alpha_{n+1}^Z(fu, v, fu) *_{n-1}^Z (fu *_{n-1}^Z q^\dagger) *_{n-1}^Z \rho_{n+1}^Z(fu))$$

to r in Z . Since f is an ω -equifibration, we can lift $q^\dagger$ to an equivalence $(n+2)$ -cell $\bar{q}^\dagger: \bar{q}' \rightarrow \bar{q}$. Finally, we observe that the domain of the above equivalence $(n+3)$ -cell is in the image of f , and we take $\bar{r}$ to be the codomain of its lift through f . $\square$

We then check that $\mathcal{K}_{\mathcal{H}}$ satisfies (K2) by showing an analogue of Proposition 4.3.4.3.

Proposition A.6.4.

- (i) If a marked weak ω -category (X, tX) is $\mathcal{K}_{\mathcal{H}}$ -injective, then all cells in tX are equivalences in the underlying weak ω -category X .
- (ii) For any weak ω -category X , the marked weak ω -category $X^{\natural}$ is $\mathcal{K}_{\mathcal{H}}$ -injective. ┘

Proof. If a marked weak ω -category (X, tX) is $\mathcal{K}_{\mathcal{H}}$ -injective, then we have $tX \subseteq \Phi(tX)$, where Φ is defined in Definition 4.2.0.2. This proves (i).

(ii) follows from Proposition A.6.3. □

Finally, that $\mathcal{K}_{\mathcal{H}}$ satisfies (K3) is obvious.

Applying the construction of Section 4.3.3 to $\mathcal{K}_{\mathcal{H}}$, we obtain a set $J_{\mathcal{H}}$ of strict ω -functors characterising the ω -equifibrations.

Free strict ω -category monad

In this chapter, we will recollect about strict ω -categories from [Lei04], and rearrange its basic constructions from the viewpoint of the familial representation.

B.1. Preliminaries on infinitary distributive and lextensive categories

This section fixes terminology and establishes an infinitary counterpart of the standard characterisations of distributive and lextensive categories. Working with the category of families $\mathbf{Fm}(\mathbf{E})$ —the canonical coproduct completion— we package distributivity and lextensivity as left-exactness properties of the coproduct functor $\coprod -: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$. For lextensivity we prove that, for a category with finite limits and small coproducts, the following are equivalent: (i) the sliced-coproduct adjunction is an equivalence for all families; (ii) small coproducts are disjoint and stable under pullback; and (iii) $\coprod -$ preserves finite limits. The novelty is a direct infinitary proof of (ii) $\Leftrightarrow$ (iii) that avoids appeal to general cofibration theory used in [GL12]; the argument reduces finite-limit preservation to the preservation of four template pullback squares in $\mathbf{Fm}(\mathbf{E})$.

Definition B.1.1. A category $\mathbf{E}$ is *infinitary distributive* if it has finite products and small coproducts that distribute over finite products: i.e., for any finite family of sets $(I_j)_{j \in J}$ and a family $(X_{j,i})_{j \in J, i \in I_j}$ of objects in $\mathbf{E}$, the canonical morphism

$$\lambda\langle u, \xi \rangle \cdot (\lambda j. \langle u(j), \xi(j) \rangle): \coprod_{u \in \prod_{j \in J} I_j} \prod_{j \in J} X_{j, u(j)} \rightarrow \prod_{j \in J} \prod_{i \in I_j} X_{j, i}$$

is an isomorphism. $\lrcorner$

For each category $\mathbf{E}$, let us denote by $\mathbf{Fm}(\mathbf{E})$ for the category of (small) families of objects in $\mathbf{E}$: the objects of $\mathbf{Fm}(\mathbf{E})$ are families $(X_j)_{j \in J}$ of objects in $\mathbf{E}$ indexed by a small set J , and a morphism $(f_0, f): (X_j)_{j \in J} \rightarrow (Y_k)_{k \in K}$ consists of a function $f_0: J \rightarrow K$ and a family $f = (f_j: X_j \rightarrow Y_{f_0(j)})_{j \in J}$ of morphisms in $\mathbf{E}$. This category $\mathbf{Fm}(\mathbf{E})$ is the coproduct completion of $\mathbf{E}$, and in particular, $\mathbf{E}$ has small coproducts if and only if the unit $\mathbf{E} \rightarrow \mathbf{Fm}(\mathbf{E})$ has a left adjoint $\coprod: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$, which sends a family to its coproduct.

The following characterisation is not only well known but also easy to see. See, for example, [Coc93].

Proposition B.1.2. Let $\mathbf{E}$ be a category with finite products and small coproducts. The following are equivalent.

- (i) $\mathbf{E}$ is infinitary distributive.
- (ii) The functor $E \times -: \mathbf{E} \rightarrow \mathbf{E}$ is connected for each $E \in \mathbf{E}$.
- (iii) The functor $\coprod: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$ preserves finite products. $\lrcorner$

Now we turn to infinitary lextensive categories. Observe that if we have a category $\mathbf{E}$ with small coproducts and (certain) pullbacks, any slice of the coproduct functor $\coprod_{j \in J} -: [J, \mathbf{E}] \rightarrow \mathbf{E}$ is a left adjoint:

$$\coprod_{j \in J} - / (X_j)_{j \in J}: \prod_{j \in J} \mathbf{E} / X_j \xrightarrow{\hookrightarrow} \mathbf{E} / \coprod_{j \in J} X_j : (\iota_j^* -)_{j \in J} \quad (\text{B.1.2.1})$$

for each small family $(X_j)_{j \in J}$ of objects in $\mathbf{E}$, whose right adjoint sends a morphism $f: A \rightarrow \coprod_{j \in J} X_j$ in $\mathbf{E}$ to the family $(\iota_j^*(A, f))_{j \in J}$ of morphisms in $\mathbf{E}$ obtained by pulling back (A, f) along the coproduct injections.

Definition B.1.3. A category $\mathbf{E}$ is *infinitary lextensive* if it has finite limits and small coproducts and the adjunction (B.1.2.1) is an adjoint equivalence for each small family $(X_j)_{j \in J}$ of objects in $\mathbf{E}$.

We write $\infty\text{-LextCAT}$ for the huge 2-category of infinitary lextensive categories, functors preserving *pullbacks* (not necessarily all finite limits) and small coproducts, and natural transformations. We write $\infty\text{-LextCAT}$ for the locally wide sub-2-category of $\infty\text{-LextCAT}$ consisting of cartesian natural transformations. $\lrcorner$

There is a characterisation of infinitary lextensive categories similar to Proposition B.1.2. To state it, we need to recall two notions on coproducts.

Definition B.1.4. Let $\mathbf{E}$ be a category with finite limits and small coproducts.

- Small coproducts are *stable under pullbacks* if for each morphism $f: X \rightarrow Y$ in $\mathbf{E}$, the pullback functor $f^*: \mathbf{E}/Y \rightarrow \mathbf{E}/X$ along f preserves small coproducts.
- A coproduct $\coprod_{j \in J} X_j$ in $\mathbf{E}$ is *disjoint* if there are the following pullback squares for each $i, k \in J$.

$$\begin{array}{ccc} X_i & \xrightarrow{\text{Id}} & X_i \\ \text{Id} \downarrow & \lrcorner & \downarrow \iota_i \\ X_i & \xrightarrow{\iota_i} & \coprod_{j \in J} X_j \end{array}, \quad \begin{array}{ccc} \emptyset & \xrightarrow{!} & X_k \\ ! \downarrow & \lrcorner & \downarrow \iota_k \\ X_i & \xrightarrow{\iota_i} & \coprod_{j \in J} X_j \end{array} \quad (i \neq k)$$

⌋

Now the statement is as follows.

Proposition B.1.5. Let $\mathbf{E}$ be a category with finite limits and small coproducts. Then the following are equivalent.

- $\mathbf{E}$ is infinitary lextensive.
- Small coproducts in $\mathbf{E}$ are disjoint and stable under pullbacks.
- The coproduct functor $\mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$ preserves finite limits.

⌋

The equivalence of the first two conditions is well-known; see, for example, [CLW93]. The equivalence of the last two conditions is observed in [GL12, §5.3] for the finitary case; the proof there relies on a very general result by [RW86].

Here, we give a direct proof for the infinitary version. To this end, we first rephrase the several conditions recalled above in terms of pullback squares and their preservation by the coproduct functor $\coprod: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$. The proofs for the following rephrasings are straightforward.

Lemma B.1.6. Let $\mathbf{E}$ be a category with finite limits and small coproducts. Consider the following classes of pullback squares in $\mathbf{Fm}(\mathbf{E})$. Functions labelled by ι are subset inclusions.

- ($J \in \mathbf{Set}$ and $(b_j: B_j \rightarrow Y)_{j \in J}$ is a coproduct diagram in $\mathbf{E}$.)

$$\begin{array}{ccc} (A_j)_{j \in J} & \xrightarrow{(\text{Id}_J, (f_j)_{j \in J})} & (B_j)_{j \in J} \\ (!_J, (a_j)_{j \in J}) \downarrow & \lrcorner & \downarrow (!_J, (b_j)_{j \in J}) \\ (X)_{* \in 1} & \xrightarrow{(\text{Id}_1, f)} & (Y)_{* \in 1} \end{array}$$

- ($J \in \mathbf{Set}$ and $i \in J$.)

$$\begin{array}{ccc} (A_j)_{j \in \{i\}} & \xrightarrow{(\iota, \text{Id})} & (A_j)_{j \in J} \\ (\text{Id}_{\{i\}}, u_j) \downarrow & \lrcorner & \downarrow (\text{Id}_J, (u_j)_{j \in J}) \\ (X_j)_{j \in \{i\}} & \xrightarrow{(\iota, \text{Id})} & (X_j)_{j \in J} \end{array}$$

- ($J \in \mathbf{Set}$.)

$$\begin{array}{ccc} (A_j)_{j \in J} & \xrightarrow{(\text{Id}_J, (f_j)_{j \in J})} & (B_j)_{j \in J} \\ (!_J, (a_j)_{j \in J}) \downarrow & \lrcorner & \downarrow (!_J, (b_j)_{j \in J}) \\ (X)_{* \in 1} & \xrightarrow{(\text{Id}_1, f)} & (Y)_{* \in 1} \end{array}$$

- ($J \in \mathbf{Set}$ and $i, k \in J$.)

$$\begin{array}{ccc} (X_j)_{j \in \{i\} \cap \{k\}} & \xrightarrow{(\iota, \text{Id})} & (X_j)_{j \in \{k\}} \\ (\iota, \text{Id}) \downarrow & \lrcorner & \downarrow (\iota, \text{Id}) \\ (X_j)_{j \in \{i\}} & \xrightarrow{(\iota, \text{Id})} & (X_j)_{j \in J} \end{array}$$

Then the coproduct functor $\coprod: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$ preserves pullbacks of the form

- B.1.6.(i) if and only if the counit of each adjunction (B.1.2.1) is an isomorphism.
- B.1.6.(ii) if and only if the unit of each adjunction (B.1.2.1) is an isomorphism.
- B.1.6.(iii) if and only if small coproducts in $\mathbf{E}$ are stable under pullbacks.
- B.1.6.(iv) if and only if small coproducts in $\mathbf{E}$ are disjoint. $\lrcorner$

Lemma B.1.7. Let $\mathbf{E}$ be a category with finite limits and small coproducts. Then for the coproduct functor $\coprod: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$,

- (i) preservation of pullbacks of the form B.1.6.(i) is equivalent to that of B.1.6.(iii), and
- (ii) preservation of pullbacks of the form B.1.6.(ii) implies that of B.1.6.(iv). $\lrcorner$

Proof. Pullbacks of the form B.1.6.(i) are special cases of those of the form B.1.6.(iii), so suppose that $\coprod$ – preserves pullbacks of the form B.1.6.(i), and we show that $\coprod$ – preserves pullbacks of the form B.1.6.(iii). To this end, let us first take the factorisation of the right vertical arrow $(!_J, (b_j)_{j \in J})$ in B.1.6.(iii) through a coproduct diagram as in the right vertical part of the following diagram. Then take the pullback for the lower part of the factorisation, which results in the following diagram. By the pasting lemma, the top square is also a pullback, and hence by the preservation of pullbacks of the form B.1.6.(i), the outer square is sent to a pullback square since the lower square is sent to a pullback square by definition of pullbacks in $\mathbf{Fm}(\mathbf{E})$.

$$\begin{array}{ccc}
 (A_j)_{j \in J} & \longrightarrow & (B_j)_{j \in J} \\
 (!_J, (\bar{a}_j)_{j \in J}) \downarrow & & \downarrow (!_J, (\iota_j)_{j \in J}) \\
 (C)_{* \in 1} & \longrightarrow & (\coprod_{j \in J} B_j)_{* \in 1} \\
 \downarrow & \lrcorner & \downarrow (\text{Id}, (b_j)) \\
 (X)_{* \in 1} & \xrightarrow{(\text{Id}, f)} & (Y)_{* \in 1}
 \end{array}
 \quad \begin{array}{c} \curvearrowright \\ (!_J, (b_j)_{j \in J}) \end{array}$$

Next, suppose that $\coprod$ – preserves pullbacks of the form B.1.6.(ii), and we show that it preserves pullbacks of the form B.1.6.(iv). Consider the following pullback square in $\mathbf{Fm}(\mathbf{E})$, which is a special case of B.1.6.(ii), and hence it is sent to a pullback square by $\coprod$ –.

$$\begin{array}{ccc}
 (\coprod_{l \in \{j\} \cap \{k\}} X_l)_{j \in \{i\}} & \xrightarrow{(\iota, \text{Id})} & (\coprod_{l \in \{j\} \cap \{k\}} X_l)_{j \in J} \\
 \downarrow & \lrcorner & \downarrow \\
 (X_j)_{j \in \{i\}} & \xrightarrow{(\iota, \text{Id})} & (X_j)_{j \in J}
 \end{array}$$

Here, the right vertical arrow is the family of images of $(\iota, \text{Id}): (X_l)_{l \in \{j\} \cap \{k\}} \rightarrow (X_l)_{l \in \{j\}}$ under $\coprod$ –. It is easy to see that the image of the above square under $\coprod$ – coincides with that of the square B.1.6.(iv), which completes the proof. $\square$

Now we are ready to prove Proposition B.1.5.

Proof (of Proposition B.1.5). By Lemma B.1.6, it is obvious that B.1.5.(iii) implies both B.1.5.(ii) and B.1.5.(i). Moreover, Lemmas B.1.6 and B.1.7 show B.1.5.(i) $\Rightarrow$ B.1.5.(ii).

Finally, we show B.1.5.(ii) $\Rightarrow$ B.1.5.(iii). Suppose small coproducts in $\mathbf{E}$ are stable under pullbacks and disjoint. Since $\coprod$ – always preserves the terminal object, it suffices to show that $\coprod$ – preserves pullbacks. To this end, observe that any pullback square in $\mathbf{Fm}(\mathbf{E})$ is of the following form:

$$\begin{array}{ccc}
 (W_{i,j})_{i \in I, j \in J, f(i)=g(j)} & \xrightarrow{(\text{prl}, (u_{i,j})_{i,j})} & (Y_j)_{j \in J} \\
 (\text{prl}, (v_{i,j})_{i,j}) \downarrow & \lrcorner & \downarrow (g, (g_j)_{j \in J}) \\
 (X_i)_{i \in I} & \xrightarrow{(f, (f_i)_{i \in I})} & (Z_k)_{k \in K}
 \end{array} \tag{B.1.7.1}$$

For these data, we have the following pullback squares in $\mathbf{Fm}(\mathbf{E})$ for each $i \in I$ and $j \in J$.

$$\begin{array}{ccccc}
 (W_{i,j})_{k \in \{f(i)\} \cap \{g(j)\}} & \xrightarrow{(\text{Id}, u_{i,j})} & (Y_j)_{k \in \{f(i)\} \cap \{g(j)\}} & \xrightarrow{(\iota, \text{Id})} & (Y_j)_{k \in \{g(j)\}} \\
 (\text{Id}, v_{i,j}) \downarrow & \lrcorner & (a) & \lrcorner & (b) \downarrow (\text{Id}, g_j) \\
 (X_i)_{k \in \{f(i)\} \cap \{g(j)\}} & \xrightarrow{(\text{Id}, f_i)} & (Z_k)_{k \in \{f(i)\} \cap \{g(j)\}} & \xrightarrow{(\iota, \text{Id})} & (Z_k)_{k \in \{g(j)\}} \\
 (\iota, \text{Id}) \downarrow & \lrcorner & (b) & \lrcorner & (c) \downarrow (\iota, \text{Id}) \\
 (X_i)_{k \in \{f(i)\}} & \xrightarrow{(\text{Id}, f_i)} & (Z_k)_{k \in \{f(i)\}} & \xrightarrow{(\iota, \text{Id})} & (Z_k)_{k \in K}
 \end{array}$$

The pullback square labelled by (a) is preserved by $\coprod$ – because the indexing sets of its vertices are the same and are singletons or empty. Those labelled by (b) are also preserved by $\coprod$ – because of stability of the initial object (i.e. the empty coproduct) under pullback. Finally, the pullback square labelled by (c) is preserved by $\coprod$ – because of the disjointness. Thus the outer pullback square is also preserved by $\coprod$. Applying pullback-stability twice to its image, we obtain a pullback square in $\mathbf{E}$ that coincides with the image of B.1.7.1 under $\coprod$, which completes the proof. $\square$

B.2. Cartesian sharp construction

Given a category $\mathbf{E}$ with finite products, we can consider the category $\mathbf{E}\text{-Cat}$ of $\mathbf{E}$ -enriched categories. Forgetting the structure of categories, we have a forgetful functor $\mathbf{E}\text{-Cat} \rightarrow \mathbf{E}\text{-Gph}$ to the category of $\mathbf{E}$ -enriched graphs [Wol74]. This functor is monadic if $\mathbf{E}$ is an infinitary distributive category [Kel80, Theorem 23.4], and the induced monad is cartesian if $\mathbf{E}$ is an infinitary lextensive category [CFP18]. Moreover, if we have a good monad on $\mathbf{E}$, then the category $(T\text{-Alg})\text{-Cat}$ is monadic over $\mathbf{E}\text{-Gph}$, which results in a construction $T \mapsto T^\sharp$ of a monad on $\mathbf{E}\text{-Gph}$ from a monad on $\mathbf{E}$ ([Lei04, Appendix F]). The free ω -category monad is a fixed point of this operation by [CL19, Theorem 4.14].

In this section, we revisit these results from the viewpoint of the characterisation of infinitary lextensive categories observed in the previous section. We believe almost all of the contents in this section are known and can be found in the literature (e.g., [Wol74; Kel80; Lei04; CFP18; CL19]), so the objective of this section is to organise them in a coherent manner. Moreover, we give the 2-functorial aspect of the construction, which seems to be missing in the literature.

Notation B.2.1. We introduce some notations about lists and non-empty lists.

- The empty list is denoted by nil .
- Given a list $\mathbf{x}$, we write $\mathbf{x}.\text{length}$ for the length of $\mathbf{x}$.
- For each $0 \leq i < \mathbf{x}.\text{length}$, we write $\mathbf{x}[i] \in S$ for the i -th element of $\mathbf{x}$.
- We write S^+ for the set of all non-empty finite lists of elements of S .
- Given a list $\mathbf{x}$, we write $\mathbf{x}.\text{lind}$ for the last index of $\mathbf{x}$, i.e., $\mathbf{x}.\text{lind} = \mathbf{x}.\text{length} - 1$.
- For each $a, b \in S$, we write $S^+\langle a, b \rangle$ for the subset of S^+ consisting of all lists $\mathbf{x}$ such that $\mathbf{x}[0] = a$ and $\mathbf{x}[\mathbf{x}.\text{lind}] = b$.
- For each function $f: S \rightarrow A$ and each pair of elements $s, s' \in S$, we write $f_{s,s'}^+: S^+\langle s, s' \rangle \rightarrow S^+\langle fs, fs' \rangle$ for the function sending a list $\mathbf{x}$ to $(f(\mathbf{x}[i]))_{i \leq \mathbf{x}.\text{lind}}$.
- For each non-empty list $\mathbf{x} \in S^+$, the concatenation of lists defines a function¹

$$\text{concat}: \prod_{i < \mathbf{x}.\text{lind}} S^+\langle \mathbf{x}[i], \mathbf{x}[i+1] \rangle \rightarrow S^+\langle \mathbf{x}[0], \mathbf{x}[\mathbf{x}.\text{lind}] \rangle.$$

$\lrcorner$

We use the notation C^* to denote the list object of an object C in any category (if exists) other than $\mathbf{Set}$. The following lemma is obvious but fundamental for later arguments.²

¹Given a set S , consider its coskeleton graph $S \times S \rightrightarrows S$, and take its free category. Then the objects of this category are the elements of S and the hom-set $\text{Hom}(a, b)$ is precisely $S^+\langle a, b \rangle$. The concatenation is precisely the composition in this category.

