

GLOBAL INTUITIONISTIC LOGIC AND ITS SEMANTIC COMPLETENESS

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GI, Global Intuitionistic logic, is an intuitionistic modal predicate logic which was first studied in the form of a sequent calculus in Takeuti-Titani[2]. Later another version of GI was studied in Titani[3]. The goal of this paper is to prove the semantic completeness of Titani's GI with respect to complete Heyting algebras with a unary operation $\Box$ called a "globalization."

We note here that Ono[1] contains completeness theorems for several propositional sequent calculi similar to the propositional part of Titani's GI.

1 Syntax of GI

1.1 Language L of GI

1.1.1 Symbols of L

- (1) Individual constants: $c_0, c_1, c_2, \dots$
- (2) Free variables: $a_0, a_1, a_2, \dots$
- (3) Bound variables: $x_0, x_1, x_2, \dots$
- (4) Predicate constants with n argument places ($n=1, 2, 3, \dots$): $R^n_0, R^n_1, R^n_2, \dots$
- (5) Logical symbols: $\neg, \wedge, \vee, \rightarrow, \forall, \exists, \Box$
- (6) Punctuation symbols: $(,), ,$ (comma)

1.1.2 Well-formed formulas (wffs) of L

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Individual constants and free variables are called “terms.”

- (1) If $t_1, \dots, t_n$ are terms and R^n is a predicate constant with n argument places, then $R^n(t_1, \dots, t_n)$ is a wff.
- (2) If A and B are wffs, so are $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\neg A$, and $\Box A$.
- (3) If $A(t)$ is a wff with a term t and x is a bound variable, then $\forall x A(x)$ and $\exists x A(x)$ are wffs, where $A(x)$ is obtained from $A(t)$ by replacing each occurrence of t in $A(t)$ with x .
- (4) Wffs are obtained only by the above (1)–(3).

As usual, sentences are those wffs with no free variables. In what follows, we will consider only sentences.

1.1.3 $\Box$ -closed sentences of L

- (1) If A is a sentence, then $\Box A$ is a $\Box$ -closed sentence.
- (2) If A and B are $\Box$ -closed sentences, so are $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\neg A$.
- (3) If $A(c)$ is a $\Box$ -closed sentence with an individual constant c , then $\forall x A(x)$ and $\exists x A(x)$ are $\Box$ -closed sentences, where $\forall x A(x)$ and $\exists x A(x)$ are formed as in 1.1.2,(3).
- (4) $\Box$ -closed sentences are obtained only by the above (1)-(3)

1.1.4 Sequents of L

If $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_n$ are sentences, then

$$A_1, A_2, \dots, A_m \Rightarrow B_1, B_2, \dots, B_n \quad (m, n \geq 0)$$

is a sequent of L .

We use Greek capital letters $\Gamma, \Delta, \Pi, \Lambda, \Gamma_0, \Gamma_1, \dots$ to denote finite sequences of sentences separated by commas. We also use $\bar{\Gamma}, \bar{\Delta}, \dots$ to denote finite sequences of $\Box$ -closed sentences separated by commas.

1.2 Formal proofs in GI

The system GI contains axioms and a group of rules of inference, which consists of (1) structural rules and (2) logical rules.

1.2.1 Axioms of GI: any sequents of the form : $A \Rightarrow A$, where A is a sentence.

1.2.2 The structural rules of GI

$$\begin{array}{l} \text{Thinning: } \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} , \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \quad \text{Contraction: } \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} , \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \\ \\ \text{Interchange: } \frac{\Gamma, A, B, \Pi \Rightarrow \Delta}{\Gamma, B, A, \Pi \Rightarrow \Delta} , \quad \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda} \quad \text{Cut: } \frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \end{array}$$