²One can prove this lemma by hand, but there is a more conceptual way to check it: as mentioned in the previous footnote, the assignment $S \mapsto (S^+\langle a, b \rangle)_{a, b \in S}$ can be considered as a composite of the coskeleton $\mathbf{Set} \rightarrow \mathbf{Gph}$ and the free category monad $\mathbf{Gph} \rightarrow \mathbf{Gph}$. The former is a right adjoint functor, and hence it preserves limits. The latter preserves pullbacks, and the multiplication and unit for the monad, which give the concatenation of lists, are cartesian natural transformations.

Lemma B.2.2. For each function $f: S \rightarrow A$ and each $\mathbf{x} \in S^+$, the following square is a pullback square in **Set**.

$$\begin{array}{ccc} \prod_{i < \mathbf{x}. \text{bind}} S^+ \langle \mathbf{x}[i], \mathbf{x}[i+1] \rangle & \xrightarrow{\prod_{i < \mathbf{x}. \text{bind}} f_{\mathbf{x}[i], \mathbf{x}[i+1]}^+} & \prod_{i < \mathbf{x}. \text{bind}} A^+ \langle f(\mathbf{x}[i]), f(\mathbf{x}[i+1]) \rangle \\ \text{concat} \downarrow & \lrcorner & \downarrow \text{concat} \\ S^+ \langle \mathbf{x}[0], \mathbf{x}[\mathbf{x}. \text{bind}] \rangle & \xrightarrow{f_{\mathbf{x}[0], \mathbf{x}[\mathbf{x}. \text{bind}]}^+} & A^+ \langle f(\mathbf{x}[0]), f(\mathbf{x}[\mathbf{x}. \text{bind}]) \rangle \end{array}$$

On the other hand, if we have a pullback square in **Set** as on the left below, then the induced square on the right below is also a pullback square for each list $\mathbf{x} \in A^+$.

$$\begin{array}{ccc} A \xrightarrow{u} C & & \prod_{i < \mathbf{x}. \text{bind}} A^+ \langle \mathbf{x}[i], \mathbf{x}[i+1] \rangle \xrightarrow{\prod u^+} \prod_{i < \mathbf{x}. \text{bind}} C^+ \langle u(\mathbf{x}[i]), u(\mathbf{x}[i+1]) \rangle \\ v \downarrow \lrcorner \downarrow g & \mapsto & \prod v^+ \downarrow \lrcorner \downarrow \prod g^+ \\ B \xrightarrow{f} D & & \prod_{i < \mathbf{x}. \text{bind}} B^+ \langle v(\mathbf{x}[i]), v(\mathbf{x}[i+1]) \rangle \xrightarrow{\prod f^+} \prod_{i < \mathbf{x}. \text{bind}} D^+ \langle f(v(\mathbf{x}[i])), f(v(\mathbf{x}[i+1])) \rangle \end{array}$$

Definition B.2.3. Let $\mathbf{E}$ be a category. An **E-enriched graph** or just **E-graph** is a pair $(X_0, X(-, -))$ of a set X_0 and a functor $X(-, -): X_0 \times X_0 \rightarrow \mathbf{E}$ from the discrete category. An **E-graph morphism** from $(X_0, X(-, -))$ to $(Y_0, Y(-, -))$ consists of a function $f_0: X_0 \rightarrow Y_0$ and a natural transformation

$$f_{-, -}: X(-, -) \Rightarrow Y(f_0(-), f_0(-)): X_0 \times X_0 \rightarrow \mathbf{E}.$$

We often omit subscripts when the meaning is clear from the context. We denote the category of **E-graphs** and **E-graph morphisms** by **E-Gph**.

Definition B.2.4. Given a functor $F: \mathbf{E} \rightarrow \mathbf{E}'$, we define a functor $F\text{-Gph}: \mathbf{E}\text{-Gph} \rightarrow \mathbf{E}'\text{-Gph}$ by the post-composition of F with the hom-functor: i.e., for each **E-enriched graph** $X = (X_0, X(-, -))$, we define $F\text{-Gph}(X)$ by $(X_0, F(X(-, -)))$. This assignment $--\text{Gph}$ defines a 2-functor $\mathcal{CAT} \rightarrow \mathcal{CAT}$ in an obvious way.

Remark B.2.5. Let $\alpha: F \Rightarrow G: \mathbf{E} \rightarrow \mathbf{E}'$ be a natural transformation. Then we have a natural transformation $\alpha\text{-Gph}: F\text{-Gph} \Rightarrow G\text{-Gph}: \mathbf{E}\text{-Gph} \rightarrow \mathbf{E}'\text{-Gph}$, and $\alpha\text{-Gph}$ is cartesian if α is so.

Definition B.2.6. Let $\mathbf{E}$ be a category with finite products. An **E-enriched category** or just an **E-category** is an **E-enriched graph** X equipped with two families of morphisms $(\text{comp}_{x,y,z}: X(x,y) \times X(y,z) \rightarrow X(x,z))_{x,y,z \in X_0}$ and $(\text{id}_x: 1 \rightarrow X(x,x))_{x \in X_0}$ in $\mathbf{E}$ satisfying associativity and identity axioms: i.e., for each $E \in \mathbf{E}$, the pair $(\mathbf{E}(E, \text{comp}), \mathbf{E}(E, \text{id}))$ endows the graph $(\mathbf{E}(E, X(x,y)))_{x,y \in X_0}$ with a category structure.

Given two **E-enriched categories** X and Y , an **E-enriched functor** from X to Y is an **E-enriched graph morphism** $f: X \rightarrow Y$ preserving the composition and identity morphisms, i.e., for each $x, y, z \in X_0$, we have $f_{x,y} \times f_{y,z} \circ \text{comp}_{f_x, f_y, f_z} = \text{comp}_{f_x, f_y, f_z} \circ f_{x,z}$ and $\text{id}_x \circ f_{x,x} = \text{id}_{f_x}$. We denote the category of **E-enriched categories** and **E-enriched functors** by **E-Cat**.

Lemma B.2.7. If $\mathbf{E}$ has small coproducts, then so does **E-Gph**. If $\mathbf{E}$ and $\mathbf{E}'$ are categories with small coproducts and a functor $F: \mathbf{E} \rightarrow \mathbf{E}'$ preserves them, then so is the functor $F\text{-Gph}: \mathbf{E}\text{-Gph} \rightarrow \mathbf{E}'\text{-Gph}$. If $\mathbf{E}$ is moreover infinitary distributive, small coproducts are created by the forgetful functor $\mathbf{E}\text{-Cat} \rightarrow \mathbf{E}\text{-Gph}$.

Proof. If $\mathbf{E}$ has coproduct, the Grothendieck fibration $\mathbf{E}\text{-Gph} \rightarrow \mathbf{Set}$ is a bifibration that has opfibred coproducts. Therefore by a folklore fact of opfibrations, the coproduct in the total category **E-Gph** is calculated using coproducts in **Set** and each fibre. The functor $F\text{-Gph}: \mathbf{E}\text{-Gph} \rightarrow \mathbf{E}'\text{-Gph}$ is an opfibred functor over **Set** if F preserves small coproducts, and hence $F\text{-Gph}$ preserves small coproducts.

In detail, the coproduct is calculated as follows. Let S be a set and suppose that $\mathbf{E}$ has S -coproducts and initial object $\emptyset$. Then given a family $(X_s)_{s \in S}$ of **E-graphs**, the coproduct $\coprod_{s \in S} X_s$ is defined by $(\coprod_{s \in S} X_s)_0 = \coprod_{s \in S} (X_s)_0$ and

$$\left(\coprod_{s \in S} X_s \right) ((s, x), (s', x')) = \coprod_{s_0 \in \{s\} \cap \{s'\}} X_{s_0}(x, x'). \quad (\text{B.2.7.1})$$

Now using the distributivity,³ the product of composable homs is calculated as follows.

$$\coprod_{s_0 \in \{s\} \cap \{s'\}} X_{s_0}(x, x') \times \coprod_{s'_0 \in \{s'\} \cap \{s''\}} X_{s'_0}(x', x'') \cong \coprod_{s_0 \in \{s\} \cap \{s'\} \cap \{s''\}} X_{s_0}(x, x') \times X_{s_0}(x', x'')$$

³Indeed, here we only need the distributivity for coproducts indexed by propositions (sets with at most one elements).

Using this identification, we can define the composition and identity for $\coprod_{s \in S} X_s$ in the obvious way, if we are given an $\mathbf{E}$ -category structure on each X_s . This indeed gives the coproduct in $\mathbf{E}\text{-Cat}$, which is obviously created by the (conservative) forgetful functor $\mathbf{E}\text{-Cat} \rightarrow \mathbf{E}\text{-Gph}$. $\square$

Lemma B.2.8. Let $\mathbf{E}$ be a category. Let $\mathbf{I}$ be a small category. Then $\mathbf{E}\text{-Gph}$ has $\mathbf{I}$ -limits if and only if $\mathbf{E}$ has them. Moreover, given any functor $F: \mathbf{E} \rightarrow \mathbf{E}'$, if F preserves $\mathbf{I}$ -limits then the functor $F\text{-Gph}$ also preserves them. $\lrcorner$

Proof. The forgetful functor $\mathbf{E}\text{-Gph} \rightarrow \mathbf{Set}$ is a fibration such that $\mathbf{E}$ has $\mathbf{I}$ -limits if and only if the fibration $\mathbf{E}\text{-Gph} \rightarrow \mathbf{Set}$ has fibred $\mathbf{I}$ -limits. Now the statement follows from a folklore fact that limits in the total category of a fibration is calculated using reindexing functors and limits in the base category and each fibre. Observe here that the functor $F\text{-Gph}: \mathbf{E}\text{-Gph} \rightarrow \mathbf{E}'\text{-Gph}$ is a fibred functor over $\mathbf{Set}$. $\square$

Example B.2.9. Let $\mathbf{E}$ be a category with pullbacks. Then by Lemma B.2.8, the category $\mathbf{E}\text{-Gph}$ has pullbacks. A square

$$\begin{array}{ccc} (W(w, w'))_{w, w' \in W_0} & \xrightarrow{(u_0, (u_{w, w'})_{w, w' \in W_0})} & (Y(y, y'))_{y, y' \in Y_0} \\ (v_0, (v_{w, w'})_{w, w' \in W_0}) \downarrow & & \downarrow (g_0, (g_{y, y'})_{y, y' \in Y_0}) \\ (X(x, x'))_{x, x' \in X_0} & \xrightarrow{(f_0, (f_{x, x'})_{x, x' \in X_0})} & (Z(z, z'))_{z, z' \in Z_0} \end{array} \quad (\text{B.2.9.1})$$

is a pullback square if and only if the underlying square in $\mathbf{Set}$ is a pullback square, and for each $w, w' \in W_0$, the following square is a pullback square in $\mathbf{E}$.

$$\begin{array}{ccc} W(w, w') & \xrightarrow{u_{w, w'}} & Y(y, y') \\ v_{w, w'} \downarrow & \lrcorner & \downarrow g_{y, y'} \\ X(x, x') & \xrightarrow{f_{x, x'}} & Z(z, z') \end{array}$$

Here we write $(x, x'), (y, y'), (z, z')$ for the images of (w, w') under $v_0, u_0, v_0 \circ f_0$ respectively. $\lrcorner$

Proposition B.2.10. The category $\mathbf{E}\text{-Gph}$ is infinitary lextensive if $\mathbf{E}$ is so. $\lrcorner$

Proof. Consider a pullback square in $\mathbf{Fm}(\mathbf{E}\text{-Gph})$:

$$\begin{array}{ccc} ((W_h(w, w'))_{w, w' \in (W_h)_0})_{h \in H} & \longrightarrow & ((Y_j(y, y'))_{y, y' \in (Y_j)_0})_{j \in J} \\ \downarrow & \lrcorner & \downarrow \\ ((X_i(x, x'))_{x, x' \in (X_i)_0})_{i \in I} & \longrightarrow & ((Z_k(z, z'))_{z, z' \in (Z_k)_0})_{k \in K} \end{array} \quad (\text{B.2.10.1})$$

We show that this pullback is preserved by the coproduct functor $\coprod -: \mathbf{Fm}(\mathbf{E}\text{-Gph}) \rightarrow \mathbf{E}\text{-Gph}$.

Since $\mathbf{Set}$ is obviously infinitary lextensive, we have the following pullback square in $\mathbf{Set}$, which shows that the image under $\coprod$ is a pullback square at the level of objects.

$$\begin{array}{ccc} \coprod_{h \in H} (W_h)_0 & \xrightarrow{u} & \coprod_{j \in J} (Y_j)_0 \\ v \downarrow & \lrcorner & \downarrow g \\ \coprod_{i \in I} (X_i)_0 & \xrightarrow{f} & \coprod_{k \in K} (Z_k)_0 \end{array}$$

Now it remains to show the hom-part of the coproduct of (B.2.10.1) is a pullback square in $\mathbf{E}$. To this end, for the above diagram in $\mathbf{Set}$, suppose we are given the following elements.

$$\begin{array}{ccc} (h, w) \xrightarrow{u} (j, y) & (h', w') \xrightarrow{u} (j', y') & \\ v \downarrow & \downarrow g & v \downarrow \quad \downarrow g \\ (i, x) \xrightarrow{f} (k, z) & (i', x') \xrightarrow{f} (k', z') & \end{array}$$

Observe that, by the description of pullback square in **E-Gph** (Example B.2.9), there is a pullback square in **Fm(E)** of the following form.

$$\begin{array}{ccc} (W_{h_0}(w, w'))_{h_0 \in \{h\} \cap \{h'\}} & \longrightarrow & (Y_{j_0}(y, y'))_{j_0 \in \{j\} \cap \{j'\}} \\ \downarrow & \lrcorner & \downarrow \\ (X_{i_0}(x, x'))_{i_0 \in \{i\} \cap \{i'\}} & \longrightarrow & (Z_{k_0}(z, z'))_{k_0 \in \{k\} \cap \{k'\}} \end{array}$$

By definition of coproducts in **E-Gph** (Lemma B.2.7 and (B.2.7.1)), it suffices to show that the above pullback is preserved by coproduct $\coprod - : \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$, which follows from the infinitary distributivity of **E**. $\square$

Construction B.2.11. Let **E** be an infinitary distributive category. We define a monad $\text{Id}_{\mathbf{E}}^{\sharp}$ on **E-Gph** as follows.

- $\text{Id}_{\mathbf{E}}^{\sharp}$ does nothing on objects; i.e., for each $X \in \mathbf{E-Gph}$, we have $\text{Id}_{\mathbf{E}}^{\sharp}(X)_0 = X_0$. Moreover, the multiplication and unit for $\text{Id}_{\mathbf{E}}^{\sharp}$ are identity-on-objects.
- Suppose we are given $X \in \mathbf{E-Gph}$ and $a, b \in X_0$.
 - The hom-object for $\text{Id}_{\mathbf{E}}^{\sharp}(X)$ is defined as follows.

$$\text{Id}_{\mathbf{E}}^{\sharp}(X)(a, b) = \coprod_{\mathbf{x} \in X_0^+ \langle a, b \rangle} \prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1])$$

- There is a canonical inclusion

$$X(a, b) \xrightarrow{(ab, -)} \prod_{\mathbf{x} \in X_0^+ \langle a, b \rangle} \prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) = \text{Id}_{\mathbf{E}}^{\sharp}(X)(a, b)$$

for the summand indexed by $ab \in X_0^+ \langle a, b \rangle$, which defines the hom-part of the unit $\eta_X : X \rightarrow \text{Id}_{\mathbf{E}}^{\sharp}(X)$.

- By the infinitary distributivity of **E**, we have the following isomorphism

$$\text{Id}_{\mathbf{E}}^{\sharp}(\text{Id}_{\mathbf{E}}^{\sharp}(X))(a, b) \cong \prod_{\substack{\mathbf{x} \in X_0^+ \langle a, b \rangle \\ \mathbf{u} \in \prod_{i < \mathbf{x}. \text{lind}} X_0^+ \langle \mathbf{x}[i], \mathbf{x}[i+1] \rangle}} \prod_{\substack{i < \mathbf{x}. \text{lind} \\ j < \mathbf{u}(i). \text{lind}}} X(\mathbf{u}(i)[j], \mathbf{u}(i)[j+1]), \quad (\text{B.2.11.1})$$

which are natural in $X \in \mathbf{E-Gph}$. The multiplication $\mu_X : \text{Id}_{\mathbf{E}}^{\sharp}(\text{Id}_{\mathbf{E}}^{\sharp}(X)) \rightarrow \text{Id}_{\mathbf{E}}^{\sharp}(X)$ acts on the hom-object via the concatenation (see Notation B.2.1)

$$\prod_{\mathbf{x} \in X_0^+ \langle a, b \rangle} \prod_{i < \mathbf{x}. \text{lind}} X_0^+ \langle \mathbf{x}[i], \mathbf{x}[i+1] \rangle \rightarrow X_0^+ \langle a, b \rangle. \quad (\text{B.2.11.2})$$

Proposition B.2.12. Let **E** be an infinitary distributive category. Then the forgetful functor $\mathbf{E-Cat} \rightarrow \mathbf{E-Gph}$ is monadic and the induced monad is $\text{Id}_{\mathbf{E}}^{\sharp}$. $\square$

Proof. Suppose we are given $X \in \mathbf{E-Gph}$. We set the identity $\text{id}_a : 1 \rightarrow \text{Id}_{\mathbf{E}}^{\sharp}(X)(a, a)$ to be the coproduct injection at $a \in X_0^+ \langle a, a \rangle$:

$$1 \cong \prod_{i < 0} X(a[i], a[i+1]) \hookrightarrow \prod_{\mathbf{x} \in X_0^+ \langle a, a \rangle} \prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1])$$

On the other hand, we define the composition $\text{comp}_{a,b,c} : \text{Id}_{\mathbf{E}}^{\sharp}(X)(a, b) \times \text{Id}_{\mathbf{E}}^{\sharp}(X)(b, c) \rightarrow \text{Id}_{\mathbf{E}}^{\sharp}(X)(a, c)$ as the one induced from the infinitary distributivity of **E** and the canonical isomorphism

$$\prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) \times \prod_{j < \mathbf{y}. \text{lind}} X(\mathbf{y}[j], \mathbf{y}[j+1]) \cong \prod_{i < (\mathbf{x} \mathbin{++} \mathbf{y}). \text{lind}} X(\mathbf{x} \mathbin{++} \mathbf{y}[i], \mathbf{x} \mathbin{++} \mathbf{y}[i+1])$$

for each $\mathbf{x} \in X_0^+ \langle a, b \rangle$, $\mathbf{y} \in X_0^+ \langle b, c \rangle$, where $\mathbf{x} \mathbin{++} \mathbf{y}$ is the concatenation as in (B.2.11.2). It is a routine to check that these data satisfy the axioms of an **E**-category.

We now show that the **E**-category $\text{Id}_{\mathbf{E}}^{\sharp}X$ equipped with the unit $\eta_X : X \rightarrow \text{Id}_{\mathbf{E}}^{\sharp}X$ (Construction B.2.11) is free on X . Let Y be an **E**-category and $f : X \rightarrow Y$ be an **E**-graph morphism. There exists a unique **E**-functor $\bar{f} : \text{Id}_{\mathbf{E}}^{\sharp}X \rightarrow Y$ such that $\eta_X \circ \bar{f} = f$ in **E-Gph**. Indeed, it must satisfies the following conditions.

- (1) $\bar{f}_0$ is equal to f_0 .

(2) For each $\mathbf{x} \in X_0^+(a, b)$, the morphism $\bar{f}_{a,b}$ restricts the following composite:

$$\prod_{i < n} X(\mathbf{x}[i], \mathbf{x}[i+1]) \xrightarrow{\prod_{i < n} f_{\mathbf{x}[i], \mathbf{x}[i+1]}} \prod_{i < n} Y(f_0(\mathbf{x}[i]), f_0(\mathbf{x}[i+1])) \xrightarrow{\text{comp}} Y(f_0(a), f_0(b))$$

Here, the latter morphism labelled by ‘comp’ is the n -fold composition in Y , which is uniquely determined by axioms of an $\mathbf{E}$ -category.