1.2.3 The logical rules of GI

$$\begin{array}{l} \wedge \Rightarrow: \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} , \quad \frac{A, \Gamma \Rightarrow \Delta}{B \wedge A, \Gamma \Rightarrow \Delta} \quad \Rightarrow \wedge: \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \\ \\ \vee \Rightarrow: \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \quad \Rightarrow \vee: \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} , \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, B \vee A} \\ \\ \rightarrow \Rightarrow: \frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Lambda}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \quad \Rightarrow \rightarrow: \frac{A, \Gamma \Rightarrow \bar{\Delta}, B}{\Gamma \Rightarrow \bar{\Delta}, A \rightarrow B} \\ \\ \neg \Rightarrow: \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \quad \Rightarrow \neg: \frac{A, \Gamma \Rightarrow \bar{\Delta}}{\Gamma \Rightarrow \bar{\Delta}, \neg A} \\ \\ \forall \Rightarrow: \frac{A(c), \Gamma \Rightarrow \Delta}{\forall x A(x), \Gamma \Rightarrow \Delta} , \text{ where } c \text{ is an arbitrary individual constant.} \\ \\ \Rightarrow \forall: \frac{\Gamma \Rightarrow \bar{\Delta}, A(c)}{\Gamma \Rightarrow \bar{\Delta}, \forall x A(x)} , \text{ where } c \text{ is an individual constant not occurring in the lower sequent.} \\ \\ \exists \Rightarrow: \frac{A(c), \Gamma \Rightarrow \Delta}{\exists x A(x), \Gamma \Rightarrow \Delta} , \text{ where } c \text{ is an individual constant not occurring in the lower sequent.} \\ \\ \Rightarrow \exists: \frac{\Gamma \Rightarrow \Delta, A(c)}{\Gamma \Rightarrow \Delta, \exists x A(x)} , \text{ where } c \text{ is an arbitrary individual constant.} \end{array}$$

$$\Box \Rightarrow: \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \quad \Rightarrow \Box: \frac{\bar{\Gamma} \Rightarrow \bar{\Delta}, A}{\bar{\Gamma} \Rightarrow \bar{\Delta}, \Box A}$$

When a sequent $\Gamma \Rightarrow \Delta$ is provable in GI, we write $\vdash \Gamma \Rightarrow \Delta$.

1.3 Theorems (i.e., Provable sequents) in GI

- (1) $\Rightarrow \Box A \vee \neg \Box A$
- (2) $\Box A \Rightarrow A$
- (3) $\Box(A \rightarrow B) \Rightarrow (\Box A \rightarrow \Box B)$
- (4) $\Box \neg A \Rightarrow \neg \Box A$
- (5) $\Box(A \wedge B) \Rightarrow (\Box A \wedge \Box B)$
- (6) $(\Box A \wedge \Box B) \Rightarrow \Box(A \wedge B)$
- (7) $\Box A \vee \Box B \Rightarrow \Box(A \vee B)$
- (8) $\bar{A} \Rightarrow \Box \bar{A}$, for any $\Box$ -closed sentence $\bar{A}$
- (9) $\neg \neg \bar{A} \Rightarrow \bar{A}$, for any $\Box$ -closed sentence $\bar{A}$
- (10) $\neg \bar{A} \rightarrow B \Rightarrow \bar{A} \vee B$, for any $\Box$ -closed sentence $\bar{A}$
- (11) $\Rightarrow \bar{A} \vee \neg \bar{A}$, for any $\Box$ -closed sentence $\bar{A}$
- (12) $\Box(A \rightarrow B) \wedge \Box(B \rightarrow C) \Rightarrow \Box(A \rightarrow C)$
- (12) $(\Box A \rightarrow \Box B) \Rightarrow \Box(\Box A \rightarrow B)$
- (13) $\Box(\Box A \rightarrow B) \Rightarrow \Box(\Box A \rightarrow \Box B)$
- (14) $\Box \forall x(A \rightarrow B(x)) \Rightarrow \Box(A \rightarrow \forall x B(x))$
- (15) $\Box \forall x(A(x) \rightarrow B) \Rightarrow \Box(\exists x A(x) \rightarrow B)$
- (16) $\forall x \Box A(x) \Rightarrow \Box \forall x A(x)$.

2 Semantics of GI

We now introduce structures for the language L, which we will call “complete Heyting algebras with a globalization (cHags, for short).”

2.1 cHag interpretations

Let $\mathcal{D}$ be a nonempty set and $L(\mathcal{D})$ be the extended language obtained from L by adding a new individual constant $\bar{d}$ for each member d of $\mathcal{D}$. By a cHag interpretation for $L(\mathcal{D})$, we mean a triple $\langle \mathcal{D}, H, \llbracket \cdot \rrbracket \rangle$ such that :

(1) H is a complete Heyting algebra with a globalization $\Box$:

$$H = \langle H, \wedge, \vee, \rightarrow, \neg, \Box, 0, 1, \bigwedge, \bigvee \rangle,$$

where $\Box$ is a unary operation on H satisfying the following conditions: for each $a, b \in H$ and for each indexed set $\{a_i\}_i \subseteq H$,

$$G1 \quad \Box a \leq a$$

$$G2 \quad (\Box a \rightarrow \Box b) \leq \Box(\Box a \rightarrow b)$$

$$G3 \quad \bigwedge_i \Box a_i \leq \Box \bigwedge_i a_i$$

$$G4 \quad \text{If } \Box a \leq b, \text{ then } \Box a \leq \Box b$$

$$G5 \quad \Box a \vee \neg \Box a = 1.$$

(2) $\llbracket \cdot \rrbracket$ is a map from the constants of $L(\mathcal{D})$ such that

(i) $\llbracket c \rrbracket \in \mathcal{D}$ for each individual constant c of L

(ii) $\llbracket \bar{d} \rrbracket = d \in \mathcal{D}$ for each $d \in \mathcal{D}$

(iii) $\llbracket R^n \rrbracket$ is a function: $\mathcal{D}^n \rightarrow H$ for each predicate constant R^n with n argument places.

(3) The symbol $\llbracket \cdot \rrbracket$ is also used to denote the truth value of a sentence of $L(\mathcal{D})$:

(i) Let R^n be a predicate constant with n argument-places and let $t_1, \dots, t_n$ be individual constants of $L(\mathcal{D})$. Then

$$\llbracket R^n(t_1, \dots, t_n) \rrbracket = \llbracket R^n \rrbracket(\llbracket t_1 \rrbracket, \dots, \llbracket t_n \rrbracket) \in H.$$

(ii) For sentences of $L(\mathcal{D})$ containing logical symbols, their truth values are determined by:

$$\llbracket A \wedge B \rrbracket \cong \llbracket A \rrbracket \wedge \llbracket B \rrbracket$$

$$\llbracket A \vee B \rrbracket \triangleq \llbracket A \rrbracket \vee \llbracket B \rrbracket$$

$$\llbracket A \rightarrow B \rrbracket \triangleq \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$$

$$\llbracket \neg A \rrbracket \triangleq \neg \llbracket A \rrbracket$$

$$\llbracket \forall x A(x) \rrbracket \triangleq \bigwedge_{d \in \mathcal{D}} \llbracket A(\bar{d}) \rrbracket$$

$$\llbracket \exists x A(x) \rrbracket \triangleq \bigvee_{d \in \mathcal{D}} \llbracket A(\bar{d}) \rrbracket$$

$$\llbracket \Box A \rrbracket \triangleq \Box \llbracket A \rrbracket,$$

where $\wedge, \vee, \rightarrow, \neg, \bigwedge, \bigvee$, and $\Box$ in the right-hand side of $\triangleq$ are the operations on H .

Note: When c is an individual constant of L and $\llbracket c \rrbracket = d \in \mathcal{D}$, we have $\llbracket A(c) \rrbracket = \llbracket A(\bar{d}) \rrbracket$.

2.2 Validity

(1) A sentence A of $L(\mathcal{D})$ is valid in a cHag interpretation $\langle \mathcal{D}, H, \llbracket \rrbracket \rangle$, if $\llbracket A \rrbracket = 1$ for every $\llbracket \rrbracket$.

(2) The truth value of a sequent of $L(\mathcal{D})$ is defined as follows:

$$\llbracket A_1, A_2, \dots, A_m \Rightarrow B_1, B_2, \dots, B_n \rrbracket \triangleq \llbracket A_1 \wedge A_2 \wedge \dots \wedge A_m \rightarrow B_1 \vee B_2 \vee \dots \vee B_n \rrbracket$$

$$\llbracket A_1, A_2, \dots, A_m \Rightarrow \rrbracket \triangleq \llbracket \neg(A_1 \wedge A_2 \wedge \dots \wedge A_m) \rrbracket$$

$$\llbracket \Rightarrow B_1, B_2, \dots, B_n \rrbracket \triangleq \llbracket B_1 \vee B_2 \vee \dots \vee B_n \rrbracket.$$

$$\llbracket \Rightarrow \rrbracket \triangleq \llbracket A \wedge \neg A \rrbracket \text{ for any sentence } A.$$

Let $A_1, A_2, \dots, A_m \Rightarrow B_1, B_2, \dots, B_n$ be a sequent of $L(\mathcal{D})$. Then it is valid in a cHag interpretation $\langle \mathcal{D}, H, \llbracket \rrbracket \rangle$, if $\llbracket A_1, A_2, \dots, A_m \Rightarrow B_1, B_2, \dots, B_n \rrbracket = 1$ for every $\llbracket \rrbracket$.

Also, sequent $A_1, A_2, \dots, A_m \Rightarrow B_1, B_2, \dots, B_n$ of L is valid, in symbol, $\models A_1, A_2, \dots, A_m \Rightarrow B_1, B_2, \dots, B_n$, if $A_1, A_2, \dots, A_m \Rightarrow B_1, B_2, \dots, B_n$ is valid in every cHag interpretation.

Now the following two propositions are immediate:

Proposition 2.2.1. Let $H = \langle H, \wedge, \vee, \rightarrow, \neg, \Box, 0, 1, \bigwedge, \bigvee \rangle$ be a cHag.