These conditions obviously show the uniqueness of such $\mathbf{E}$ -functor $\bar{f}$. It is also obvious that they define (at least) an $\mathbf{E}$ -graph morphism $\bar{f}$. One can readily check that such graph morphism $\bar{f}$ preserves the identity and composition, and hence it is the desired $\mathbf{E}$ -functor.

Applying the above construction to the case when $X = \text{Id}_{\mathbf{E}}^\sharp X$, $Y = \text{Id}_{\mathbf{E}}^\sharp X$, and $f = \text{Id}$, we show that the multiplication μ_X coincides with the one defined in Construction B.2.11.

It remains to show the monadicity. Now the comparison functor $K: \mathbf{E}\text{-Cat} \rightarrow \text{Id}_{\mathbf{E}}^\sharp\text{-Alg}$ is described as follows.

- For each $\mathbf{E}$ -category Y , the underlying $\mathbf{E}$ -graph of KY is the same as that of Y . The structure morphism $\alpha_{KY}: \text{Id}_{\mathbf{E}}^\sharp Y \rightarrow Y$ of an $\text{Id}_{\mathbf{E}}^\sharp$ -algebra is constructed as 2 with $f: Y \rightarrow Y$ is the identity graph morphism on Y .
- For each $\mathbf{E}$ -functor $g: Y \rightarrow Y'$, one can easily check f preserves the $\text{Id}_{\mathbf{E}}^\sharp$ -algebra structures because the α_Y and $\alpha_{Y'}$ are defined by iterated use of concatenations in Y and Y' respectively.

Given an $\text{Id}_{\mathbf{E}}^\sharp$ -algebra (Z, α_Z) , restricting α_Z to the summands whose indices are sequences of length 0 and 2, we obtain identity and composition for Z , and this indeed defines an $\mathbf{E}$ -category. Obviously, KZ coincides with Z by the coherence, and this shows (essential) surjectivity of K . By the same reason, an $\mathbf{E}$ -graph morphism becomes an $\mathbf{E}$ -functor when it preserves the structures of $\text{Id}_{\mathbf{E}}^\sharp$ -algebras, which shows the fully faithfulness of K . $\square$

By Lemma B.2.7, we conclude the following.

Corollary B.2.13. $\text{Id}_{\mathbf{E}}^\sharp$ preserves small coproducts for each infinitary distributive category $\mathbf{E}$. $\lrcorner$

Corollary B.2.14. For each infinitary distributive category $\mathbf{E}$, the forgetful functor $\mathbf{E}\text{-Cat} \rightarrow \mathbf{E}\text{-Gph}$ is faithful and creates small coproducts and limits. In particular, $\mathbf{E}\text{-Cat}$ is infinitary distributive, and it is infinitary lexensive if $\mathbf{E}$ is so. $\lrcorner$

The next proposition is the point at which the assumption that $\mathbf{E}$ is infinitary lexensive becomes indispensable in our development.

Proposition B.2.15 ([CFP18, Theorem 3.7]). Let $\mathbf{E}$ be an infinitary lexensive category. Then the monad $\text{Id}_{\mathbf{E}}^\sharp$ is cartesian. $\lrcorner$

Proof. Suppose we are given a pullback square (B.2.9.1) in $\mathbf{E}\text{-Gph}$. Since $\text{Id}_{\mathbf{E}}^\sharp$ does nothing on objects, the image of (B.2.9.1) under $\text{Id}_{\mathbf{E}}^\sharp$ is a pullback square at the level of objects. Now it remains to show that the hom-part of the image of (B.2.9.1) under $\text{Id}_{\mathbf{E}}^\sharp$ is a pullback square in $\mathbf{E}$. To this end, we observe that the following square is a pullback square in $\mathbf{Fm}(\mathbf{E})$ for each $w, w' \in W_0$. Here, we write $(x, x'), (y, y'), (z, z')$ for the images of (w, w') under $v_0, u_0, v_0 \circ f_0$ respectively.

$$\begin{array}{ccc} (\prod_{i < \mathbf{w}. \text{lind}} W(\mathbf{w}[i], \mathbf{w}[i+1]))_{\mathbf{w} \in W_0^+(w, w')} & \longrightarrow & (\prod_{i < \mathbf{y}. \text{lind}} Y(\mathbf{y}[i], \mathbf{y}[i+1]))_{\mathbf{y} \in Y_0^+(y, y')} \\ \downarrow & \lrcorner & \downarrow \\ (\prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]))_{\mathbf{x} \in X_0^+(x, x')} & \longrightarrow & (\prod_{i < \mathbf{z}. \text{lind}} Z(\mathbf{z}[i], \mathbf{z}[i+1]))_{\mathbf{z} \in Z_0^+(z, z')} \end{array}$$

Indeed, this follows from Lemma B.2.2, the description of pullbacks (Example B.2.9) in $\mathbf{E}\text{-Gph}$, and that pullbacks commute with products. By sending it by $\coprod -: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$, the infinitary distributivity of $\mathbf{E}$ concludes that the hom-part of the image of (B.2.9.1) under $\text{Id}_{\mathbf{E}}^\sharp$ is a pullback square in $\mathbf{E}$.

Now it remains to show the multiplication and unit of $\text{Id}_{\mathbf{E}}^\sharp$ are cartesian natural transformations. We only show the cartesianness of the multiplication, as the other case is similar and easier. Suppose that we are given a morphism $f: X \rightarrow Y$ in $\mathbf{E}\text{-Gph}$. The hom-part of the naturality square for the multiplication at f is, through

the isomorphism obtained by using the infinitary distributivity of $\mathbf{E}$ (B.2.11.1), identified with the image of the following square under $\coprod: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$.

$$\begin{array}{ccc} \left(\prod_{\substack{i < \mathbf{x}. \text{lind} \\ j < \mathbf{u}(i). \text{lind}}} X(\mathbf{u}(i)[j], \mathbf{u}(i)[j+1]) \right) & \xrightarrow{\substack{\mathbf{x} \in X_0^+(a,b) \\ \mathbf{u} \in \prod_{i < \mathbf{x}. \text{lind}} X_0^+(\mathbf{x}[i], \mathbf{x}[i+1])}} & \left(\prod_{\substack{i < \mathbf{y}. \text{lind} \\ j < \mathbf{v}(i). \text{lind}}} Y(\mathbf{v}(i)[j], \mathbf{v}(i)[j+1]) \right) \\ \downarrow & & \downarrow \\ \left(\prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) \right)_{\mathbf{x} \in X_0^+(a,b)} & \xrightarrow{\quad} & \left(\prod_{i < \mathbf{y}. \text{lind}} Y(\mathbf{y}[i], \mathbf{y}[i+1]) \right)_{\mathbf{y} \in Y_0^+(f(a), f(b))} \end{array}$$

Here, the horizontal morphisms are induced from f in an obvious way, and each component of the vertical morphisms is a canonical isomorphism rearranging indices of the products, and hence the square above is a pullback square by Lemma B.2.2. Now infinitary distributivity of $\mathbf{E}$ concludes the desired cartesianness. $\square$

Lemma B.2.16. Let $F: \mathbf{A} \rightarrow \mathbf{B}$ be a functor between finitely complete categories that preserves pullbacks. Let I be a finite set. Observe that I -products are captured as the right adjoints of the diagonal functors $\text{const}: \mathbf{A} \rightarrow \mathbf{A}^I$, $\text{const}: \mathbf{B} \rightarrow \mathbf{B}^I$. Then the mate with respect to those adjunctions of the following commutative square is cartesian.

$$\begin{array}{ccc} \mathbf{A}^I & \xrightarrow{F^I} & \mathbf{B}^I \\ \text{const} \uparrow & = & \uparrow \text{const} \\ \mathbf{A} & \xrightarrow{F} & \mathbf{B} \end{array}$$

⌋

Proof. The commutative square above is the composite of the following commutative squares:

$$\begin{array}{ccccc} \mathbf{A}^I & \xrightarrow{(F/1)^I} & (\mathbf{B}/F1)^I & \xrightarrow{(F1)_!^I} & \mathbf{B}^I \\ \text{const} \uparrow & & \uparrow \text{const} & & \uparrow \text{const} \\ \mathbf{A} & \xrightarrow{F/1} & \mathbf{B}/F1 & \xrightarrow{(F1)_!} & \mathbf{B} \end{array}$$

where $F/1: \mathbf{A} \rightarrow \mathbf{B}/F1$ is the functor obtained from F by slicing over 1, and $(F1)_!: \mathbf{B}/F1 \rightarrow \mathbf{B}$ is the forgetful (domain) functor. Since $F/1$ preserves I -products, the mate of the left square is an isomorphism. Thus, it suffices to show that the mate of the right square is cartesian. The desired mate, however, is the mate of a square of the following type, with respect to the adjunctions $(F1)_! \dashv (F1 \times -)$ and $(F1)_!^I \dashv (F1 \times -)^I$.

$$\begin{array}{ccc} (\mathbf{B}/F1)^I & \xleftarrow{(F1)^{*I}} & \mathbf{B}^I \\ \Pi \downarrow & \cong & \downarrow \Pi \\ \mathbf{B}/F1 & \xleftarrow{(F1)^*} & \mathbf{B} \end{array}$$

One can easily observe that the units and counits of these adjunctions are cartesian, which completes the proof. $\square$

Construction B.2.17. Let $\mathbf{E}, \mathbf{E}'$ be infinitary lextensive categories and $F: \mathbf{E} \rightarrow \mathbf{E}'$ be a functor that preserves small coproducts. We define a natural transformation $\text{Id}_F^\sharp: \text{Id}_{\mathbf{E}}^\sharp \circ F\text{-}\mathbf{Gph} \Rightarrow F\text{-}\mathbf{Gph} \circ \text{Id}_{\mathbf{E}}^\sharp$ as follows. Let $X \in \mathbf{E}\text{-}\mathbf{Gph}$ be an $\mathbf{E}$ -graph, and let $a, b \in X_0$ be objects of X .

- The function $((\text{Id}_F^\sharp)_X)_0: F\text{-}\mathbf{Gph}(\text{Id}_{\mathbf{E}}^\sharp(X))_0 \rightarrow \text{Id}_{\mathbf{E}}^\sharp(F\text{-}\mathbf{Gph}(X))_0$ is the identity on X_0 .
- The morphism $((\text{Id}_F^\sharp)_X)_{a,b}: F\text{-}\mathbf{Gph}(\text{Id}_{\mathbf{E}}^\sharp(X))(a, b) \rightarrow \text{Id}_{\mathbf{E}}^\sharp(F\text{-}\mathbf{Gph}(X))(a, b)$ is, through the identification

$$F \left(\coprod_{\mathbf{x} \in X_0^+(a,b)} \prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) \right) \cong \coprod_{\mathbf{x} \in X_0^+(a,b)} F \left(\prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) \right) \quad (\text{B.2.17.1})$$

obtained by using the preservation of small coproducts by F , defined as the image (under $\coprod -: \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$) of the family of canonical morphisms

$$\left(F \left(\prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) \right) \rightarrow \prod_{i < \mathbf{x}. \text{lind}} F(X(\mathbf{x}[i], \mathbf{x}[i+1])) \right)_{\mathbf{x} \in X_0^+(a,b)}$$

that is obtained as components of the mate of the commutative square $F \circ \text{const} = \text{const} \circ F^{\mathbf{x}. \text{lind}} : \mathbf{E} \rightarrow \mathbf{E}^{\mathbf{x}. \text{lind}}$ with respect to $\text{const} \dashv \prod_{i < \mathbf{x}. \text{lind}}$. Note here that such mate is cartesian by Lemma B.2.16 if F preserves pullbacks, and an isomorphism if F preserves finite products. $\lrcorner$

Remark B.2.18. By the last remark of the second item of Construction B.2.17, in the setting there, the transformation $\text{ld}_F^\sharp$ is an isomorphism if F preserves finite products. On the other hand, by the characterisation Proposition B.1.5 of infinitary lextensive categories, this transformation $\text{ld}_F^\sharp$ is cartesian if F preserves pullbacks. $\lrcorner$

Lemma B.2.19. In the setting of Construction B.2.17, the pair $(F\text{-}\mathbf{Gph}, \text{ld}_F^\sharp)$ defines an oplax monad morphism of type $(\mathbf{E}, \text{ld}_F^\sharp) \rightarrow (\mathbf{E}', \text{ld}_{F'}^\sharp)$. $\lrcorner$

Proof. We only check the compatibility with the multiplication, as the unit case is similar and easier. Let X be an $\mathbf{E}$ -graph and $a, b \in X_0$. Since all the transformations we consider are identity-on-objects, it suffices to show suitable commutativity at the level of hom-objects. The two morphisms we want to compare are, through the identifications (B.2.11.1) and (B.2.17.1) coming from the infinitary distributivity and the preservation of small coproducts by F , described as the images under $\llbracket : \mathbf{Fm}(\mathbf{E}) \rightarrow \mathbf{E}$ of the two composites from top-left to bottom-right in the diagram below. Thus, it suffices to show that the square commutes in $\mathbf{Fm}(\mathbf{E})$.

$$\begin{array}{ccc}
 \left(F \prod_{\substack{i < \mathbf{x}. \text{lind} \\ j < \mathbf{u}(i). \text{lind}}} X(\mathbf{u}(i)[j], \mathbf{u}(i)[j+1]) \right)_{\substack{\mathbf{x} \in X_0^+(a,b) \\ \mathbf{u} \in \prod_{i < \mathbf{x}. \text{lind}} X_0^+(\mathbf{x}[i], \mathbf{x}[i+1])}} & \xrightarrow{(\text{concat}, -)} & (F \prod_{i < \mathbf{x}. \text{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]))_{\mathbf{x} \in X_0^+(a,b)} \\
 \downarrow (\text{Id}, -) & & \downarrow (\text{Id}, -) \\
 \left(\prod_{\substack{i < \mathbf{x}. \text{lind} \\ j < \mathbf{u}(i). \text{lind}}} F X(\mathbf{u}(i)[j], \mathbf{u}(i)[j+1]) \right)_{\substack{\mathbf{x} \in X_0^+(a,b) \\ \mathbf{u} \in \prod_{i < \mathbf{x}. \text{lind}} X_0^+(\mathbf{x}[i], \mathbf{x}[i+1])}} & \xrightarrow{(\text{concat}, -)} & (\prod_{i < \mathbf{x}. \text{lind}} F X(\mathbf{x}[i], \mathbf{x}[i+1]))_{\mathbf{x} \in X_0^+(a,b)}
 \end{array} \tag{B.2.19.1}$$

Let us explain each component of the morphisms in the diagram above.

- Each component of the left and right vertical morphisms is a component of the mate of the commutative squares

$$\begin{array}{ccc}
 \prod_{i < \mathbf{x}. \text{lind}} \mathbf{E}^{\mathbf{u}(i). \text{lind}} & \xrightarrow{\prod_{i < \mathbf{x}. \text{lind}} F^{\mathbf{u}(i). \text{lind}}} & \prod_{i < \mathbf{x}. \text{lind}} \mathbf{E}'^{\mathbf{u}(i). \text{lind}} \\
 \uparrow \prod_{i < \mathbf{x}. \text{lind}} \text{const} & & \uparrow \prod_{i < \mathbf{x}. \text{lind}} \text{const} \\
 \mathbf{E}^{\mathbf{x}. \text{lind}} & & \mathbf{E}'^{\mathbf{x}. \text{lind}} \\
 \uparrow \text{const} & & \uparrow \text{const} \\
 \mathbf{E} & \xrightarrow{F} & \mathbf{E}'
 \end{array} \quad \text{and} \quad \begin{array}{ccc}
 \mathbf{E}^{\mathbf{x}. \text{lind}} & \xrightarrow{F^{\mathbf{x}. \text{lind}}} & \mathbf{E}'^{\mathbf{x}. \text{lind}} \\
 \uparrow \text{const} & & \uparrow \text{const} \\
 \mathbf{E} & \xrightarrow{F} & \mathbf{E}'
 \end{array}$$

with respect to the adjunctions

$$\text{const} \circ \prod_{i < \mathbf{x}. \text{lind}} \text{const} \dashv \prod_{i < \mathbf{x}. \text{lind}} \left(\prod_{j < \mathbf{u}(i). \text{lind}} - \right) \circ \prod_{i < \mathbf{x}. \text{lind}} - \quad \text{and} \quad \text{const} \dashv \prod_{i < \mathbf{x}. \text{lind}} - \tag{B.2.19.2}$$

respectively.

- Each component of the bottom horizontal morphism is a component of the mate of the commutative square

$$\begin{array}{ccc}
 \prod_{i < \mathbf{x}. \text{lind}} \mathbf{E}'^{\mathbf{u}(i). \text{lind}} & \xrightarrow{\cong} & \mathbf{E}'^{\text{concat}(\mathbf{u}). \text{lind}} \\
 \uparrow \text{const} \circ \prod_{i < \mathbf{x}. \text{lind}} \text{const} & & \uparrow \text{const} \\
 \mathbf{E}' & \xrightarrow{\quad \quad \quad} & \mathbf{E}'
 \end{array}$$

with respect to the adjunctions (B.2.19.2). Here, the function concat is the concatenation of lists observed in (B.2.11.2), and the top horizontal isomorphism is the canonical one rearranging indices. Since this is an isomorphism, the resulting mate is an isomorphism.

For the top horizontal morphism, we can argue similarly using $\mathbf{E}$ instead of $\mathbf{E}'$, but consider the images under F of the components.

Now the commutativity of (B.2.19.1) follows from the obvious commutative square

$$\begin{array}{ccc} \prod_{i < \mathbf{x}. \text{.lind}} \mathbf{E}^{\mathbf{u}(i). \text{.lind}} & \xrightarrow{\cong} & \mathbf{E}^{\text{concat}(\mathbf{u}). \text{.lind}} \\ \prod_{i < \mathbf{x}. \text{.lind}} F^{\mathbf{u}(i). \text{.lind}} \downarrow & & \downarrow F^{\text{concat}(\mathbf{u}). \text{.lind}} \\ \prod_{i < \mathbf{x}. \text{.lind}} \mathbf{E}'^{\mathbf{u}(i). \text{.lind}} & \xrightarrow{\cong} & \mathbf{E}'^{\text{concat}(\mathbf{u}). \text{.lind}}, \end{array}$$

using the uniqueness of the mates. $\square$

Proposition B.2.20. Let us write $\mathbf{mnd}_o(\mathcal{K})$ for the 2-category of monads and oplax monad morphisms in a 2-category $\mathcal{K}$. The assignment $\mathbf{E} \mapsto (\mathbf{E}\text{-}\mathbf{Gph}, \text{Id}_{\mathbf{E}}^\sharp)$ (Construction B.2.11), together with Construction B.2.17 on 1-cells, extends to 2-functors making the following diagram commute.

$$\begin{array}{ccc} \infty\text{-}\mathcal{LextCAT} & \xrightarrow{(-\mathbf{Gph}, \text{Id}_{-}^\sharp)} & \mathbf{mnd}_o(\infty\text{-}\mathcal{LextCAT}) \\ \uparrow & & \uparrow \\ \infty\text{-}\mathcal{LextCART} & \xrightarrow{(-\mathbf{Gph}, \text{Id}_{-}^\sharp)} & \mathbf{mnd}_o(\infty\text{-}\mathcal{LextCART}) \\ \downarrow & & \downarrow \\ \mathcal{CAT} & \xrightarrow{-\mathbf{Gph}} & \mathcal{CAT} \end{array} \quad (\text{B.2.20.1})$$

Here, the upward vertical arrows are the inclusions, and the downward vertical arrows are the forgetful 2-functors. $\lrcorner$

Proof. By Proposition B.2.10, Lemmas B.2.7, B.2.8 and B.2.19, Corollary B.2.13, and Remarks B.2.5 and B.2.18, considering the local faithfulness of the forgetful 2-functor $\mathbf{mnd}_o(\mathcal{K}) \rightarrow \mathcal{K}$, it remains to show

- the 1-functoriality of this assignment and that
- for each 2-cell $\alpha: F \Rightarrow G: \mathbf{E} \rightarrow \mathbf{E}'$, the natural transformation $\alpha\text{-}\mathbf{Gph}: F\text{-}\mathbf{Gph} \Rightarrow G\text{-}\mathbf{Gph}: \mathbf{E}\text{-}\mathbf{Gph} \rightarrow \mathbf{E}'\text{-}\mathbf{Gph}$ defines a modification of oplax monad morphisms of type $(F\text{-}\mathbf{Gph}, \text{Id}_F^\sharp) \Rightarrow (G\text{-}\mathbf{Gph}, \text{Id}_G^\sharp)$.