Then the following hold: for each $a, b \in H$ and each indexed set $\{a_i\}_i \subseteq H$,

- (1) If $a \leq b$, then $\Box a \leq \Box b$
- (2) $\Box a = \Box \Box a$
- (3) $\Box a \wedge \Box b = \Box(\Box a \wedge \Box b)$
- (4) $\Box(a \wedge b) = \Box a \wedge \Box b$
- (5) $\Box a \vee \Box b = \Box(\Box a \vee \Box b)$
- (6) $\Box a \vee \Box b \leq \Box(a \vee b)$
- (7) $\Box a \rightarrow \Box b = \Box(\Box a \rightarrow \Box b)$
- (8) $\Box(a \rightarrow b) \leq (\Box a \rightarrow \Box b)$
- (9) $\neg \Box a = \Box \neg \Box a$
- (10) $\bigwedge_i \Box a_i = \Box \bigwedge_i \Box a_i$
- (11) $\bigvee_i \Box a_i = \Box \bigvee_i \Box a_i$
- (12) $\Box 0 = 0$ and $\Box 1 = 1$.

Proposition 2.2.2. For each cHag interpretation and for each $\Box$ -closed sentence $\bar{A}$ of $L(\Box)$,

- (1) $\Box \llbracket \bar{A} \rrbracket = \llbracket \bar{A} \rrbracket$
- (2) $\llbracket \bar{A} \rrbracket \vee \neg \llbracket \bar{A} \rrbracket = 1$
- (3) If $\llbracket \bar{A} \rrbracket \leq \llbracket B \rrbracket$, then $\llbracket \bar{A} \rrbracket \leq \Box \llbracket B \rrbracket$, where B is a sentence of $L(\Box)$.

Theorem 2.2.3.(The Soundness Theorem for GI) Let $\Gamma \Rightarrow \Delta$ be a sequent of L such that $\vdash \Gamma \Rightarrow \Delta$. Then $\models \Gamma \Rightarrow \Delta$.

Proof: Induction on the length of the proof $\vdash \Gamma \Rightarrow \Delta$.

Theorem 2.2.4.(The Completeness Theorem for GI) Let $\Rightarrow \Gamma$ be a sequent of L such that $\models \Rightarrow \Gamma$. Then $\vdash \Rightarrow \Gamma$.

Proof: We prove that $\not\models \bar{\Gamma}_1 \Rightarrow \bar{\Delta}_1$ implies $\not\vdash \bar{\Gamma}_1 \Rightarrow \bar{\Delta}_1$. Then this shows

as a special case that $\nVdash \Rightarrow \Box A$ implies $\nVdash \Rightarrow \Box A$, where A is the disjunction of all the sentences in Γ . Since $(\vdash \Rightarrow \Box A \text{ iff } \vdash \Rightarrow A)$ and $(\models \Rightarrow \Box A \text{ iff } \models \Rightarrow A)$, we can obtain: $\nVdash \Rightarrow A$ implies $\nVdash \Rightarrow A$, i.e., $\nVdash \Rightarrow \Gamma$ implies $\nVdash \Rightarrow \Gamma$.

We now show in three steps that $\nVdash \bar{P} \Rightarrow \bar{Q}$ implies $\nVdash \bar{P} \Rightarrow \bar{Q}$, where $\bar{P}$ and $\bar{Q}$ are respectively the conjunction of all the sentences in $\bar{\Gamma}_1$ and the disjunction of all the sentences in $\bar{\Delta}_1$. Let $\mathfrak{D}$ be the set of all individual constants of L and $L(\mathfrak{D})$ be the same as L . We sometimes regard $L(\mathfrak{D})$ as the set of sentences of $L(\mathfrak{D})$.

Step 1: The construction of a Ha (Heyting algebra)

Definition 1: Let $A, B \in L(\mathfrak{D})$. Set

- (1) $A \leq B \Leftrightarrow \vdash A, \bar{P}, \neg \bar{Q} \Rightarrow B$
- (2) $A \equiv B \Leftrightarrow (A \leq B \text{ and } B \leq A)$
- (3) $\llbracket A \rrbracket \triangleq \{ B \in L(\mathfrak{D}) : A \equiv B \}$
- (4) $H \triangleq \{ \llbracket A \rrbracket : A \in L(\mathfrak{D}) \}$
- (5) $\llbracket A \rrbracket \leq \llbracket B \rrbracket \Leftrightarrow A \leq B$.

Then the relation $\equiv$ is an equivalence relation on $L(\mathfrak{D})$ and the relation $\leq$ on H is well-defined. The following three lemmas are immediate:

Lemma 2: For each $A, B \in L(\mathfrak{D})$,

- (1) $A \in \llbracket A \rrbracket$
- (2) $A \equiv B \text{ iff } \llbracket A \rrbracket = \llbracket B \rrbracket$
- (3) $A \not\equiv B \text{ iff } \llbracket A \rrbracket \cap \llbracket B \rrbracket = \emptyset$
- (4) $\llbracket B \rrbracket \leq \llbracket A \rightarrow A \rrbracket = \llbracket \bar{P} \rrbracket$
- (5) $\llbracket A \wedge \neg A \rrbracket = \llbracket \bar{Q} \rrbracket \leq \llbracket B \rrbracket$
- (6) $\llbracket \bar{P} \rrbracket \neq \llbracket \bar{Q} \rrbracket$.

Lemma 3: Let $\llbracket A \rrbracket, \llbracket B \rrbracket \in H$. Then the g.l.b of $\llbracket A \rrbracket$ and $\llbracket B \rrbracket$, i.e. $\llbracket A \rrbracket \wedge \llbracket B \rrbracket$ exists and equals $\llbracket A \wedge B \rrbracket$. The l.u.b. of $\llbracket A \rrbracket$ and $\llbracket B \rrbracket$, i.e. $\llbracket A \rrbracket \vee \llbracket B \rrbracket$ exists and equals $\llbracket A \vee B \rrbracket$. The pseudo-complement of $\llbracket A \rrbracket$ relative to $\llbracket B \rrbracket$, i.e. $\llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ exists and equals $\llbracket A \rightarrow B \rrbracket$. Also $0 = \llbracket \bar{Q} \rrbracket = \llbracket A \wedge \neg A \rrbracket$ and $1 = \llbracket \bar{P} \rrbracket = \llbracket A \rightarrow A \rrbracket$ for any sentence A of $L(\mathfrak{D})$. Thus $\langle H, \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle$ is a Ha, where $\neg \llbracket A \rrbracket \triangleq \llbracket A \rrbracket \rightarrow 0$, which means $\neg \llbracket A \rrbracket = \llbracket A \rightarrow A \wedge \neg A \rrbracket = \llbracket \neg A \rrbracket$.

Lemma 4: For each $\forall x A(x), \exists x A(x) \in L(\mathfrak{D})$,

$$\llbracket \forall x A(x) \rrbracket = \bigwedge_{c \in \mathfrak{D}} \llbracket A(c) \rrbracket \quad \text{and} \quad \llbracket \exists x A(x) \rrbracket = \bigvee_{c \in \mathfrak{D}} \llbracket A(c) \rrbracket.$$

Definition 5: Set $\Box \llbracket A \rrbracket \triangleq \llbracket \Box A \rrbracket$ for each $\llbracket A \rrbracket \in H$.

From this definition we can obtain

Lemma 6: For every $A, B, A(c), \bar{A}$ ($\Box$ -closed) in $L(\mathfrak{D})$, the following hold:

- (1) $\Box \llbracket \bar{A} \rrbracket = \llbracket \bar{A} \rrbracket$
- (2) $\Box 1 = 1$ and $\Box 0 = 0$
- (3) $\llbracket \bar{A} \rrbracket \wedge \llbracket \neg \bar{A} \rrbracket = 0$ and $\llbracket \bar{A} \rrbracket \vee \llbracket \neg \bar{A} \rrbracket = 1$
- (4) $G1_H: \Box \llbracket A \rrbracket \leq \llbracket A \rrbracket$

$$G2_H: \Box \llbracket A \rrbracket \rightarrow \Box \llbracket B \rrbracket \leq \Box (\Box \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket)$$

$$G3_H: \bigwedge_{c \in \mathfrak{D}} \Box \llbracket A(c) \rrbracket \leq \Box \bigwedge_{c \in \mathfrak{D}} \llbracket A(c) \rrbracket,$$

$$\text{i.e., } \llbracket \forall x \Box A(x) \rrbracket \leq \llbracket \Box \forall x A(x) \rrbracket$$

$$G4_H: \text{If } \Box \llbracket A \rrbracket \leq \llbracket B \rrbracket, \text{ then } \Box \llbracket A \rrbracket \leq \Box \llbracket B \rrbracket.$$

$$G5_H: \Box \llbracket A \rrbracket \vee \neg \Box \llbracket A \rrbracket = 1.$$

Thus $\langle H, \wedge, \vee, \rightarrow, \neg, \Box, 0, 1 \rangle$ is a Ha with a globalization in the sense that G3 of a cHag holds in the form of $G3_H$.

Step 2: The construction of a new Ha

Definition 7: Let $\Box H \triangleq \{ \llbracket \bar{A} \rrbracket : \bar{A} \text{ is a } \Box\text{-closed sentence of } L(\mathfrak{D}) \}$.