The 1-functoriality is obvious from the Construction B.2.17 because taking mates is compatible with compositions of squares. Thus, we show the latter modification condition. By the Construction B.2.17, this reduces to showing that for each $\mathbf{E}$ -graph X and $\mathbf{x} \in X_0^+$, the following diagram commutes. Here, the top and bottom horizontal morphisms are those used in Construction B.2.17: i.e., components of the mates of suitable commutative squares.

$$\begin{array}{ccc} F \left(\prod_{i < \mathbf{x}. \text{.lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) \right) & \longrightarrow & \prod_{i < \mathbf{x}. \text{.lind}} F(X(\mathbf{x}[i], \mathbf{x}[i+1])) \\ F \prod_{i < \mathbf{x}. \text{.lind}} \alpha_X(\mathbf{x}[i], \mathbf{x}[i+1]) \downarrow & & \downarrow \prod_{i < \mathbf{x}. \text{.lind}} \alpha_X(\mathbf{x}[i], \mathbf{x}[i+1]) \\ G \left(\prod_{i < \mathbf{x}. \text{.lind}} X(\mathbf{x}[i], \mathbf{x}[i+1]) \right) & \longrightarrow & \prod_{i < \mathbf{x}. \text{.lind}} G(X(\mathbf{x}[i], \mathbf{x}[i+1])) \end{array} \quad (\text{B.2.20.2})$$

This follows from the following obvious identity of transformations

$$\begin{array}{ccc} \begin{array}{ccc} & \xrightarrow{G^{\mathbf{x}. \text{.lind}}} & \\ \mathbf{E}^{\mathbf{x}. \text{.lind}} & \xrightarrow{\alpha^{\mathbf{x}. \text{.lind}} \uparrow} & \mathbf{E}'^{\mathbf{x}. \text{.lind}} \\ \text{const} \uparrow & \xrightarrow{F^{\mathbf{x}. \text{.lind}}} & \uparrow \text{const} \\ \mathbf{E} & \xrightarrow{F} & \mathbf{E}' \end{array} & = & \begin{array}{ccc} & \xrightarrow{G^{\mathbf{x}. \text{.lind}}} & \\ \mathbf{E}^{\mathbf{x}. \text{.lind}} & \xrightarrow{=} & \mathbf{E}'^{\mathbf{x}. \text{.lind}} \\ \text{const} \uparrow & \xrightarrow{G} & \uparrow \text{const} \\ \mathbf{E} & \xrightarrow{F} & \mathbf{E}' \end{array} \end{array}$$

since the top-right composite of (B.2.20.2) is a component of the mate of the left-hand side square above, while the bottom-left composite is a component of the mate of the right-hand side square above. $\square$

Using the above 2-functorial construction, we obtain a 2-functorial alternative of [CL19, Definition 4.8] taking care of cartesianess of transformations, which enhances Leinster's $T^\sharp$ in [Lei04, Proposition F.1.1]. Before stating it, let us recall some basic facts and notations on 2-categories and formal monad theory.

Fact B.2.21. By $\mathcal{K}$, we mean a (possibly huge) 2-category. As noted in Proposition B.2.20, we write $\mathbf{mnd}_o(\mathcal{K})$ for the 2-category of monads, oplax monad morphisms, and modifications. We also write $\mathbf{mnd}_s(\mathcal{K})$ for the sub 2-category of $\mathbf{mnd}_o(\mathcal{K})$ spanned by strong monad morphisms.

- (i) The assignment $\mathcal{K} \mapsto \mathbf{mnd}_o(\mathcal{K})$ is 2-functorial in an obvious way.

- (ii) A *distributive law* in $\mathcal{K}$ is precisely a monad in $\mathbf{mnd}_o(\mathcal{K})$, and taking *composition* of distributive laws assembles into a 2-functor of the type

$$\mathbf{mnd}_o(\mathbf{mnd}_o(\mathcal{K})) \longrightarrow \mathbf{mnd}_o(\mathcal{K})$$

in a natural way. $\lrcorner$

Now we are ready to define the sharp construction in a 2-functorial way.

Definition B.2.22 (cf. [Lei04, Proposition F.1.1]). Applying the 2-functorial assignment B.2.21.(i) to the top and mid horizontal 2-functors in (B.2.20.1) and post-compose them with the 2-functor B.2.21.(ii) for $\mathcal{K} = \infty\text{-LextCAT}$ and $\mathcal{K} = \infty\text{-LextCAT}$, we obtain an endo-2-functor

$$(\mathbf{E}, T) \mapsto (\mathbf{E}\text{-}\mathbf{Gph}, T^\sharp): \mathbf{mnd}_o(\infty\text{-LextCAT}) \longrightarrow \mathbf{mnd}_o(\infty\text{-LextCAT})$$

that restricts to an endo-2-functor on $\mathbf{mnd}_o(\infty\text{-LextCAT})$. We call them the *sharp construction*. $\lrcorner$

Remark B.2.23. Considering Remark B.2.18, the endo-2-functors in Definition B.2.22 give rise to another endo-2-functor on the locally full sub-2-category of $\mathbf{mnd}_s(\infty\text{-LextCAT})$ spanned by monads in $\infty\text{-LextCAT}$ and strong monad morphisms whose underlying functors preserve finite limits. $\lrcorner$

Remark B.2.24. Unpacking Definition B.2.22, we obtain the following explicit description of the sharp construction. See also [Lei04, Appendix F].

Let $\mathbf{E}$ be a infinitary lextensive category and T be a monad on $\mathbf{E}$ in $\infty\text{-LextCAT}$. The monad $T^\sharp$ on $\mathbf{E}\text{-}\mathbf{Gph}$ is described as follows:

- (i) $(T^\sharp X)_0 = X_0$
- (ii) For each $a, b \in X_0$,

$$(T^\sharp X)(a, b) = \coprod_{\vec{x} \in X_0^+ \langle a, b \rangle} \prod_{i < \vec{x}. \text{lind}} T(X(\vec{x}[i], \vec{x}[i+1])).$$

In particular, we have

$$(T^\sharp 1)_0 = 1 = \{*\}, \quad (T^\sharp 1)(*, *) = \coprod_{n \in \mathbb{N}} (T1)^n = (T1)^*, \quad (\text{B.2.24.1})$$

where $(-)^*$ is the list object in $\mathbf{E}$. Moreover,

$$(T^\sharp(T^\sharp 1))_0 = 1, \quad T^\sharp(T^\sharp 1)(*, *) = \coprod_{n \in \mathbb{N}} \left(T \left(\prod_{m \in \mathbb{N}} (T1)^m \right) \right)^n \cong \coprod_{n \in \mathbb{N}} \left(\prod_{m \in \mathbb{N}} T((T1)^m) \right)^n.$$

- (iii) Consider the following morphisms in $\mathbf{E}$.

$$\begin{aligned} T^\sharp(T^\sharp 1)(*, *) &\cong \coprod_{n \in \mathbb{N}} \left(\prod_{m \in \mathbb{N}} T((T1)^m) \right)^n \longrightarrow \coprod_{n \in \mathbb{N}} \left(\prod_{m \in \mathbb{N}} (TT1)^m \right)^n \\ &\xrightarrow{\coprod_n (\coprod_m (\mu_1^T)^m)^n} \coprod_{n \in \mathbb{N}} \left(\prod_{m \in \mathbb{N}} (T1)^m \right)^n_c = T1^{**} \longrightarrow T1^* = T^\sharp 1(*, *) \end{aligned}$$

Here, the first morphism is the canonical one induced by the universality of $(-)^m$, and the last morphism is the concatenation of lists. The composite gives the multiplication $\mu_1^{T^\sharp}$ at 1 for the monad $T^\sharp$, while the unit $\eta_1^{T^\sharp}: 1 \longrightarrow T^\sharp 1$ is given by the composite

$$1(*, *) = 1 \xrightarrow{\eta_1^T} T1 \longrightarrow T1^* = T^\sharp 1(*, *)$$

where the latter function is the unit for the free monoid monad. $\lrcorner$

Our construction is different from [CL19, Definition 4.8] in the following points:

- We work in a 2-categorical level, which means the construction is 2-functorial.
- We are taking care of cartesianness of transformations, while [CL19] does not.
- Objects of our 2-categories are infinitary lextensive categories, while [CL19] works with infinitary distributive categories.
- Morphisms in the category of monads in [CL19] are lax monad morphisms with underlying functors preserving finite products and small coproducts, while we work with oplax monad morphisms whose underlying functors preserves pullbacks and small coproducts.

However, since those two constructions are identical on objects up to isomorphism, we can safely use [CL19, Theorem 4.15] to some extent, and obtain the following theorem. Before stating it, let us recall, for example, from [Lei04, Section F.2], that there is an equivalence $\omega\text{-}\mathbf{Gph} \xrightarrow{\sim} \mathbf{GSet}$ from the category of ω -graphs (i.e., iterated enriched graphs) to the category of globular set (see also Remark B.3.5). This in particular induces an equivalence $\mathbf{GSet}\text{-}\mathbf{Gph} \xrightarrow{\sim} \mathbf{GSet}$, which in turn coincides with that we will observe in Proposition B.3.1 in more general setting (see Remark B.3.4).

Theorem B.2.25 (cf. [CL19, Theorem 4.15]). Let $\mathbf{T}$ be the free strict ω -category monad on $\mathbf{GSet}$. Then there is an equivalence $(\mathbf{GSet}\text{-}\mathbf{Gph}, \mathbf{T}^\sharp) \xrightarrow{\sim} (\mathbf{GSet}, \mathbf{T})$ of monads whose underlying functor $\mathbf{GSet}\text{-}\mathbf{Gph} \xrightarrow{\sim} \mathbf{GSet}$ is the above equivalence. $\lrcorner$

The following follows from several known results like [Lei04, Proposition F.1.1] and [CL19, Lemma 4.11], but we also give a proof from our perspective.

Corollary B.2.26 (cf. [Lei04, Proposition F.1.1], [CL19, Lemma 4.11]). Suppose we are given an infinitary lextensive category $\mathbf{E}$ equipped with a monad T in $\infty\text{-}\mathcal{LextCAT}$ on it. Then the forgetful functor $(T\text{-}\mathbf{Alg})\text{-}\mathbf{Cat} \rightarrow \mathbf{E}\text{-}\mathbf{Gph}$ is monadic and the induced monad is $T^\sharp$. $\lrcorner$

Proof. Observe that the forgetful functor $T\text{-}\mathbf{Alg} \rightarrow \mathbf{E}$ creates small coproducts and pullbacks, and hence $T\text{-}\mathbf{Alg}$ is infinitary lextensive. Moreover, by the Construction B.2.11 and Proposition B.2.12, the monad $\mathrm{Id}_{T\text{-}\mathbf{Alg}}^\sharp$ on $T\text{-}\mathbf{Alg}\text{-}\mathbf{Gph}$ lifts the monad $\mathrm{Id}_{\mathbf{E}}^\sharp$ on $\mathbf{E}\text{-}\mathbf{Gph}$ along the forgetful functor, which means that the following square commutes and the multiplications and units are compatible.

$$\begin{array}{ccc} (T\text{-}\mathbf{Alg})\text{-}\mathbf{Gph} & \xrightarrow{\mathrm{Id}_{T\text{-}\mathbf{Alg}}^\sharp} & (T\text{-}\mathbf{Alg})\text{-}\mathbf{Gph} \\ \downarrow & & \downarrow \\ \mathbf{E}\text{-}\mathbf{Gph} & \xrightarrow{\mathrm{Id}_{\mathbf{E}}^\sharp} & \mathbf{E}\text{-}\mathbf{Gph} \end{array}$$

On the other hand, an algebra for the monad $T\text{-}\mathbf{Gph}$ on $\mathbf{E}\text{-}\mathbf{Gph}$ is precisely a $T\text{-}\mathbf{Alg}$ -enriched graph. Therefore, it suffices to show that the lax monad morphism $(\mathbf{E}\text{-}\mathbf{Gph}, T\text{-}\mathbf{Gph}) \rightarrow (\mathbf{E}\text{-}\mathbf{Gph}, T\text{-}\mathbf{Gph})$ induced by the above square coincides with $(\mathrm{Id}_{\mathbf{E}}^\sharp, \mathrm{Id}_T^\sharp)$. To this end, let us write $U: T\text{-}\mathbf{Alg} \rightleftarrows \mathbf{E} : F$ for the free-forgetful adjunction. It suffices to show that the transformation $\mathrm{Id}_F^\sharp$ coincides with the mate of the above square. By the Construction B.2.17, this is reduced to showing that for each $X \in \mathbf{E}\text{-}\mathbf{Gph}$ and $\mathbf{x} \in X_0^+$, the following diagram commutes.

$$\begin{array}{ccc} F(\prod_{i < \mathbf{x}. \mathrm{lind}} X(\mathbf{x}[i], \mathbf{x}[i+1])) & \xrightarrow{\quad} & \prod_{i < \mathbf{x}. \mathrm{lind}} F(X(\mathbf{x}[i], \mathbf{x}[i+1])) \\ \downarrow F \prod_{i < \mathbf{x}. \mathrm{lind}} \eta_{X(\mathbf{x}[i], \mathbf{x}[i+1])} & & \uparrow \varepsilon_{\prod_{i < \mathbf{x}. \mathrm{lind}} F X(\mathbf{x}[i], \mathbf{x}[i+1])} \\ F(\prod_{i < \mathbf{x}. \mathrm{lind}} U F(X(\mathbf{x}[i], \mathbf{x}[i+1]))) & \xrightarrow{\cong} & F U(\prod_{i < \mathbf{x}. \mathrm{lind}} F(X(\mathbf{x}[i], \mathbf{x}[i+1]))) \end{array} \quad (\text{B.2.26.1})$$

Here, the top horizontal morphism is a component of the mate of the square on the left below with respect to the adjunctions associated to functors named $\mathbf{const}$ in the diagram. On the other hand, the composite of the path of morphisms is a component of the following natural transformation: First take the mate for the square on the right below with respect to the adjunctions $\mathbf{const} \dashv \prod_{i < \mathbf{x}. \mathrm{lind}}$, and then take the mate for the resulting square with respect to the adjunctions $F \dashv U, F^{\mathbf{x}. \mathrm{lind}} \dashv U^{\mathbf{x}. \mathrm{lind}}$ again.

$$\begin{array}{ccccc} \mathbf{E}^{\mathbf{x}. \mathrm{lind}} & \xrightarrow{F^{\mathbf{x}. \mathrm{lind}}} & T\text{-}\mathbf{Alg}^{\mathbf{x}. \mathrm{lind}} & \xrightarrow{T\text{-}\mathbf{Alg}^{\mathbf{x}. \mathrm{lind}}} & \mathbf{E}^{\mathbf{x}. \mathrm{lind}} \\ \uparrow \mathbf{const} & & \uparrow \mathbf{const} & & \uparrow \mathbf{const} \\ \mathbf{E} & \xrightarrow{F} & T\text{-}\mathbf{Alg} & \xrightarrow{U} & \mathbf{E} \end{array}$$

Now the commutation of (B.2.26.1) follows from the obvious fact that the left hand square above is the mate of the right hand square with respect to the adjunctions $F \dashv U, F^{\mathbf{x}. \mathrm{lind}} \dashv U^{\mathbf{x}. \mathrm{lind}}$. $\square$

B.3. Familial sharp construction

Proposition B.3.1. If $\mathbf{E}$ is a presheaf category, then so is the category $\mathbf{E}\text{-}\mathbf{Gph}$. Moreover, given a shape $\mathbf{C}$ of $\mathbf{E}$, the shape $\mathbf{G}[\mathbf{C}]$ of $\mathbf{E}\text{-}\mathbf{Gph}$ is given by the collage of the constant copresheaf $\Delta(\{\sigma_0, \tau_0\}): \mathbf{C} \rightarrow \mathbf{Set}$, viewed as a profunctor $\{0\} \times \mathbf{C} \rightarrow \mathbf{Set}$: that is, the objects of $\mathbf{G}[\mathbf{C}]$ form the disjoint union $\{0\} \sqcup \{c+1 \mid c \in \mathrm{Obj} \mathbf{C}\}$; ⁴

⁴Here, $c+1$ is the copy of an object $c \in \mathbf{C}$ in $\mathbf{G}[\mathbf{C}]$.

the morphisms are those coming from $\mathbf{C}$, denoted $f + 1$ for each $f \in \mathbf{Mor} \mathbf{C}$; and, for each $c \in \mathbf{C}$, we have $\mathbf{G}[\mathbf{C}](0, c + 1) = \{\sigma_0, \tau_0\}$, subject to the relation $\sigma_0 \circ (f + 1) = \tau_0 \circ (f + 1)$.

The equivalence

$$\mathbf{PshG}[\mathbf{C}] \xrightarrow{\sim} \mathbf{PshC}\text{-}\mathbf{Gph} \quad (\text{B.3.1.1})$$

sends a presheaf $X: \mathbf{G}[\mathbf{C}]^{\text{op}} \rightarrow \mathbf{Set}$ to an $\mathbf{PshC}$ -graph $(X_0, X(-, -))$, where $X(a, b)$ is defined as the pullback

$$\begin{array}{ccc} X(a, b) & \longrightarrow & (- + 1)^{\text{op}} \circ X \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow{(a, b)} & \Delta(X_0 \times X_0) \end{array}$$

in $\mathbf{PshC}$. Here, $(- + 1): \mathbf{C} \rightarrow \mathbf{G}[\mathbf{C}]$ is the inclusion defining the collage $\mathbf{G}[\mathbf{C}]$. $\lrcorner$

Proof. Since $\mathbf{E}$ is an infinitary distributive category, for each set S , there is an equivalence

$$[S \times S, \mathbf{E}] \simeq \mathbf{E} / \coprod_{S \times S} 1 = \mathbf{E} / \Delta(S \times S),$$

which is natural in $S \in \mathbf{Set}$. Here, $\Delta: \mathbf{Set} \rightarrow \mathbf{E}$ is the ‘diagonal’ functor, i.e., the left adjoint to the global sections functor $\mathbf{E}(1, -): \mathbf{E} \rightarrow \mathbf{Set}$. This implies that the category $\mathbf{E}\text{-}\mathbf{Gph}$ is the *Artin glueing* [CJ95, Section 4] of the functor $\Delta \times \Delta: \mathbf{Set} \rightarrow \mathbf{E}$ (defined by $S \mapsto \Delta(S) \times \Delta(S)$), which is the nerve of the profunctor $\{0\} \times \mathbf{C} \rightarrow \mathbf{Set}$ defined above. By [CJ95, Lemma 4.9], the category $\mathbf{E}\text{-}\mathbf{Gph}$ is a presheaf category and its shape is the collage of this profunctor, namely $\mathbf{G}[\mathbf{C}]$ as described. $\square$

Remark B.3.2. Let $F: \mathbf{C} \rightarrow \mathbf{C}'$ be a functor between small categories. Then F induces a functor $\mathbf{G}[F]: \mathbf{G}[\mathbf{C}] \rightarrow \mathbf{G}[\mathbf{C}']$ defined by $0 \mapsto 0$, $c + 1 \mapsto F(c) + 1$ for each $c \in \mathbf{C}$, and $\sigma_c, \tau_c \mapsto \sigma_{F(c)}, \tau_{F(c)}$. In the same way, a natural transformation $\alpha: F \Rightarrow F': \mathbf{C} \rightarrow \mathbf{C}'$ induces a natural transformation $\mathbf{G}[\alpha]: \mathbf{G}[F] \Rightarrow \mathbf{G}[F']$ defined by $0 \mapsto \text{Id}_0$ and $c + 1 \mapsto \alpha_c + 1$ for each $c \in \mathbf{C}$. This assignment defines a 2-functor $\mathbf{G}[-]: \mathbf{Cat} \rightarrow \mathbf{Cat}$, and it is compatible with the 2-functor $\text{--}\mathbf{Gph}: \mathbf{CAT} \rightarrow \mathbf{CAT}$ through the contravariant presheaf construction $\mathbf{Psh}(-): \mathbf{Cat}^{\text{coop}} \rightarrow \mathbf{CAT}$ and the equivalence (B.3.1.1). $\lrcorner$

Proposition B.3.3. The functor $\mathbf{G}[-]: \mathbf{Cat} \rightarrow \mathbf{Cat}$ is a *parametric left 2-adjoint*: i.e., it lifts through the forgetful functor $\mathbf{Cat}_* \rightarrow \mathbf{Cat}$ from the 2-category $\mathbf{Cat}_*$ of pointed categories, and the resulting functor is a left 2-adjoint. $\lrcorner$

Proof. Note that $\mathbf{G}[\emptyset]$ is the terminal category $\{0\}$, and the slice of $\mathbf{G}[-]$ under $\emptyset$ gives a functor $\mathbf{G}[-]_{\emptyset/}: \mathbf{Cat} \rightarrow \mathbf{Cat}_*$. It suffices to show that this functor is a left adjoint. For each pointed category $\mathbf{D} = (\mathbf{D}, d_0)$, we can construct a category $\mathbf{C}$ as the category of elements for the copresheaf $\mathbf{D}(d_0, -)^2: \mathbf{D} \rightarrow \mathbf{Set}$ obtained by powering the representable copresheaf $\mathbf{D}(d_0, -)$ by 2. Let us write $p: \mathbf{C} \rightarrow \mathbf{D}$ for the discrete opfibration. Define the counit $\mathbf{G}[\mathbf{C}] \rightarrow \mathbf{D}$ to be the pointed functor induced from the morphism $\Delta(\{\sigma, \tau\}) \rightarrow \mathbf{D}(d_0, p(-))$ in $[\mathbf{C}, \mathbf{Set}]$ sending σ and τ to a and b respectively, for each $(x, (a, b)) \in \text{Obj}(\mathbf{C}) = \coprod_{x \in \mathbf{D}} \mathbf{D}(d_0, x)^2$. It is routine to verify that these data indeed give the cofree object on $\mathbf{D}$ with respect to the functor $\mathbf{G}[-]_{\emptyset/}: \mathbf{Cat} \rightarrow \mathbf{Cat}_*$. $\square$

Remark B.3.4. Starting from $\mathbf{G}_{-1} = \emptyset$, we can inductively obtain categories $\mathbf{G}_n$ by $\mathbf{G}_{n+1} = \mathbf{G}[\mathbf{G}_n]$, and this coincides with the categories defined in Definition 1.1.1.1. The inclusion $\mathbf{G}_n \hookrightarrow \mathbf{G}_{n+1}$ is the image of the unique functor $!: \mathbf{G}_{-1} \rightarrow \mathbf{G}_0$ under the n -fold iteration of $\mathbf{G}[-]: \mathbf{Cat} \rightarrow \mathbf{Cat}$. Moreover, the category $\mathbf{G}$ of globes is the colimit of the sequence $\mathbf{G}_{-1} \hookrightarrow \mathbf{G}_0 \hookrightarrow \dots \hookrightarrow \mathbf{G}_{n-1} \hookrightarrow \mathbf{G}_n \hookrightarrow \dots$ obtained in this way.

By Proposition B.3.3, the 2-functor $\mathbf{G}[-]$ preserves all connected (conical) colimits. Therefore, this expression of $\mathbf{G}$ equips it with the structure of the initial $\mathbf{G}[-]$ -algebra, by Adamek’s fixed point theorem. In particular, the algebra structure $\mathbf{G}[\mathbf{G}] \cong \mathbf{G}$ is an isomorphism, and through this isomorphism, we identify $\mathbf{G}[\mathbf{G}]$ with $\mathbf{G}$. This identification indeed means that the formal symbol 0 and the formal addition $- + 1$ in $\mathbf{G}[\mathbf{G}]$ are interpreted as the zero and successor of natural numbers underlying $\mathbf{G}$.

Under this identification, the equivalence (B.3.1.1) gives an equivalence $\mathbf{GSet}\text{-}\mathbf{Gph} \simeq \mathbf{GSet}$, and this coincides with the equivalence appearing in Theorem B.2.25. $\lrcorner$

Remark B.3.5. Remark B.3.4 explains how $\mathbf{G}$ is obtained as a colimit of the iterated application of $\mathbf{G}[-]$, and hence we obtain the category $\mathbf{GSet}$ as the limit of the sequence $1 \leftarrow \mathbf{Set} \leftarrow \dots \leftarrow \mathbf{PshG}_{n-1} \leftarrow \mathbf{PshG}_n \leftarrow \dots$. On the other hand, the category $\omega\text{-}\mathbf{Gph}$ of ω -graphs is defined as the limit of an equivalent sequence $1 \leftarrow \mathbf{Set} \leftarrow \dots \leftarrow (n-1)\text{-}\mathbf{Gph} \leftarrow n\text{-}\mathbf{Gph} \leftarrow \dots$ obtained by iterating the enriched graph construction $\text{--}\mathbf{Gph}$ starting from the terminal category 1. However, this does not show the equivalence $\mathbf{GSet} \simeq \omega\text{-}\mathbf{Gph}$ immediately, since we must first check that those limits are *bilimits* in $\mathbf{CAT}$. These can be checked directly by computing the pseudo-limits of those sequences, but we can also verify this by observing that $\mathbf{PshG}_{n-1} \rightarrow \mathbf{PshG}_n$ and $(n-1)\text{-}\mathbf{Gph} \leftarrow n\text{-}\mathbf{Gph}$ are isofibrations for each $n \geq 0$. $\lrcorner$

Fact B.3.6 ([CJ95]). Let A, B be small categories and $F: \mathbf{Psh}A \rightarrow \mathbf{Psh}B$ be a functor between the presheaf categories. Then the following are equivalent:

- F is a *parametric right adjoint*, i.e., the sliced functor $F_{/1}: \mathbf{Psh}A \rightarrow \mathbf{Psh}B/F1$ over the terminal object $1 \in \mathbf{Psh}A$ is a right adjoint.
- There is a functor $\widehat{(-)}: B/F1 \rightarrow \mathbf{Psh}A$ such that the slice $F_{/1}$ is isomorphic to the nerve of $\widehat{(-)}$.

A functor F satisfying these equivalent conditions is called *familial*. By virtue of the second item, F is completely determined by the pair $(F1, \widehat{(-)})$ of a presheaf on B and a functor $\widehat{(-)}: B/F1 \rightarrow \mathbf{Psh}A$ from the category of elements. We call this pair the *familial representation* of F . $\lrcorner$

The above fact implies that familial representations and familial functors are essentially the same notion. This observation can be enhanced to an equivalence of categories.

Definition B.3.7. Let A, B be small categories. We write $\mathcal{Fam}(A, B)$ for the category of familial functors $\mathbf{Psh}A \rightarrow \mathbf{Psh}B$ and natural transformations between them. In other words, $\mathcal{Fam}(A, B)$ is the Grothendieck construction of the functor

$$X \mapsto \mathcal{RA}(\mathbf{Psh}A, \mathbf{Psh}B/X): \mathbf{Psh}B^{\text{op}} \xrightarrow{(\mathbf{Psh}B)/-} \mathcal{RA} \xrightarrow{\mathcal{RA}(\mathbf{Psh}A, -)} \mathcal{CAT},$$

where $\mathcal{RA}$ is the huge 2-category of categories, right adjoints, and natural transformations. $\lrcorner$

From this perspective, cartesian morphisms between familial functors can be characterized as follows. The proof is straightforward from the pasting lemma for pullback squares and the definition of cartesian morphisms in a Grothendieck fibration.

Lemma B.3.8. In the situation of Definition B.3.7, a morphism in $\mathcal{Fam}(A, B)$ is cartesian with respect to the Grothendieck fibration over $\mathbf{Psh}B$ if and only if the corresponding natural transformation is cartesian. $\lrcorner$

We write $\mathcal{Fam}_{\text{cart}}$ for the locally wide sub-2-category of $\mathcal{Fam}$ consisting of cartesian morphisms.

Definition B.3.9. Let A and B be small categories. We define a category $\mathcal{FamR}(A, B)$ as the Grothendieck construction of the functor

$$X \mapsto [B/X, \mathbf{Psh}A]^{\text{op}}: \mathbf{Psh}B^{\text{op}} \xrightarrow{(B/-)^{\text{op}}} \mathbf{Cat}^{\text{op}} \xrightarrow{[-, \mathbf{Psh}A]^{\text{op}}} \mathcal{CAT}$$

We write $\mathcal{FamR}_{\text{cart}}(A, B)$ for the wide subcategory of $\mathcal{FamR}(A, B)$ consisting of cartesian morphisms with respect to the Grothendieck fibration. $\lrcorner$

The following are well-known and obvious from the folklore results on the nerve construction.

Lemma B.3.10. The nerve construction gives an equivalence

$$[B/X, \mathbf{Psh}A]^{\text{op}} \simeq \mathcal{RA}(\mathbf{Psh}A, \mathbf{Psh}B/X)$$

for each $X \in \mathbf{Psh}B$. These equivalences are pseudo-natural in X and hence induce equivalences

$$\mathcal{FamR}(A, B) \simeq \mathcal{Fam}(A, B), \quad \mathcal{FamR}_{\text{cart}}(A, B) \simeq \mathcal{Fam}_{\text{cart}}(A, B)$$

over $\mathbf{Psh}B$. $\lrcorner$

Remark B.3.11. In particular, by Lemma B.3.8, Lemma B.3.10 gives another characterisation of cartesian morphisms between familial functors: a natural transformation $\alpha: F \rightarrow G$ between familial functors $F, G: \mathbf{Psh}A \rightarrow \mathbf{Psh}B$ is cartesian if and only if the induced morphism $B/\alpha_1 \circ \widehat{(-)}_G \Rightarrow \widehat{(-)}_F: B/F1 \rightarrow \mathbf{Psh}A$ is an isomorphism, where $(F1, \widehat{(-)}_F)$ and $(G1, \widehat{(-)}_G)$ are the familial representations of F and G respectively.

This is observed e.g. in [GH18, Proposition 3.8] and [Sha25, Proposition 1.7], and is used in Lemma 1.1.3.1. $\lrcorner$

Remark B.3.12. Let A, B, C be small categories. Through the equivalence observed in Lemma B.3.10, we have a *composition* of familial representations:

$$\mathcal{FamR}(A, B) \times \mathcal{FamR}(B, C) \rightarrow \mathcal{FamR}(A, C)$$

This sends a pair $(F1, \widehat{(-)}_F) \in \mathcal{FamR}(A, B)$, $(G1, \widehat{(-)}_G) \in \mathcal{FamR}(B, C)$ to a familial representation $(H1, \widehat{(-)}_H) \in \mathcal{FamR}(A, C)$ defined as follows.

- $H1$ is the image under the projection $\mathbf{PshC}/G1 \rightarrow \mathbf{PshC}$ of the presheaf $\mathbf{PshB}(\widehat{(-)}_G, F1)$. In other words, the category of elements $C/H1$ fits into the following pullback square:

$$\begin{array}{ccc} C/H1 & \longrightarrow & \mathbf{PshB}/F1 \\ \downarrow & \lrcorner & \downarrow \text{projection} \\ C/G1 & \xrightarrow{(\widehat{-})_G} & \mathbf{PshB} \end{array}$$

- The functor $\widehat{(-)}_H: C/H1 \rightarrow \mathbf{PshA}$ is the composite of the top horizontal arrow in the above pullback square followed by the left Kan extension $\mathbf{PshB}/F1 \rightarrow \mathbf{PshA}$ of $\widehat{(-)}_F: B/F1 \rightarrow \mathbf{PshA}$ along the yoneda embedding $B/F1 \rightarrow \mathbf{PshB}/F1$. For each $(c, \pi, \phi) \in C/H1$ with $\pi \in G1_c$ and $\phi \in \mathbf{PshB}(\widehat{(c, \pi)}_G, F1)$, we have the following expression:

$$(c, \pi, \phi)_H = \operatorname{colim}_{(b, \rho) \in B/(\widehat{c, \pi})_G} (b, \phi_b(\rho))_F$$

On the other hand, the *identity familial representation* on A is given by $(1, \mathfrak{y})$, where 1 is the terminal A -presheaf and $\mathfrak{y}: A/1 = A \rightarrow \mathbf{PshA}$ is the yoneda embedding. $\lrcorner$

Definition B.3.13. Let A be a small category. A *familial monad* on a presheaf category $\mathbf{PshA}$ is a cartesian monad whose underlying endofunctor is familial. $\lrcorner$

Lemma B.3.14. Let A, B be small categories and $F: \mathbf{PshA} \rightarrow \mathbf{PshB}$ be a familial functor whose familial representation is given by $(F1, \widehat{(-)})$. Let I be a small category. Then the following are equivalent:

- F preserves I -colimits.
- For each $\pi \in B/F1$, the representable functor $\mathbf{PshA}(\widehat{\pi}, -): \mathbf{PshA} \rightarrow \mathbf{Set}$ preserves I -colimits.

In particular, the functor F preserves small coproducts if and only if each $\widehat{\pi}$ is *connected*. $\lrcorner$

Proof. Since small colimits are computed pointwise in any presheaf category, the latter condition is equivalent to saying that the nerve functor $F_{/1}: \mathbf{PshA} \rightarrow \mathbf{PshB}/F1$ preserves I -colimits. This is in turn equivalent to the former condition since the slice projection $\mathbf{PshB}/F1 \rightarrow \mathbf{PshB}$ creates all colimits. $\square$

Definition B.3.15. A functor between presheaf categories is *connectedly familial* if it is familial and preserves small coproducts. A *connectedly familial monad* is a cartesian monad whose underlying endofunctor is connectedly familial. We write $\mathbf{CFMonC}$ for the category of connectedly familial monads on $\mathbf{PshC}$ and cartesian (icon-)monad morphisms between them. $\lrcorner$

Definition B.3.16. Let $\mathbf{E}$ be a category. An *n -pointed $\mathbf{E}$ -enriched graph* is an $\mathbf{E}$ -enriched graph X equipped with n elements $x_0, \dots, x_{n-1} \in X_0$. We write $\mathbf{E-Gph}_{n*}$ for the category of n -pointed $\mathbf{E}$ -enriched graphs and $\mathbf{E}$ -graph morphisms preserving points. $\lrcorner$

When $n = 1$ we write $\mathbf{E-Gph}_*$, and when $n = 2$ we write $\mathbf{E-Gph}_{**}$. Observe that the equivalence (B.3.1.1) extends to an equivalence $\mathbf{Psh}_{n*}G[C] \xrightarrow{\sim} \mathbf{PshC-Gph}_{n*}$, when $G[C]$ is pointed by 0 .

Proposition B.3.17. Suppose that $\mathbf{E}$ is a category with an initial object. There is an adjunction

$$\mathbf{E-Gph}_{**} \xrightleftharpoons{\quad} \mathbf{E}$$

such that the right adjoint sends a 2-pointed graph (X, a, b) to the hom $X(a, b)$, which we call the *hom* functor. We write $\underline{\Sigma}: \mathbf{E} \rightarrow \mathbf{E-Gph}_{**}$ for the left adjoint following the forgetful functor $\mathbf{E-Gph}_{**} \rightarrow \mathbf{E-Gph}$, which we call the *suspension* functor.

In particular, when we take $\mathbf{E} = \mathbf{PshC}$ for a small category C , the equivalence (B.3.1.1) induces an adjunction

$$\mathbf{Psh}_{**}G[C] \xrightleftharpoons{\quad} \mathbf{PshC} \tag{B.3.17.1}$$

whose left adjoint following the forgetful functor $\mathbf{Psh}_{**}G[C] \rightarrow \mathbf{PshG}[C]$ is also called the *suspension* functor, and denoted by $\underline{\Sigma}$ as well. $\lrcorner$

Proof. Let $E \in \mathbf{E}$ be an object. Define a $\mathbf{E}$ -graph $\underline{\Sigma}E$ by $(\underline{\Sigma}E)_0 = \{\star, \star'\}$ and

$$\underline{\Sigma}E(x, y) = \begin{cases} E & \text{if } x = \star \text{ and } y = \star' \\ \emptyset & \text{otherwise.} \end{cases}$$

One can easily check that $(\underline{\Sigma}E, (\star, \star'))$ defines a free object on E with respect to the hom-functor $\mathbf{E-Gph}_{**} \rightarrow \mathbf{E}$. $\square$

Remark B.3.18. When $\mathbf{C}$ is a fixed point of $\mathbf{G}[-]$ such as $\mathbf{G}$ observed in Remark B.3.4, then the adjunction (B.3.17.1) induces an adjunction of the form

$$\mathrm{hom}: \mathbf{Psh}_{**}\mathbf{C} \xleftrightarrow{\perp} \mathbf{Psh}\mathbf{C} : \underline{\Sigma}.$$

This gives the usual hom-suspension adjunction for globular sets observed in Remark 1.1.1.9. $\lrcorner$

Definition B.3.19. Let $\mathbf{D} = (\mathbf{D}, 0)$ be a pointed small category. Then we define a category $\mathbf{Psh}_{n*}\mathbf{D}$ of n -pointed presheaves over $\mathbf{D}$ as the under category $\coprod_{i < n} \mathbf{D}(0)/\mathbf{Psh}\mathbf{D}$. In other words, an n -pointed presheaf is a presheaf $X: \mathbf{D}^{\mathrm{op}} \rightarrow \mathbf{Set}$ equipped with n elements $x_0, \dots, x_{n-1} \in X_0$. $\lrcorner$

Lemma B.3.20. Let $\mathbf{E}$ be a category with finite colimits, which is pointed by an object $\bullet \in \mathbf{E}$. We write $\mathbf{E}_{n*}$ for the total category of the n -fold fibred power of the fibration $\bullet/\mathbf{E} \rightarrow \mathbf{E}$ over $\mathbf{E}$. Then there is an adjunction

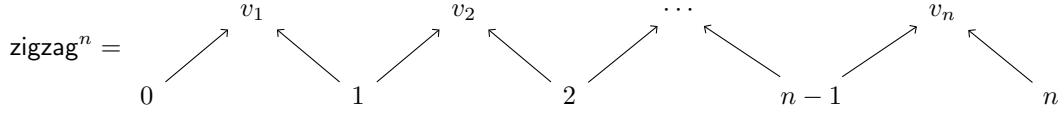
$$\mathbf{E}_{(n+1)*} \xleftrightarrow{\perp} (\mathbf{E}_{2*})^n$$

whose right adjoint sends $(X, \mathbf{x})$ to the finite family $(X, (\mathbf{x}[i], \mathbf{x}[i+1]))_{i < n}$, where we think of $\mathbf{x} \in \mathbf{E}(\bullet, X)^{n+1}$ as a (non-empty) list of length $n+1$.

In particular, when we take $\mathbf{E} = \mathbf{Psh}\mathbf{D}$ for a pointed small category $\mathbf{D} = (\mathbf{D}, d_0)$ and $\bullet = \mathbf{D}(d_0)$, the adjunction is of the following form:

$$\mathbf{Psh}_{n+1*}\mathbf{D} \xleftrightarrow{\perp} (\mathbf{Psh}_{**}\mathbf{D})^n \quad (\text{B.3.20.1})$$

Proof. Let $\mathbf{zigzag}^n$ be the category freely generated from the barycentric subdivision of the Hasse diagram of the poset $[n]$:



On the other hand, let $\mathbf{cocone}^n$ be the shape category for the cocone over the discrete category with $n+1$ objects. In other words, $\mathbf{cocone}^n$ is the category obtained by freely adding the terminal object to the discrete category with $n+1$ objects (denoted by $n+1$). Observe that the following identity transformation

$$\begin{array}{ccc} n+1 & \hookrightarrow & \mathbf{cocone}^n \\ \parallel & \searrow \mathrm{Id} & \uparrow i \\ n+1 & \hookrightarrow & \mathbf{zigzag}^n \end{array}$$

exhibits $n+1$ as the comma category of the upper-right corner. Here, horizontal functors are the inclusions and the functor $i: \mathbf{zigzag}^n \rightarrow \mathbf{cocone}^n$ sends each v_{k+1} ($k < n$) to the vertex of the cocone. Therefore, considering the (pointwise) left Kan extension along i , we obtain an adjunction

$$i^*: [\mathbf{zigzag}^n, \mathbf{E}] \xleftrightarrow{\perp} [\mathbf{cocone}^n, \mathbf{E}] : i_!$$

over $\mathbf{E}^{n+1}$ through the canonical projections. The desired adjunction is the fibre of this adjunction at the constant $(\bullet, \dots, \bullet) \in \mathbf{E}^{n+1}$. $\square$

Construction B.3.21. Given a small category $\mathbf{C}$, consider the following functors.

$$(\mathbf{Psh}\mathbf{C})^n \xrightarrow[\underline{\Sigma}^n]{(\text{B.3.17.1})} (\mathbf{Psh}_{**}\mathbf{G}[\mathbf{C}])^n \xrightarrow{(\text{B.3.20.1})} \mathbf{Psh}_{n+1*}\mathbf{G}[\mathbf{C}] \rightarrow \mathbf{Psh}_{**}\mathbf{G}[\mathbf{C}]$$

This assembles into a functor $\mathbf{Br}: (\mathbf{Psh}\mathbf{C})^* \rightarrow \mathbf{Psh}_{**}\mathbf{G}[\mathbf{C}]$, from the list category $(\mathbf{Psh}\mathbf{C})^* := \coprod_{n \in \mathbb{N}} (\mathbf{Psh}\mathbf{C})^n$.