Then $\langle \Box H, \wedge^H, \vee^H, \rightarrow^H, \neg^H, \Box^H, 0^H, 1^H \rangle$, or simply $\Box H$, is a sublattice of H and a Ba (Boolean algebra) since $\Box H$ is a distributive lattice with 0 and 1 and for each $\llbracket \bar{A} \rrbracket \in \Box H$, $\llbracket \bar{A} \rrbracket \wedge \neg \llbracket \bar{A} \rrbracket = 0$ and $\llbracket \bar{A} \rrbracket \vee \neg \llbracket \bar{A} \rrbracket = 1$. It also holds that

$$\bigwedge^{\Box H}_{c \in \mathcal{D}} \llbracket \bar{A}(c) \rrbracket = \bigwedge^{H}_{c \in \mathcal{D}} \llbracket \bar{A}(c) \rrbracket = \llbracket \forall x \bar{A}(x) \rrbracket \text{ and}$$

$$\bigvee^{\Box H}_{c \in \mathcal{D}} \llbracket \bar{A}(c) \rrbracket = \bigvee^{H}_{c \in \mathcal{D}} \llbracket \bar{A}(c) \rrbracket = \llbracket \exists x \bar{A}(x) \rrbracket.$$

Definition 8: Let B be a Ba and let (Q) be a set of infinite joins and meets in B as follows:

$$a_s = \bigvee^{B}_{t \in Ts'} a_{s,t} \quad (s \in S') \text{ and}$$

$$b_s = \bigwedge^{B}_{t \in Ts''} b_{s,t} \quad (s \in S''),$$

where two sets S' and S'' are at most countable.

Lemma 9 (Rasiowa & Sikorski's Theorem): Let B and (Q) be as in Definition 8. Then there exists a maximal filter ∇ of B such that

$$\forall s \in S' \quad (a_s \in \nabla \Rightarrow \exists t \in Ts' \quad a_{s,t} \in \nabla) \text{ and}$$

$$\forall s \in S'' \quad ((\forall t \in Ts'' \quad b_{s,t} \in \nabla) \Rightarrow b_s \in \nabla).$$

Such a filter is called a "Q-filter." From now on, we will use ∇ to denote the Q-filter in $\Box H$, where (Q) is the set of all infinite meets of the form $\bigwedge^{\Box H}_{c \in \mathcal{D}} \llbracket \bar{A}(c) \rrbracket$ and all infinite joins of the form $\bigvee^{\Box H}_{c \in \mathcal{D}} \llbracket \bar{A}(c) \rrbracket$.

Definition 10: For each $\llbracket A \rrbracket, \llbracket B \rrbracket \in H$, set

$$(1) \quad \llbracket A \rrbracket \leq \llbracket B \rrbracket \text{ iff } (\llbracket A \rrbracket \multimap \llbracket B \rrbracket) \in \nabla,$$

$$\text{where } \llbracket A \rrbracket \multimap \llbracket B \rrbracket \triangleq \Box(\llbracket A \rrbracket \rightarrow \llbracket B \rrbracket)$$

$$(2) \quad \llbracket A \rrbracket \sim \llbracket B \rrbracket \text{ iff } (\llbracket A \rrbracket \leq \llbracket B \rrbracket \text{ and } \llbracket B \rrbracket \leq \llbracket A \rrbracket).$$

Then the following lemma is immediate:

Lemma 11: For each $\llbracket A \rrbracket, \llbracket B \rrbracket, \llbracket C \rrbracket \in H$,

$$(1) \quad \llbracket A \rrbracket \sim \llbracket B \rrbracket \text{ iff } (\llbracket A \rrbracket \multimap \llbracket B \rrbracket \wedge \llbracket B \rrbracket \multimap \llbracket A \rrbracket) \in \nabla$$

- (2) $\llbracket A \rrbracket \leq \llbracket A \rrbracket$
- (3) $\llbracket A \rrbracket \leq \llbracket B \rrbracket$ and $\llbracket B \rrbracket \leq \llbracket C \rrbracket$ implies $\llbracket A \rrbracket \leq \llbracket C \rrbracket$
- (4) $\sim$ is an equivalence relation on H .