- The functors marked with (B.3.17.1) and (B.3.20.1) are the left adjoint parts of the adjunctions referred there.
- The last functor forgets points other than the first and last points. $\lrcorner$

Remark B.3.22. Let $\mathbf{C}$ be a category and X be a presheaf on $\mathbf{C}$. For each $n \in \mathbb{N}$, the universality of n -th power induces a functor $\mathbf{C}/X^n \rightarrow (\mathbf{C}/X)^n$, which assembles into a functor

$$\mathbf{C}/X^* \rightarrow (\mathbf{C}/X)^*. \quad (\text{B.3.22.1})$$

Here, we write $(-)^*$ for the list objects in $\mathbf{Psh}\mathbf{C}$ and $\mathbf{Cat}$ (see also Construction B.3.28). $\lrcorner$

Definition B.3.23. Let $\mathbf{C}$ be a small category. The functor $+1: \mathbf{C} \rightarrow \mathbf{G}[\mathbf{C}]$ induces a functor $(+1)^* := (+1)^{\text{op}} \circ -: \mathbf{PshG}[\mathbf{C}] \rightarrow \mathbf{PshC}$. Considering the unique family of morphisms $\Delta(\{\sigma, \tau\}) \rightarrow \mathbf{Set}(\emptyset, \mathbf{C}(c, -))$ in $[\mathbf{C}, \mathbf{Set}]$ that are natural in $c \in \mathbf{C}^{\text{op}}$, the universality of $\mathbf{G}[\mathbf{C}]$ induces a profunctor $\mathbf{C}^{\text{op}} \times \mathbf{G}[\mathbf{C}] \rightarrow \mathbf{Set}$. We write $\text{shift}: \mathbf{PshC} \rightarrow \mathbf{PshG}[\mathbf{C}]$ for the nerve of this profunctor, which gives a section of $(+1)^*$. This sends a presheaf X to the presheaf $\text{shift}(X)$ defined by

- $\text{shift}(X)_{c+1} = X_c$, and
- $\text{shift}(X)_0 = 1$.

The following lemma is obvious from the above definition of shift .

Lemma B.3.24. For each presheaf X , the category of elements $\mathbf{G}[\mathbf{C}]/\text{shift}(X)$ is isomorphic to $\mathbf{G}[\mathbf{C}/X]$. In particular, $\mathbf{G}[-]$ preserves discrete fibrations.

By this lemma, the universality of $\mathbf{G}[\mathbf{C}/X]$ as a collage immediately gives the following isomorphism of categories for each pointed category $\mathbf{D} = (\mathbf{D}, d_0)$ and each presheaf X over $\mathbf{C}$:

$$+1 \circ -: [(\mathbf{G}[\mathbf{C}]/\text{shift}(X), 0), (\mathbf{PshD}, \mathfrak{z}(d_0))] \cong [\mathbf{C}/X, \mathbf{Psh}_{**}\mathbf{D}] \quad (\text{B.3.24.1})$$

Here the left-hand side is the subcategory of the functor category $[\mathbf{G}[\mathbf{C}]/\text{shift}(X), \mathbf{PshD}]$ consisting of functors and natural transformations respecting the distinguished points.

Construction B.3.25. Let $\mathbf{C}$ be a small category and T be a monad on $\mathbf{PshC}$ whose underlying endofunctor is connectedly familial and whose familial representation is given by $(T1, \widehat{-})$. Then, by Definition B.2.22, the sharp construction gives a monad $T^\sharp$ on $\mathbf{PshG}[\mathbf{C}]$.

- Applying Proposition B.3.1 to Remark B.2.24, we obtain a canonical isomorphism $T^\sharp 1 \cong \text{shift}(T1^*)$.
- We define a functor $\widehat{(-)}: \mathbf{G}[\mathbf{C}]/T^\sharp 1 \rightarrow \mathbf{PshG}[\mathbf{C}]$ as the transpose through (B.3.24.1) of the composite

$$\mathbf{C}/T1^* \xrightarrow{(\text{B.3.22.1})} (\mathbf{C}/T1)^* \xrightarrow{\widehat{(-)}^*} (\mathbf{PshC})^* \xrightarrow{\text{Br}} \mathbf{Psh}_{**}\mathbf{G}[\mathbf{C}]$$

More explicitly, for each $c \in \mathbf{C}$, $n \in \mathbb{N}$, and an element $\vec{\pi} = (\pi_{i+1})_{i < n} \in (T1)_c^n \subset (T^\sharp 1)_{c+1}$, the object $\widehat{(c+1, \vec{\pi})}$ is defined as the colimit of the following zigzag diagram in $\mathbf{PshG}[\mathbf{C}]$:

$$\begin{array}{ccccccc} & \widehat{\Sigma(c, \pi_1)} & & \widehat{\Sigma(c, \pi_2)} & & \cdots & & \widehat{\Sigma(c, \pi_n)} \\ & \swarrow^* & \searrow_{*'} & \swarrow^* & \searrow_{*'} & \swarrow^* & \searrow_{*'} & \swarrow^* & \searrow_{*'} \\ \mathfrak{z}(0) & & \mathfrak{z}(0) & & \mathfrak{z}(0) & & \cdots & & \mathfrak{z}(0) \end{array} \quad (\text{B.3.25.1})$$

We write $\vec{x}$ for the $n+1$ points of $\widehat{(c+1, \vec{\pi})}$ induced from the above zigzag diagram. If $n = 0$, we define $\widehat{(c+1, *)}$ to be $\mathfrak{z}(0)$, while for the unique element $* \in (T^\sharp 1)_0$, we set $\widehat{(0, *)} = \mathfrak{z}(0)$ as well.

Proposition B.3.26. Let $\mathbf{E}$ be a presheaf category and T be a monad on $\mathbf{E}$ that preserves small coproducts. If the underlying endofunctor of T is familial, then so is $T^\sharp$. When we fix a shape $\mathbf{C}$ of $\mathbf{E}$, the familial representation $(T^\sharp 1, \widehat{-})$ is given by Construction B.3.25.

Proof. By Definition B.2.22, the monad $T^\sharp$ is cartesian. Therefore, it suffices to check that the endofunctor $T^\sharp$ satisfies the conditions in Fact B.3.6. One can easily check that $T^\sharp$ preserves all wide pullbacks as shown in [Lei04, Proposition F.1.1], but we will prove this more directly by showing that the data $(T^\sharp 1, \widehat{-})$ indeed gives the familial representation of $T^\sharp$.

Let us first observe that, in general, given an $n+1$ -pointed presheaf $(D, \mathbf{d})$ over a pointed category $\mathbf{D}$, there is a family of isomorphisms

$$\coprod_{\mathbf{x} \in X_0^{n+1}} \mathbf{Psh}_{n+1*}\mathbf{D}((D, \mathbf{d}), (X, \mathbf{x})) \cong \mathbf{PshD}(D, X) \quad (\text{B.3.26.1})$$

natural in $X \in \mathbf{PshD}$.⁵

⁵This follows from the more general fact that any projection from an under category is a left *multi-adjoint*. See, for example, [Osm21, Section 1].

We obviously have a bijection $(T^\sharp X)_0 \cong \mathbf{PshG}[C](\widehat{(0, *)}, X)$ natural in $X \in \mathbf{PshG}[C]$, while there are the following bijections over $T^\sharp 1_{c+1}$ natural in $X \in \mathbf{PshG}[C]$, and these conclude the proof. See Remark B.2.24 for the fact that the first set gives the value of $T^\sharp X$ at $c+1 \in G[C]$.

$$\begin{aligned}
\coprod_{\mathbf{x} \in X_0^+} \coprod_{i < \mathbf{x}.lind} T(X(\mathbf{x}[i], \mathbf{x}[i+1]))_c &\cong \coprod_{\mathbf{x} \in X_0^+} \coprod_{i < \mathbf{x}.lind} \coprod_{\pi \in T1_c} \mathbf{PshC}(\widehat{(c, \pi)}, X(\mathbf{x}[i], \mathbf{x}[i+1])) \\
&\cong \coprod_{n \in \mathbb{N}} \coprod_{\mathbf{x} \in X_0^{n+1}} \coprod_{(\pi_{i+1})_{i < n} \in T1_c^n} \coprod_{i < n} \mathbf{PshC}(\widehat{(c, \pi_{i+1})}, X(\mathbf{x}[i], \mathbf{x}[i+1])) \\
&\cong \coprod_{n \in \mathbb{N}} \coprod_{\mathbf{x} \in X_0^{n+1}} \coprod_{(\pi_{i+1})_{i < n} \in T1_c^n} \coprod_{i < n} \mathbf{Psh}_{**}G[C](\widehat{(\underline{\Sigma}(c, \pi_{i+1}), (\star, \star'))}, (X, (\mathbf{x}[i], \mathbf{x}[i+1]))) \\
&\cong \coprod_{n \in \mathbb{N}} \coprod_{\tilde{\pi} \in T1_c^n} \coprod_{\mathbf{x} \in X_0^{n+1}} \mathbf{Psh}_{n+1*}G[C](\widehat{((c+1, \tilde{\pi}), \star)}, (X, \mathbf{x})) \\
&\cong \coprod_{\tilde{\pi} \in T^\sharp 1_{c+1}} \mathbf{PshG}[C](\widehat{(c+1, \tilde{\pi})}, X)
\end{aligned}$$

The first bijection is the universal property of the familial representation of T , the second is just the distributivity of the coproduct over the product, the third is the universality of the suspension, the fourth follows from the adjunction (B.3.20.1), and the last is (B.3.26.1). $\square$

Definition B.3.27. Thanks to Proposition B.3.26, for each small category C , the sharp construction restricts to a functor $(-)^{\sharp}: \mathbf{CFMonC} \rightarrow \mathbf{CFMonG}[C]$.

We call this the *familial sharp construction*. $\lrcorner$

In the remainder of this section, we explain how multiplication of $T^\sharp$ is computed in terms of the familial representation $(T1, \widehat{-})$ of T . In particular, we give an explicit description of the familial representation of $T^\sharp T^\sharp$. To this end, we first introduce the following general construction that enhances the list construction.

Construction B.3.28. Let $\mathbf{E}$ be a category with finite limits and countable colimits. We define an endofunctor $(-)^{\otimes}: \mathbf{E}^2 \rightarrow \mathbf{E}^2$ as follows.

- Given such a category $\mathbf{E}$, we have the *list* endofunctor $(-)^{*}_{\mathbf{E}}$ on $\mathbf{E}$ defined as

$$(-)^{*}_{\mathbf{E}} = \coprod_{n \in \mathbb{N}} (-)^n$$

induced from the polynomial $1 \leftarrow \coprod_{n \in \mathbb{N}} n \rightarrow \mathbb{N} \rightarrow 1$ in **Set**.

- For each object $e \in \mathbf{E}$, there is an endofunctor $(-)^{*}_{\mathbf{E}/e}: \mathbf{E}/e \rightarrow \mathbf{E}/e$ since $\mathbf{E}/e$ has finite limits and countable colimits. Moreover, this is lax natural as an endomorphism on the 2-functor $e \mapsto \mathbf{E}/e: \mathbf{E} \rightarrow \mathbf{CAT}$ whose Grothendieck construction is the codomain opfibration. Therefore, we have an endofunctor $(-)^{\bar{*}}$ on $\mathbf{E}^2$ over $\mathbf{E}$.
- Finally, we define $(-)^{\otimes}$ as the composite

$$\mathbf{E}^2 \xrightarrow{(-)^{\bar{*}}} \mathbf{E}^2 \xrightarrow{((-)^{*}_{\mathbf{E}})^2} \mathbf{E}^2.$$

Note that if we have a functor $F: \mathbf{E} \rightarrow \mathbf{E}'$ that preserves finite limits and countable colimits, then we have a natural isomorphism $F \circ (-)^{*}_{\mathbf{E}} \cong (-)^{*}_{\mathbf{E}'} \circ F$. In particular, if the category $\mathbf{E}$ is a presheaf category, $(-)^{\otimes}$ is calculated pointwise, because the evaluation functor $\text{ev}_c: \mathbf{E} \rightarrow \mathbf{Set}$ at each representable object c is continuous and cocontinuous.

We call the object $\text{dom}(f^{\otimes}) \in \mathbf{E}$ the *2-list* of $f \in \mathbf{E}^2$. $\lrcorner$

Remark B.3.29. In the case when $\mathbf{E} = \mathbf{Set}$, the functor $(-)^{\otimes}: \mathbf{Set}^2 \rightarrow \mathbf{Set}^2$ constructed in Construction B.3.28 is described as follows. For each morphism $f: X \rightarrow Y$ in **Set**, the function $f^{\otimes}$ is defined as the obvious projection of the following form forgetting the second component.

$$\{(\mathbf{y}, \mathbf{u}) \in Y^* \times X^{**} \mid \mathbf{y}.length = \mathbf{u}.length, \forall i < \mathbf{y}.length \forall j < \mathbf{u}[i].length, f(\mathbf{u}[i][j]) = \mathbf{y}[i]\} \rightarrow Y^*$$

A pair $(\mathbf{y}, \mathbf{u})$ satisfying the condition in the domain is called a *2-list* of f . In other words, if we consider $f^{\otimes}$ as a graph $(f^{\otimes}, f^{\otimes}): \text{dom}(f^{\otimes}) \rightrightarrows Y^*$, it is precisely the underlying graph of the free strict monoidal category on the graph $(f, f): X \rightrightarrows Y$.⁶ $\lrcorner$

⁶Since a strict monoidal category is precisely a 2-category with a single object, one can rephrase this as the free 2-category construction on a 2-graph with one object and coincident source/target maps. Viewed this way, iterating Construction B.3.28

Lemma B.3.30. For each small category $\mathbf{C}$, observe that the assignment $T \mapsto (T! : TT1 \rightarrow T1)$ extends to a functor $\overline{\text{ev}}_1 : \mathbf{CFMonC} \rightarrow (\mathbf{PshC})^2$. There is an isomorphism filling the following square.

$$\begin{array}{ccc} \mathbf{CFMonC} & \xrightarrow{(-)^\sharp} & \mathbf{CFMonG}[\mathbf{C}] \\ \overline{\text{ev}}_1 \downarrow & \cong & \downarrow \overline{\text{ev}}_1 \\ (\mathbf{PshC})^2 & \xrightarrow{(-)^{\otimes \sharp \text{shift}^2}} & (\mathbf{PshG}[\mathbf{C}])^2 \end{array}$$

⌋

Proof. By Remark B.2.24.(i), it suffices to find a family of isomorphisms

$$\left(\begin{array}{c} TT1 \\ \downarrow T! \\ T1 \end{array} \right)^{\otimes} \xrightarrow{\sim} \left(\begin{array}{c} (T^\sharp T^\sharp 1)_{-+1} \\ \downarrow (T^\sharp!)_{-+1} \\ (T^\sharp 1)_{-+1} \end{array} \right)$$

in $(\mathbf{PshC})^2$ that is natural in $T \in \mathbf{CFMonC}$. Now (B.2.24.1) shows that we already have such isomorphisms at the level of codomains. Let T be a connectedly familial monad on $\mathbf{PshC}$. For each $c \in \mathbf{C}$, we have

$$\begin{aligned} (T^\sharp T^\sharp 1)_{c+1} &\cong \coprod_{n \in \mathbb{N}} \left(\coprod_{m \in \mathbb{N}} T((T1)^m)_c \right)^n && \text{(Remark B.2.24.(ii))} \\ &\cong \coprod_{n \in \mathbb{N}} \left(\coprod_{m \in \mathbb{N}} \coprod_{\pi \in (T1)_c} \mathbf{PshC} \left(\widehat{(c, \pi)}, (T1)^m \right) \right)^n && \text{(familial representation of } T) \\ &\cong \coprod_{n \in \mathbb{N}} \left(\coprod_{m \in \mathbb{N}} \coprod_{\pi \in (T1)_c} \mathbf{PshC} \left(\widehat{(c, \pi)}, T1 \right)^m \right)^n && \text{(universality of finite powers)} \\ &\cong \coprod_{n \in \mathbb{N}} \left(\coprod_{\pi \in (T1)_c} \coprod_{m \in \mathbb{N}} ((T!)_c^{-1}(\pi))^m \right)^n \\ &\cong \coprod_{n \in \mathbb{N}} (\text{dom}((T!)_c^*))^n && \text{(definition of } (-)^* \text{ in Construction B.3.28)} \\ &\cong \text{dom}(T!_c^{\otimes}) \end{aligned}$$

over the canonical isomorphism $(T^\sharp 1)_{c+1} \cong (T1)_c^*$ that is natural in both $c \in \mathbf{C}$ and $T \in \mathbf{CFMonC}$. $\square$

Proposition B.3.31. Through the isomorphism Lemma B.3.30 at a connectedly familial monad T on $\mathbf{PshC}$, we have the following descriptions of the multiplication of $T^\sharp$ and the familial representation of $T^\sharp T^\sharp$. Following Remark B.3.29, we write $(\mathbf{y}, \mathbf{u})$ for an element of $(T^\sharp T^\sharp 1)_{c+1}$, where $\mathbf{y} \in T1^*$ and $\mathbf{u} \in (T1)^{**}$ have the same length.

- $\mu_1^{T^\sharp} : \text{shift}(\text{dom}(T!)^{\otimes}) \rightarrow \text{shift}(T1^*)$ is the image under shift of the following composite:

$$\text{dom}(T!_{T1})^{\otimes} \xrightarrow{\text{dom}(\mu_1^T, !_{T1})^{\otimes}} \text{dom}(!_{T1})^{\otimes} = (T1)^{**} \xrightarrow{\text{concat}} (T1)^*.$$

More explicitly, for each $c \in \mathbf{C}$, the assignment

$$(\mathbf{y}, \mathbf{u}) \mapsto \text{concat}((\mu_1^T)_c^{**}(\mathbf{u}))$$

gives the multiplication $(\mu_1^{T^\sharp})_{c+1}$ of $T^\sharp$ at $c+1 \in \mathbf{G}[\mathbf{C}]$.

- The familial representation $\widehat{(-)}_{T^\sharp T^\sharp} : \mathbf{G}[\mathbf{C}]/(T^\sharp T^\sharp 1) \rightarrow \mathbf{PshG}[\mathbf{C}]$ is obtained as the transpose through (B.3.24.1) of the composite

$$\mathbf{C}/\text{dom}(T!_{T1})^{\otimes} \rightarrow \mathbf{C}/(TT1)^* \xrightarrow{\text{(B.3.22.1)}} (\mathbf{C}/TT1)^* \xrightarrow{\widehat{(-)}_{TT}^*} (\mathbf{PshC})^* \xrightarrow{\text{Br}} \mathbf{Psh}_{**}\mathbf{G}[\mathbf{C}],$$

further yields an endofunctor on $\mathbf{E}^{[n]\text{op}}$. On this construction, we conjecture that (i) this endofunctor is a restriction of the free strict $n+1$ -category construction on $n+1$ -graphs with one object and coincident source/target maps, and (ii) an analogue of Lemma B.3.30 holds for $\overline{\text{ev}}_1 : \mathbf{CFMonC} \rightarrow (\mathbf{PshC})^{[n]\text{op}}$ that sends T to the n -chain $T \dots TT1 \xrightarrow{T \dots T!} T \dots T1 \rightarrow \dots \xrightarrow{T!} T1$.

where the first functor is induced from the composite

$$\mathrm{dom}(T!_{T_1})^{\otimes} \xrightarrow{\mathrm{dom}(\mathrm{Id}, !_{T_1})^{\otimes}} \mathrm{dom}(!_{TT_1})^{\otimes} = (TT_1)^{**} \xrightarrow{\mathrm{concat}} (TT_1)^*$$

More explicitly, for each $c \in \mathbf{C}$, an object $(\mathbf{y}, \mathbf{u}) \in (T^{\sharp}T^{\sharp}1)_{c+1}$ is sent to the colimit of the following zigzag diagram in $\mathbf{PshG}[\mathbf{C}]$ of length $\sum_{i < m} \mathbf{u}[i].\mathrm{length}$:

$$\begin{array}{ccccccc} \widehat{\Sigma(c+1, \mathbf{u}[0][0])}_{TT} & \xleftarrow{\star'} & \cdots & \xleftarrow{\star'} & \widehat{\Sigma(c+1, \mathbf{u}[0][\mathbf{u}[0].\mathrm{bind}])}_{TT} & \xleftarrow{\star'} & \cdots & \xleftarrow{\star'} & \widehat{\Sigma(c+1, \mathbf{u}[1][0])}_{TT} & \xleftarrow{\star'} & \cdots & \xleftarrow{\star'} & \widehat{\Sigma(c+1, \mathbf{u}[m][\mathbf{u}[m].\mathrm{bind}])}_{TT} \\ \uparrow \star & & \uparrow \star & & \uparrow \star & & \uparrow \star & & \uparrow \star & & \uparrow \star & & \uparrow \star \\ \mathbb{k}(0) & & \mathbb{k}(0) & & \mathbb{k}(0) & & \mathbb{k}(0) & & \mathbb{k}(0) & & \mathbb{k}(0) & & \mathbb{k}(0) \end{array} \quad \lrcorner$$

Proof. The first assertion is just a rephrasing of Remark B.2.24.(iii) via the equivalence (B.3.1.1). For the second assertion, recall from Remark B.3.11 that there is a canonical isomorphism

$$\begin{array}{ccc} \mathbf{C}/TT_1 & \xrightarrow{(-)^{TT}} & \mathbf{PshC} \\ \downarrow \mu_1^T & \cong & \uparrow (-)^T \\ \mathbf{C}/T_1 & & \end{array}$$

since μ^T is cartesian. Since we already know that $\mu^{T^{\sharp}}$ is also cartesian by Definition B.2.22, by the naturality of (B.3.22.1), it suffices to show the following diagram commutes in $\mathbf{PshC}$.

$$\begin{array}{ccccc} \mathrm{dom}(T!_{T_1})^{\otimes} & \xrightarrow{\mathrm{dom}(\mathrm{Id}, !_{T_1})^{\otimes}} & (TT_1)^{**} & \xrightarrow{\mathrm{concat}} & (TT_1)^* \\ \downarrow \mathrm{dom}(\mu_1^T, !_{T_1})^{\otimes} & & \downarrow (\mu_1^T)^{**} & & \downarrow (\mu_1^T)^* \\ \mathrm{dom}(!_{T_1})^{\otimes} & \xlongequal{\quad} & T_1^{**} & \xrightarrow{\mathrm{concat}} & (T_1)^* \end{array}$$

The right-hand square commutes by the naturality of the concatenation map. The left-hand square commutes because we have $(\mu_1^T)^{**} = \mathrm{dom}(\mu_1^T, \mathrm{Id}_1)^{\otimes}$. $\square$

B.4. An inductive description of the free strict ω -category monad

As observed in Definition B.3.27, using Propositions B.3.1 and B.3.26 and Remark B.3.2, Definition B.2.22 gives a functorial assignment that sends an object $(\mathbf{C}, T)$ to $(\mathbf{G}[\mathbf{C}], T^{\sharp})$. Now recall that the equivalence on $(\mathbf{GSet}, T)$ in Theorem B.2.25 has its underlying functor given by Proposition B.3.1 (see Remark B.3.4). This shows that T is a fixed point of Definition B.3.27 over $\mathbf{GSet}$; we are considering the case $\mathbf{C} = \mathbf{G}$ with the identification $\mathbf{G}[\mathbf{G}] \cong \mathbf{G}$ observed in Remark B.3.4.

In the remainder of this section, we show that this fixed-point property of T suffices to characterise and describe it completely. To this end, recall all the notations from Section 1.1.1 about globular sets, and the 2-list $(-)^{\otimes}$ for $\mathbf{Set}$ from Remark B.3.29.

Theorem B.4.1. Let T be any fixed point of the endofunctor $(-)^{\sharp}$ on $\mathbf{CFMonG}$. Through the isomorphism $T \cong T^{\sharp}$, Construction B.3.25 and Proposition B.3.31 give the following description of T .

- (i) $(T!)_n: (TT_1)_n \rightarrow (T_1)_n$ is $(\mathrm{Id}: 1 \rightarrow 1)^{\otimes n}$, which is the n -fold iteration of $(-)^{\otimes}$ applied to the terminal object in $\mathbf{Set}^{\rightarrow}$.
- (ii) The source and target maps of type $(T!)_{n+1} \rightarrow (T!)_n$ in $\mathbf{Set}^{\rightarrow}$ coincide with each other, which we denote by $(\partial_n^{TT_1}, \partial_n^{T_1})$, and it is inductively determined as follows.
 - $(\partial_0^{TT_1}, \partial_0^{T_1}): (T!)_1 \rightarrow (T!)_0$ is the unique one towards the terminal object.
 - $\partial_{n+1}^{T_1}(\mathbf{y}) = (\partial_n^{T_1})^*(\mathbf{y})$ for each list $\mathbf{y}$ of $(T_1)_{n+1}$. Here $(\partial_n^{T_1})^*$ is the image under the functor $(-)^*: \mathbf{Set} \rightarrow \mathbf{Set}$ of $\partial_n^{T_1}$.
 - $(\partial_{n+1}^{TT_1})(\mathbf{y}, \mathbf{u}) = ((\partial_n^{T_1})^*(\mathbf{y}), (\partial_n^{TT_1})^{**}(\mathbf{u}))$ for each 2-list $(\mathbf{y}, \mathbf{u})$ of $(T!)_{n+1}$.
- (iii) The familial representation functor $\widehat{(-)}: \mathbf{G}/T_1 \rightarrow \mathbf{GSet}$ is given as in Construction B.3.25.(ii). More explicitly, $(n+1, \widehat{\pi})$ is the colimit of the diagram (B.3.25.1) for $c = n$ as in the following diagram. and $(0, *)$ is the representable presheaf G^0 .

$$\begin{array}{ccccccc} \widehat{\Sigma(n, \pi_1)} & \xleftarrow{\star'} & \cdots & \xleftarrow{\star'} & \widehat{\Sigma(n, \pi_2)} & \xleftarrow{\star'} & \cdots & \xleftarrow{\star'} & \widehat{\Sigma(n, \pi_n)} \\ \uparrow \star & & \uparrow \star & & \uparrow \star & & \uparrow \star & & \uparrow \star \\ G^0 & & G^0 & & G^0 & & G^0 & & G^0 \end{array}$$

(see also (B.3.25.1) for $c = n$), and $\widehat{(0, *)}$ is the representable globular set G^0 . Note here that $(T1)_{n+1}$ is the same as the set of lists $(T1)_n^*$, and we see the list $\vec{\pi} = (\pi_{i+1})_{i < n}$ of elements of $(T1)_n$ as an element of $(T1)_{n+1}$.

The leftmost $\star$ and the rightmost $\star'$ in the above diagram give the parallel morphisms

$$\widehat{\sigma_{n+1}^0}, \widehat{\tau_{n+1}^0} : \widehat{\partial_0^{T1}(n+1, \vec{\pi})} = \widehat{(0, *)} \longrightarrow \widehat{(n+1, \vec{\pi})},$$

and for each $m \leq n$, the parallel morphisms

$$\widehat{\sigma_{n+1}^{m+1}}, \widehat{\tau_{n+1}^{m+1}} : \widehat{\partial_{m+1}^{T1}(n+1, \vec{\pi})} \longrightarrow \widehat{(n+1, \vec{\pi})}$$

are those induced from the universal property of the above colimit by the morphisms

$$\underline{\Sigma}(\widehat{\sigma_n^m}), \underline{\Sigma}(\widehat{\tau_n^m}) : \underline{\Sigma}(\widehat{\partial_m^{T1}(n, \pi_{i+1})}) \longrightarrow \underline{\Sigma}(\widehat{(n, \pi_{i+1})})$$

for each $i < n$, respectively.

- (iv) $(\eta_1^T)_{n+1}(*_{n+1}) \in T1_{n+1} = T1_n^*$ is the list of length 1 whose unique element is $(\eta_1^T)_n(*_n) \in T1_n$.
- (v) $(\mu_1^T)_{n+1}(\mathbf{y}, \mathbf{u}) \cdot \text{length} = \sum_{i < \mathbf{y} \cdot \text{length}} (\mathbf{u}[i] \cdot \text{length})$, and $(\mu_1^T)_{n+1}(\mathbf{y}, \mathbf{u})[i+j] = (\mu_1^T)_n(\mathbf{u}[i][j])$.

In particular, this shows that T is the unique fixed point of $(-)^{\sharp}$ on $\mathbf{CFMonG}$, which is the free strict ω -category monad $\mathbf{T}$. $\dashv$

Fact 1.1.2.3 can be obtained from the above theorem by removing descriptions concerning $\mathbf{TT1}$ from Theorems B.4.1.(i) and B.4.1.(ii) and Theorem B.4.1.(v). Although Fact 1.1.2.3 lacks some description about the multiplication, as we observed in Section 1.1, it is still enough to characterise $\mathbf{T}$, essentially because of Proposition 1.1.2.17.

However, these descriptions of the multiplication provide an explicit algorithm for computing the multiplication on any 2-pasting scheme. Recall that the procedure in Remark 1.1.3.7 yields a criterion for a pasting scheme to be the valid result of multiplying a given 2-pasting scheme, but it does not provide a way to construct such a pasting scheme (although Example 1.1.3.9 suffices for all the multiplications required in the main body of this thesis).

We conclude this appendix by unpacking these remaining parts, namely the domain part of Theorem B.4.1.(i) and Theorem B.4.1.(v). We do this in terms of 2-pasting schemes, following the notation of Section 1.1.3.

Definition B.4.2. Given two 2-pasting schemes

$$\mathfrak{n} = \mathfrak{n}^{(\mathbf{k})} = ((\mathbf{n}_i)_{i \leq r}, (\underline{\mathbf{n}}_{i+1})_{i < r}) \text{ and } \mathfrak{n}' = \mathfrak{n}'^{(\mathbf{k}')} = ((\mathbf{n}'_j)_{j \leq r'}, (\underline{\mathbf{n}}'_{j+1})_{j < r'})$$

that are not equal to $[^0]$, we define their *concatenation* $\mathfrak{n} \frown \mathfrak{n}' = ((\mathbf{l}_i)_{i \leq r+r'+1}, (\underline{\mathbf{l}}_{i+1})_{i < r+r'+1})$ of shape $\mathbf{k} \frown \mathbf{k}'$ as follows:

$$\mathbf{l}_i = \begin{cases} \mathbf{n}_i & \text{if } i \leq r \\ \mathbf{n}'_{i-r-1} & \text{if } i > r \end{cases}, \quad \underline{\mathbf{l}}_{i+1} = \begin{cases} \underline{\mathbf{n}}_{i+1} & \text{if } i < r \\ [^0] & \text{if } i = r \\ \underline{\mathbf{n}}'_{i-r} & \text{if } i > r. \end{cases}$$

Using the table notation, it is expressed as

$$\mathfrak{n} \frown \mathfrak{n}' = \begin{bmatrix} \mathbf{n}_0 & \mathbf{n}_1 & \dots & \mathbf{n}_r & [^0] & \mathbf{n}'_0 & \mathbf{n}'_1 & \dots & \mathbf{n}'_{r'} \\ \underline{\mathbf{n}}_1 & \underline{\mathbf{n}}_2 & \dots & \underline{\mathbf{n}}_r & \underline{\mathbf{n}}'_1 & \underline{\mathbf{n}}'_2 & \dots & \underline{\mathbf{n}}'_{r'} \end{bmatrix}.$$

Observe that concatenation is associative as a binary operation on 2-pasting schemes other than $[^0]$. $\dashv$

Definition B.4.3. We define a function $\mathbf{br}^2$ such that, together with $\mathbf{br}$, it determines a morphism

$$(\mathbf{br}^2, \mathbf{br}) : \mathbf{shp}^{\otimes} \longrightarrow \mathbf{shp}$$

in $\mathbf{Set}^{\rightarrow}$, as follows. Suppose we are given a 2-list of $\mathbf{shp}$, which is a pair

$$((\mathbf{k}_1, \dots, \mathbf{k}_m), (\vec{\mathfrak{n}}_1, \dots, \vec{\mathfrak{n}}_m))$$

of a list of pasting schemes and a list of lists of 2-pasting schemes such that each component of $\vec{\mathfrak{n}}_{i+1}$ has $\mathbf{k}_{i+1}$ as its shape for each $i < m$. The function $\mathbf{br}^2$ sends this pair to the 2-pasting scheme of shape $\mathbf{br}[\mathbf{k}_1, \dots, \mathbf{k}_m]$ defined as follows:

- Suppose $m = 0$. Then the result is the unique 2-pasting scheme of shape $\mathbf{br}[\mathbf{nil}]$, which is

$$((\mathbf{br}[\mathbf{nil}]), \mathbf{nil}) = \begin{bmatrix} [^0] \end{bmatrix}.$$

- Suppose $m = 1$. Then $\text{br}^2[(\mathbf{k}), ((\mathfrak{n}_1, \dots, \mathfrak{n}_l))]$ is a 2-pasting scheme

$$\left[\text{br}[\mathbf{n}_{1,0}, \dots, \mathbf{n}_{l,0}] \quad \text{br}[\mathbf{n}_{1,1}, \dots, \mathbf{n}_{l,1}] \quad \text{br}[\mathbf{n}_{1,2}, \dots, \mathbf{n}_{l,2}] \quad \dots \quad \text{br}[\mathbf{n}_{1,r}, \dots, \mathbf{n}_{l,r}] \right]$$

equipped with the shape $\text{br}[\mathbf{k}]$, where r is the rank of $\mathbf{k}$ and we write $\mathfrak{n}_{j+1} = ((\mathbf{n}_{j+1,i})_{i \leq r}, (\mathbf{n}_{j+1,i+1})_{i < r})$ for each $j \leq l$. This is indeed a 2-pasting scheme because of the equation (1.1.2.11.1).

- When $m > 1$, its image under br^2 is defined as the concatenation

$$\text{br}^2[(\mathbf{k}_1), (\vec{\mathfrak{n}}_1)] \frown \text{br}^2[(\mathbf{k}_2), (\vec{\mathfrak{n}}_2)] \frown \dots \frown \text{br}^2[(\mathbf{k}_m), (\vec{\mathfrak{n}}_m)].$$

Note here that neither side is ever equal to $[0]$. $\lrcorner$

This function unpacks the domain part of Theorem B.4.1.(i) in terms of pasting schemes and 2-pasting schemes as follows.

Proposition B.4.4. br^2 and the pair (br^2, br) restrict to isomorphisms

$$\text{P}^2\text{S}_{\mathbf{k}}^* \cong \text{P}^2\text{S}_{\text{br}[\mathbf{k}]} \text{ in } \mathbf{Set} \text{ and } \text{shp}_{m-1}^{\otimes} \cong \text{shp}_m \text{ in } \mathbf{Set}^{\rightarrow}$$

respectively, for each pasting scheme $\mathbf{k}$ and each natural number $m \in \mathbb{N}$, where we set shp_{-1} to be the identity function on the empty set. $\lrcorner$

Proof. Since br is injective, so is br^2 . Moreover, given a pasting scheme $\mathbf{k}$ of rank r , br^2 restricts to a function $\text{P}^2\text{S}_{\mathbf{k}}^* \rightarrow \text{P}^2\text{S}_{\text{br}[\mathbf{k}]}^*$, where an element $(\mathbf{n}_1, \dots, \mathbf{n}_r)$ in $\text{P}^2\text{S}_{\mathbf{k}}^*$ is seen as a 2-list of 2-pasting schemes $((\mathbf{k}), ((\mathbf{n}_1, \dots, \mathbf{n}_r)))$.

Now to show the left-hand side, let us take a 2-pasting scheme $\mathfrak{m} = ((\mathbf{m}_i)_{i \leq r}, (\mathbf{m}_{i+1})_{i < r})$ of shape $\text{br}[\mathbf{k}]$, where we set $\mathbf{k} = ((k_i)_{i \leq r}, (k_{i+1})_{i < r})$. By Proposition 1.1.2.8, there exists a unique list $(\mathbf{n}_{i,j+1})_{j < l_i}$ such that $\text{br}[(\mathbf{n}_{i,j+1})_{j < l_i}] = \mathbf{m}_i$ for each $i \leq r$, and a unique list $(\mathbf{n}_{i+1,j+1})_{j < l'_i}$ such that $\text{br}[(\mathbf{n}_{i+1,j+1})_{j < l'_i}] = \mathbf{m}_{i+1}$ for each $i < r$. Then, by the equation (1.1.2.11.1), we conclude that $l_i = l'_{i+1} = l_{i+1}$ for each $i < r$ and that $\partial_{k_{i+1}}^{\text{T}1} \mathbf{n}_{i,j+1} = \mathbf{n}_{i+1,j+1} = \partial_{k_{i+1}}^{\text{T}1} \mathbf{n}_{i+1,j+1}$ for each $j < l_{i+1}$. Hence we have $\mathfrak{m} = \text{br}^2[(\mathbf{k}), ((\mathbf{n}_{1,1}, \dots, \mathbf{n}_{1,l_0}), \dots, (\mathbf{n}_{r,1}, \dots, \mathbf{n}_{r,l_r}))]$, which shows the surjectivity for the left-hand side.

It remains to show the right-hand side. For the base case, $\text{shp}_{-1}^{\otimes}$ is isomorphic to the identity function on the singleton set, which is shp_0 . For the inductive step, first observe that the function $\text{shp}_m^{\otimes}$ is isomorphic to the projection

$$\coprod_{(\mathbf{k}_{j+1})_{j < l} \in \text{PS}_m^*} \prod_{j < l} \text{P}^2\text{S}_{\mathbf{k}_{j+1}}^* \rightarrow \text{PS}_m^*.$$

On the other hand, $\text{P}^2\text{S}_{\text{br}[\mathbf{k}_1, \dots, \mathbf{k}_l]}$ is isomorphic to $\prod_{j < l} \text{P}^2\text{S}_{\text{br}[\mathbf{k}_{j+1}]}$ for any list $(\mathbf{k}_{j+1})_{j < l}$ of pasting schemes, since $[0]$ is the only 2-pasting scheme of dimension 0. Through those isomorphisms, br^2 restricted to $\text{shp}_{m-1}^{\otimes} \rightarrow \text{shp}_m$ is isomorphic to

$$\prod_{(\mathbf{k}_{j+1})_{j < l} \in \text{PS}_m^*} \prod_{j < l} \text{br}^2|_{\text{P}^2\text{S}_{\mathbf{k}_{j+1}}^*} : \prod_{(\mathbf{k}_{j+1})_{j < l} \in \text{PS}_m^*} \prod_{j < l} \text{P}^2\text{S}_{\mathbf{k}_{j+1}}^* \rightarrow \prod_{(\mathbf{k}_{j+1})_{j < l} \in \text{PS}_m^*} \prod_{j < l} \text{P}^2\text{S}_{\text{br}[\mathbf{k}_{j+1}]},$$

which is an isomorphism by the left-hand side. $\square$

In particular, the above proposition and Theorem B.4.1.(i) conclude that the function $\text{T}!_m : (\text{TT}1)_m \rightarrow (\text{T}1)_m$ is isomorphic to the projection $\prod_{\mathbf{k} \in \text{PS}_m} \text{P}^2\text{S}_{\mathbf{k}} \rightarrow \text{PS}_m$. From now on, we identify $(\text{TT}1)_m$ with $\prod_{\mathbf{k} \in \text{PS}_m} \text{P}^2\text{S}_{\mathbf{k}}$ via this isomorphism.

Moreover, in the same way as Corollary 1.1.2.10, we have the following characterisation of P^2S as the set of ‘2-fold Batanin trees’.

Corollary B.4.5. $(\text{shp} : \text{P}^2\text{S} \rightarrow \text{PS}, (\text{br}^2, \text{br}))$ is the initial algebra of the endofunctor $(-)^{\otimes} : \mathbf{Set}^{\rightarrow} \rightarrow \mathbf{Set}^{\rightarrow}$. $\lrcorner$

Now we are ready to rewrite Theorem B.4.1.(v) using br^2 , which gives an algorithm to compute the multiplication of $\mathbf{T}$ in terms of 2-pasting schemes.