Definition 12: For each $\llbracket A \rrbracket \in H$, let

$$|\llbracket A \rrbracket| \triangleq \{\llbracket B \rrbracket \in H : \llbracket A \rrbracket \sim \llbracket B \rrbracket\} \text{ and}$$

$$H^* \triangleq H/\sim \triangleq \{|\llbracket A \rrbracket| : \llbracket A \rrbracket \in H\}.$$

Then for each $|\llbracket A \rrbracket|, |\llbracket B \rrbracket| \in H^*$, set

$$|\llbracket A \rrbracket| \lesssim |\llbracket B \rrbracket| \Leftrightarrow \llbracket A \rrbracket \leq \llbracket B \rrbracket.$$

Note that $|\llbracket A \rrbracket| = |\llbracket B \rrbracket|$ iff $\llbracket A \rrbracket \sim \llbracket B \rrbracket$ and that $\lesssim$ is well-defined and is a partial order on H^* . We now list two easy lemmas.

Lemma 13: Let $|\llbracket A \rrbracket|, |\llbracket B \rrbracket| \in H^*$. Then the g.l.b. of $|\llbracket A \rrbracket|$ and $|\llbracket B \rrbracket|$, i.e. $|\llbracket A \rrbracket| \wedge^{H^*} |\llbracket B \rrbracket|$ exists and equals $|\llbracket A \rrbracket \wedge \llbracket B \rrbracket|$. The l.u.b. of $|\llbracket A \rrbracket|$ and $|\llbracket B \rrbracket|$, i.e. $|\llbracket A \rrbracket| \vee^{H^*} |\llbracket B \rrbracket|$ exists and equals $|\llbracket A \rrbracket \vee \llbracket B \rrbracket|$. The pseudo-complement of $|\llbracket A \rrbracket|$ relative to $|\llbracket B \rrbracket|$, i.e. $|\llbracket A \rrbracket| \rightarrow^{H^*} |\llbracket B \rrbracket|$ exists and equals $|\llbracket A \rrbracket \rightarrow \llbracket B \rrbracket|$. Also $0^{H^*} = |0^H|$ and $1^{H^*} = |1^H|$. Thus $\langle H^*, \wedge^{H^*}, \vee^{H^*}, \rightarrow^{H^*}, \neg^{H^*}, 0^{H^*}, 1^{H^*} \rangle$ is a Ha, where $\neg^{H^*} |\llbracket A \rrbracket| \triangleq |\llbracket A \rrbracket| \rightarrow^{H^*} 0^{H^*}$, which means $\neg^{H^*} |\llbracket A \rrbracket| = |\llbracket A \rrbracket| \rightarrow^{H^*} |0^H| = |\llbracket A \rrbracket \rightarrow 0^H| = |\neg \llbracket A \rrbracket|$.

Lemma 14: Let $\bar{A}$ be a $\square$ -closed sentence of $L(\mathfrak{D})$. Then

- (1) $\llbracket \neg \bar{A} \rrbracket \in \nabla$ iff $\llbracket \bar{A} \rrbracket \notin \nabla$
- (2) $|\llbracket \bar{A} \rrbracket| = 1^{H^*}$ iff $\llbracket \bar{A} \rrbracket \in \nabla$
- (3) $|\llbracket \bar{A} \rrbracket| = 0^{H^*}$ iff $\llbracket \bar{A} \rrbracket \notin \nabla$.
- (4) $\llbracket \bar{A} \rrbracket \in \nabla$ or $\neg \llbracket \bar{A} \rrbracket \in \nabla$, but not both.

Lemma 15: For each $\bigwedge^{H_{c \in \mathfrak{D}}} \llbracket A(c) \rrbracket$ and $\bigvee^{H_{c \in \mathfrak{D}}} \llbracket A(c) \rrbracket \in H$,

- (1) $|\bigwedge^{H_{c \in \mathfrak{D}}} \llbracket A(c) \rrbracket| = \bigwedge^{H^*_{c \in \mathfrak{D}}} |\llbracket A(c) \rrbracket|$
- (2) $|\bigvee^{H_{c \in \mathfrak{D}}} \llbracket A(c) \rrbracket| = \bigvee^{H^*_{c \in \mathfrak{D}}} |\llbracket A(c) \rrbracket|$.

Proof: Since $\vdash \forall x A(x) \Rightarrow A(c)$, we have $\vdash \Rightarrow \Box(\forall x A(x) \rightarrow A(c))$. Then $\llbracket \forall x A(x) \rrbracket \sqsupset \llbracket A(c) \rrbracket \in \nabla$, i.e. $|\llbracket \forall x A(x) \rrbracket| \lesssim |\llbracket A(c) \rrbracket|$ for each $c \in \mathcal{D}$.