Proposition B.4.6. Given a 2-list of 2-pasting schemes $((\mathbf{k}_1, \dots, \mathbf{k}_n), (\vec{\mathfrak{n}}_1, \dots, \vec{\mathfrak{n}}_n))$, we have the following equality.

$$\mu_1^{\text{T}}(\text{br}^2[(\mathbf{k}_1, \dots, \mathbf{k}_n), (\vec{\mathfrak{n}}_1, \dots, \vec{\mathfrak{n}}_n)]) = \text{br}[\mu_1^{\text{T}}(\mathfrak{n}_{1,1}), \dots, \mu_1^{\text{T}}(\mathfrak{n}_{1,l_1}), \dots, \mu_1^{\text{T}}(\mathfrak{n}_{n,1}), \dots, \mu_1^{\text{T}}(\mathfrak{n}_{n,l_n})]$$

Here, we write $\vec{\mathfrak{n}}_{i+1} = (\mathfrak{n}_{i+1,1}, \dots, \mathfrak{n}_{i+1,l_{i+1}})$ for each $i < n$. $\lrcorner$

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Index of Symbols

$\mathcal{K}\text{-AIMor}$	category of algebraically $\mathcal{K}$ -injective morphisms. Definition 1.3.6. 25–32, 63, 65, 67, 98
$\mathcal{K}\text{-AIMor}$	double category of algebraically $\mathcal{K}$ -injective morphisms. Definition 1.3.12. 27–29, 31, 32, 67
$\mathbf{AInj}$	category of algebraically injective objects. 26–28, 30, 80, 84, 85
ar^P	arity morphism for a cartesian monad P over $\mathbf{T}$. Definition 1.2.2.4. See also Notation 1.2.3.1. 15–21, 23, 30, 37–41, 43–46, 54, 55, 58, 72, 73, 96
$\partial_m^{\text{T1},n}/\partial_m^{\text{T1}}$	the source and/or target map of type $(\text{T1})_n \longrightarrow (\text{T1})_m$ for the ω -chain T1 seen as a globular set. 8, 10, 11, 13, 17, 23, 30, 38, 127–129
Br	Construction B.3.21. 95, 96, 123, 124, 126
br	bracket function for pasting schemes. Definition 1.1.2.7. 9, 10, 13, 95, 128, 129
br^2	bracket function for 2-pasting schemes. 128, 129
$\text{Cat } \mathcal{CAT}$	the 2-category of small categories and the huge 2-category of large categories. 3, 28, 111, 117, 120, 121, 125
$\mathbf{Cat}, \mathbf{CAT}$	the category of small categories and the huge category of large categories. 3, 24, 26, 30, 69, 110–114, 119, 121, 123
$\mathcal{C}^n$	the walking n -cell in $\mathbf{Wk}\omega\mathbf{Cat}_s$. Notation 1.2.2.11. 18, 25, 65, 69, 75–78, 81–83, 86, 87, 89, 93, 97, 98, 103, 135, 136
$\mathbf{CFMon } \mathbf{C}$	category of connectedly familial monads on $\mathbf{PshC}$. Definition B.3.15. 122, 125–128
$\frac{C}{F}^n$	the diagram for systems of witnesses. Definition 4.4.3.4. 88, 89, 138
$\mathcal{K}\text{-cof}$	class of $\mathcal{K}$ -cofibrations. Definition 1.3.2. 24, 28, 77, 78, 80, 93
$*_{k,n}^X, *_{k,n}^X$	(diagrammatic) composition resulting an n -cell glued at k -cells in a weak/strict ω -category X . Definition 1.2.3.8.(ii). 7, 20–24, 36–38, 40–47, 49, 52–61, 66, 69–74, 79, 80, 82, 86–89, 91, 99–104
$\mathcal{C}^n$	the walking n -cell in $\mathbf{Str}\omega\mathbf{Cat}_s$. Notation 1.2.2.11. 18, 83, 86, 91–93, 100, 136
$\mathcal{D}^n$	the walking witness for a fragment of the data of a flat equivalence n -cell, occurring in Definition 4.3.4.1, the definition of $\mathcal{F}^n$. 78, 87, 91
$\partial \mathcal{C}^n$	the boundary of $\mathcal{C}^n$. When $n > 0$, it is the walking parallel pair of $(n-1)$ -cells in $\mathbf{Wk}\omega\mathbf{Cat}_s$. Notation 1.2.2.11. 18, 25, 65, 75, 78, 82, 83, 86, 87, 91, 93, 97, 98
$\partial \mathcal{C}^n$	the walking parallel pair of $(n-1)$ -cells in $\mathbf{Str}\omega\mathbf{Cat}_s$. Notation 1.2.2.11. 18, 91–93
δ	the comultiplication of the weak ω -functor classifier comonad $\mathbf{Q}$. 63–65
$\delta_+^i(\mathbf{u}), \delta_-^i(\mathbf{u})$	pasting diagram of shape $\delta^i(\mathbf{k})$ obtained from $\mathbf{u}$ by replacing the i -th peak with its source(+) or target(−). Definition 2.4.11. 42, 43, 46
$\delta^i(\mathbf{k}), \delta^i(\mathbf{u})$	pasting scheme (or diagram) obtained from $\mathbf{k}(\mathbf{u})$ by reducing degeneracy appearing at the i -th peak. Definition 2.4.1. 38–40, 42–46, 135
∂G^n	the boundary of G^n . When $n > 0$, it is the walking parallel pair of $(n-1)$ -cells. Definition 1.1.1.5. 6, 16–18, 25, 30, 37, 38, 47, 63, 82, 86, 97, 137
$\kappa_{\mathcal{F}}$	the $\mathcal{K}_{\mathcal{F}}$ -lifting structure equipping $\mathcal{E}_{\mathcal{F}}^n$ as the algebraically $\mathcal{K}_{\mathcal{F}}$ -injective replacement of $\mathbf{mC}^n$. Definition 4.4.0.2. 80, 84
$\mathcal{E}_{\text{OR}}^{1,m}, e^m, f_p^m, f_q^m$	the diagram defining $\mathcal{E}_{\text{OR}}^1$. Definition 4.4.4.1. 89–91, 138
$\mathcal{E}_{\text{OR}}^1$	the weak version of the Ozornova–Rovelli model of walking equivalence. Definition 4.4.4.1. 89–91, 135, 136

$\frac{E}{E}$	the diagram for systems of witnesses for $\mathcal{E}_{\text{OR}}^1$. (4.4.4.3.1). 90, 91, 138
E_F^n	the strict ω -categorical reflection of $\mathcal{E}_F^n$. 92, 93
$\mathcal{E}_F^n$	the algebraically $\mathcal{K}_F$ -injective replacement of $\mathbf{mC}^n$. Definition 4.4.0.2. 70, 80, 81, 84, 85, 87, 89–93, 135, 136
$\mathcal{E}_F^{n,m}, f^m, e^m$	the presentation of $\mathcal{E}_F^n$ in Section 4.4.1. Definition 4.4.1.1. 81–85, 87, 89, 137
E-Gph	category of E -graphs in the sense of [Wol74]. 23, 85, 95, 96, 110–120, 122
ε	the counit of the weak ω -functor classifier comonad Q . v, 47, 48, 63–65, 67, 68, 95, 98
$\mathcal{F}^n$	the walking witness for the first-step data of a flat equivalence n -cell. Definition 4.3.4.1. 78–84, 86–89, 135, 138
$\mathcal{H}^n$	the walking witness for the first-step data of a half-adjoint equivalence n -cell. Definition A.6.2. 103, 138
$\mathcal{Fam}$	2-category of small categories, parametric right adjoints between their presheaves, and cartesian natural transformations. 121
$\mathcal{FamR}(A, B)$	category of familial representations between small categories A , B . Definition B.3.9. 121
c_n	the fundamental n -cell in $\mathcal{C}^n$. Definition 4.3.2.3. 76, 79–81, 83, 84, 87, 91–93, 136
$\text{fdl}_n(\mathbf{u})/\text{fdl}(\mathbf{u})$	set of full-dimensional labels for a pasting diagram u . Definition 2.4.5. 40, 41, 72, 73
$X^\flat$	marked weak ω -category $(X, \emptyset)$ without marked cells. 76–78, 80, 87
Fm	category of families (the small coproduct completion). 107–110, 112–116
T	free strict ω -category monad. 5, 7–24, 30, 37–41, 44–47, 54–56, 58, 72, 85, 91, 96, 97, 102, 119, 127–129, 135, 137
L	free weak ω -category monad. v, 5, 15–25, 30, 37–41, 43–47, 54–61, 63, 67, 72, 73, 82, 85, 95–98, 102, 138
$[\mathbf{E}, \mathbf{E}]_{\text{cart}}$	subcategory of the functor category $[\mathbf{E}, \mathbf{E}]$ consisting of pullback-preserving functors and cartesian natural transformations. 96, 97
p_g	the underlying globular map of a weak ω -functor p . 66, 67
$\Gamma(X)$	the strict ω -category/globular set of cylinders in X . Definition A.4.2. 68, 99, 100
$\nu(F)$	greatest post-fixed point of a monotone function F . 33, 36, 37, 41, 43, 52, 66, 72–74
G^n	the representable globular set. Definition 1.1.1.4. 6–8, 10, 11, 13, 15–18, 25, 30, 37, 38, 63, 76, 82, 86, 91, 95, 97, 98, 127, 128, 135, 137
G	the shape category for globular sets. 5–7, 10, 11, 13, 98, 120, 123, 127, 128
G_n	the shape category for n -globular sets. 120
GSet	the category of globular sets. v, 5–8, 11, 13–15, 17–21, 25, 30, 63, 66, 67, 76, 82, 84–86, 95–98, 119, 120, 127, 137
$G[\mathbf{C}]$	the shape for PshC -graphs. 119, 120, 122–127
$\text{in}^n: \mathbf{mC}^n \longrightarrow \mathcal{E}^n$	a $\mathcal{K}$ -cofibration towards an $\mathcal{K}$ -injective object fixed in Section 4.3.3, which turns out to define a walking equivalence n -cell. See Remark 4.3.3.1. v, 69, 70, 77, 78, 80, 81, 83, 85, 87, 89
$\mathbf{C}^{n-1} \xrightarrow{i_\star^n} \mathbf{E}_\star^n \xrightarrow{r_\star^n} \mathbf{C}^{n-1}$	a factorisation of the map $\mathbf{C}^n \longrightarrow \mathbf{C}^{n-1}$ picking out $\text{id}_n(c_n)$ by (folk cofibration, folk trivial fibration) in Section 4.5.2. 91, 92, 100
$\text{id}_{k,n}^X/\text{id}_n^X$	identity n -cell on an k -cell in a weak/strict ω -category X . 7, 20–24, 36–47, 49, 52–60, 69–74, 79, 82, 86–89, 91, 93, 99–104, 136
i_F^n	the $\mathcal{K}$ -cofibration $\mathbf{mC}^n \longrightarrow \mathcal{E}_F^n$ obtained as the unit of the algebraically $\mathcal{K}_F$ -injective replacement. Definition 4.4.0.2. 80, 81, 92, 93
$\infty\text{-Lext}\mathcal{CART}$	2-category of infinitary lextensive categories, functors preserving pullbacks and small coproducts, and cartesian natural transformations. 107, 117, 118
$\infty\text{-Lext}\mathcal{CAT}$	2-category of infinitary lextensive categories, functors preserving pullbacks and small coproducts, and natural transformations. 107, 117–119
Inj	category of injective objects. 24, 77, 78, 80

$\mathcal{I}_0$	the class of boundary inclusions $\partial G^n \rightarrow G^n$ for all $n \in \mathbb{N}$. 25, 30, 50, 63, 65, 67, 68, 75, 83, 97, 98, 137
ι^n	the boundary inclusion in GSet , StrωCat_s , and WkωCat_s . Definition 1.1.1.5 and Notation 1.2.2.11. 6, 16–18, 25, 30, 37, 38, 47, 63, 65, 75, 82, 91–93
$\mathcal{I}_o^\times$	the set of boundary inclusions $\partial G^n \rightarrow G^n$ for all $n > 0$. 25, 37, 98
J	the identity-on-objects functor WkωCat_s $\rightarrow$ WkωCat_w . 64, 67
$\bigvee$	join of a set in a poset. 33
κ_X^Q	the $\mathcal{I}$ -lifting structure on the counit ε_X^Q of the comonad Q . Definition 3.3.1.2 and Remark 3.3.1.3. 63, 65, 67, 98
$\bar{\kappa}^P$	symbol for a contraction equipped with a cartesian monad P over $\mathbf{T}$. Definition 1.2.2.5. See also Notation 1.2.3.1. 17, 18, 20, 23, 30, 37–40, 44, 55, 96, 97
$\mathcal{K}_{\mathcal{F}}$	An example of $\mathcal{K}$ in Section 4.3.3 for flat equivalences. Definition 4.3.4.1. 2, 70, 79–81, 84, 85, 91–93, 135, 136
$\mathcal{K}_{\mathcal{H}}$	An example of $\mathcal{K}$ in Section 4.3.3 for half-adjoint equivalences. Definition A.6.2. 70, 103–105
λ^m	the colimiting cocone for $\mathcal{E}_{\mathcal{F}}^{n,\omega}$ at m . Definition 4.4.1.5. 83–85
$\mathbf{x}.\text{length}$	length of a list $\mathbf{x}$. Notation B.2.1. 110, 125, 127, 128
level	level of a pasting scheme. 8–10
$\mathbf{x}.\text{lind}$	the last index of a (positive) list $\mathbf{x}$. Notation B.2.1. 110, 111, 113–119, 127
linv	function assigning to each set S of cells set of cells that are left S -inverses. 73
S^*	list object with alphabet object S . 3, 8, 10, 110, 118, 125–129
$\mathbf{mC}^n$	the walking marked n -cell in mWkωCat_s . Definition 4.3.2.3. 76–84, 87–89, 91–93, 103, 135, 136
$\bigwedge$	meet of a set in a poset. 33
$\mathbf{k}(X)$	the <i>core</i> ω -groupoid of a weak ω -category X . Proposition 2.5.2. 36, 47, 48, 95
$\mathbf{mnd}_w(\mathcal{K})$	2-category of monads and w -morphisms in $\mathcal{K}$, where $w \in \{o, l, s\}$. 117, 118, 124, 126
Mor	class (or set) of morphisms of a category. 51, 52, 66, 120
$\mathbb{N}$	the set of natural numbers. 5–10, 18, 25, 35–37, 47, 48, 51, 52, 54, 57, 59, 61–63, 66, 76, 81, 84, 86, 89, 118, 126, 129
$X^\flat$	marked weak ω -category (X, eX) where marked cells are exactly the equivalences. 76–80, 103, 105
$\ \mathbf{k}\ ^{(n)}$	number of occurrences of n in the upper row of a pasting scheme $\mathbf{k}$. Definition 2.4.5. 41, 43, 46, 73
nil	the empty list. 3, 9, 110, 128
Obj	class (or set) of objects of a category. 3, 119, 120
Ψ	endo-monotone map defining flat invertible cells in a weak ω -category. Definition 4.2.0.3. 72–74, 80
Φ	endo-monotone map defining spherical invertible cells in a weak ω -category X . Remark 2.2.2 and Definition 4.2.0.2. 36, 37, 41, 43, 50, 52, 72, 74, 105
$\mathbf{P}^2\mathbf{S}$	set of 2-pasting schemes. 13, 129
S^+	set of positive-length lists with alphabet S . Notation B.2.1. 110, 111, 113–119, 125
$S^+\langle a, b \rangle$	set of positive-length lists with alphabet S starting with a and ending with b . Notation B.2.1. 110, 111, 113–116, 118
E-Gph_{n*}	category of pointed graphs enriched over C . 122
GSet_{n*}	category of n -pointed globular sets. 6, 7, 86
Post	full sub poset of post-fixed points. 33
Pre	full sub poset of pre-fixed points. 33, 43, 72, 73
$\wp(S)$	power poset of a set S . 36, 43, 51, 52, 66, 72, 73

Psh _{n^*} D	pointed presheaf category of a pointed category D . 122–126
PS	set of pasting schemes. 8–10, 13, 23, 95, 129
Psh C	presheaf category of a category C . 7, 13, 120–127, 135, 136
$\text{pst}^X(\mathbf{u})$, $\text{pst}(\mathbf{u})$	set of S -labelled pastings in X . Definition 2.4.6. 41–45, 72–74
Q	the weak ω -functor classifier comonad. 47, 48, 63–68, 98, 135–137
rank	rank of a pasting scheme. 8
rinv	function assigning to each set S of cells set of cells that are right S -inverses. 73, 74
p_s	the classifying strict ω -functor of a weak ω -functor p . 64–68
Set , SET	the category of sets and the huge category of classes. 3, 5, 7–9, 13, 24–26, 30, 63, 76, 82, 95, 97, 98, 108, 110–112, 119, 120, 122–125, 127–129
shp	the function assigning the shape of a 2-pasting scheme. Definition 1.1.3.3. 13, 128, 129
Σ	unit for suspension-hom adjunction for weak ω -categories. 86, 87
$\underline{\Sigma}$	suspension functor for (marked) weak ω -categories. Definitions 4.4.2.4 and 4.4.2.6. 86, 87, 89–91
σ_m^n , σ^n	source n -cell in the walking m -cell, and that in the walking $(n+1)$ -cell. 5, 6, 8, 10, 11, 14, 15, 18, 76–78, 81, 87, 92, 93, 128
$\underline{\Sigma}$	suspension functor for globular sets and enriched graphs. 7, 8, 10, 95, 96, 122–125, 127, 128
σ_∂^n	source n -cell in the walking parallel pair of $(m-1)$ -cells, and that in the walking parallel pair of n -cells. Lemma 4.5.2.1. 6, 18, 91, 93
$s_m^{X,n}/s_m^X/s_m$	the source map of type $X_n \rightarrow X_m$ for a globular set X . 5–7, 11, 12, 14, 19–22, 24, 36, 38–47, 51, 55–58, 60, 63, 65, 66, 69–73, 82, 88, 97, 99, 104
$\text{spi}(\mathbf{k})$	standard pasting instruction of arity $\mathbf{k}$. Definition 1.2.3.6. 19–22, 37
Str-a-Cat _{s}	the category of strict ω -categories and strict functors. 7, 17, 18, 75, 85, 86, 89–92, 135, 137
S	successor for pasting schemes. 8–10, 13, 23
T-Alg	Eilenberg-Moore category of a monad T . 110
$t_m^{X,n}/t_m^X/t_m$	the target map of type $X_n \rightarrow X_m$ for a globular set X . 5–7, 11, 12, 14, 19–22, 24, 36, 38–47, 51, 55–58, 60, 61, 63, 65, 66, 70–73, 82, 88, 97, 99, 104
τ_m^n , τ^n	target n -cell in the walking m -cell, and that in the walking $(n+1)$ -cell. 5, 6, 8, 10, 11, 14, 15, 18, 87, 128
2	the walking morphism category. 3, 24–27, 29, 30, 32, 62, 63, 68, 125, 126
Wk-a-Cat _{s}	the category of weak a -categories and strict functors, for each $a \in \mathbb{N} \cup \{\omega\}$. 17, 18, 23, 25, 36, 38, 47, 51, 52, 62–68, 75–78, 81–83, 85, 86, 89–93, 98, 99, 102, 135, 137
Wk-ω-Cat _{w}	the category of weak ω -categories and weak functors. 47, 48, 63, 64, 66–68, 137
mWk-a-Cat _{s}	the category of marked weak ω -categories and strict functors. Definition 4.3.2.1. 76–81, 83, 84, 86–89, 93, 99, 100, 102, 103, 137
Wk-ω-Gpd _{s}	the category of weak ω -groupoids and strict functors. 36, 47
Wk-ω-Cat _{w}	the category of weak ω -groupoids and weak functors. 47, 48
$\tilde{e}_n$	$\eta_1^L(*)$, where $*$ $\in 1_n$. Notation 1.2.3.1. 18–20, 22, 23, 38, 40, 44, 46, 54, 56, 57, 59–61, 102
Ξ	the shape category for $\frac{C}{F}^n$ and $\frac{E}{E}n$. Definition 4.4.3.3. 88–90
$x_{\mathcal{H}}, y_{\mathcal{H}}, u_{\mathcal{H}}, v_{\mathcal{H}}, w_{\mathcal{H}}$	the cells in $\mathcal{H}^n$. 103
$x_{\text{OR}}, y_{\text{OR}}, u_{\text{OR}}, v_{\text{OR}}, w_{\text{OR}}$	the cells in $\mathcal{E}_{\text{OR}}^{1,0}$ and $\mathcal{E}_{\text{OR}}^{1,1}$. Definition 4.4.4.1. 89
$x_{\mathcal{F}}, y_{\mathcal{F}}, u_{\mathcal{F}}, v_{\mathcal{F}}, w_{\mathcal{F}}, p_{\mathcal{F}}, q_{\mathcal{F}}$	the cells in $\mathcal{F}^n$. Definition 4.3.4.1. 78, 79, 81, 83, 84, 86–88, 91
よ	Yoneda embedding (covariant). 122–124, 127