Now suppose $|\llbracket B \rrbracket| \lesssim |\llbracket A(c) \rrbracket|$, i.e. $\llbracket B \rrbracket \sqsupset \llbracket A(c) \rrbracket \in \nabla$ for each $c \in \mathcal{D}$. Then $\bigwedge^{\Box}{}_{c \in \mathcal{D}} (\llbracket B \rrbracket \sqsupset \llbracket A(c) \rrbracket) \in \nabla$, since ∇ is a Q-filter. Now since $\bigwedge^{\Box}{}_{c \in \mathcal{D}} (\llbracket B \rrbracket \sqsupset \llbracket A(c) \rrbracket) = \bigwedge^H{}_{c \in \mathcal{D}} (\llbracket B \rrbracket \sqsupset \llbracket A(c) \rrbracket)$, we can obtain $\bigwedge^H{}_{c \in \mathcal{D}} (\llbracket B \rrbracket \sqsupset \llbracket A(c) \rrbracket) \in \nabla$, from which we can also obtain $\Box \bigwedge^H{}_{c \in \mathcal{D}} (\llbracket B \rrbracket \rightarrow \llbracket A(c) \rrbracket) \in \nabla$ by $G3_H$. Since $\vdash \Box \forall x (B \rightarrow A(x)) \Rightarrow \Box (B \rightarrow \forall x A(x))$, we obtain $\Box (\llbracket B \rrbracket \rightarrow \bigwedge^H{}_{c \in \mathcal{D}} \llbracket A(c) \rrbracket) \in \nabla$. This means $|\llbracket B \rrbracket| \lesssim |\llbracket \forall x A(x) \rrbracket|$. The proof of (2) is similar.

Definition 16: Set $\Box^{H^*} \llbracket A \rrbracket \triangleq |\Box \llbracket A \rrbracket|$.

Now we can obtain the following three lemmas:

Lemma 17: For each $|\llbracket A \rrbracket|, |\llbracket B \rrbracket|, |\llbracket A(c) \rrbracket| \in H^*$,

$$G1_{H^*}: \Box^{H^*} \llbracket A \rrbracket \lesssim |\llbracket A \rrbracket|$$

$$G2_{H^*}: \Box^{H^*} \llbracket A \rrbracket \rightarrow^{H^*} \Box^{H^*} \llbracket B \rrbracket \lesssim \Box^{H^*} (\Box^{H^*} \llbracket A \rrbracket \rightarrow^{H^*} |\llbracket B \rrbracket|)$$

$$G3_{H^*}: \bigwedge^{H^*}{}_{c \in \mathcal{D}} \Box^{H^*} \llbracket A(c) \rrbracket \lesssim \Box^{H^*} \bigwedge^{H^*}{}_{c \in \mathcal{D}} |\llbracket A(c) \rrbracket|,$$

$$\text{i.e. } |\llbracket \forall x \Box A(x) \rrbracket| \lesssim |\llbracket \Box \forall x A(x) \rrbracket|$$

$$G4_{H^*}: \text{If } \Box^{H^*} \llbracket A \rrbracket \lesssim |\llbracket B \rrbracket|, \text{ then } \Box^{H^*} \llbracket A \rrbracket \lesssim \Box^{H^*} \llbracket B \rrbracket$$

$$G5_{H^*}: \Box^{H^*} \llbracket A \rrbracket \vee^{H^*} \neg^{H^*} \Box^{H^*} \llbracket A \rrbracket = 1^{H^*}.$$

Thus $\langle H^*, \wedge^{H^*}, \vee^{H^*}, \rightarrow^{H^*}, \neg^{H^*}, \Box^{H^*}, 0^{H^*}, 1^{H^*} \rangle$ is a Ha with a globalization in the sense that $G3$ of a cHag holds in the form of $G3_{H^*}$.

Lemma 18: The function $g: H \longrightarrow H^*$ defined by $\llbracket A \rrbracket \longmapsto |\llbracket A \rrbracket|$ is a natural homomorphism from H onto H^* and preserves not only $\Box$ but also infinite meets and joins of the form $\bigwedge^H{}_{c \in \mathcal{D}} \llbracket A(c) \rrbracket$ and $\bigvee^H{}_{c \in \mathcal{D}} \llbracket A(c) \rrbracket$, i.e.

$$g(\Box \llbracket A \rrbracket) = \Box^{H^*} g(\llbracket A \rrbracket)$$

$$g(\bigwedge^H{}_{c \in \mathcal{D}} \llbracket A(c) \rrbracket) = \bigwedge^{H^*}{}_{c \in \mathcal{D}} g(\llbracket A(c) \rrbracket) \text{ and}$$

$$g(\bigvee_{c \in \mathcal{D}} \llbracket A(c) \rrbracket) = \bigvee_{c \in \mathcal{D}} g(\llbracket A(c) \rrbracket).$$

Lemma 19: For each $\llbracket A \rrbracket \in H^*$,

$$\Box^{H^*} \llbracket A \rrbracket = \bigvee^{H^*} \{ \Box^{H^*} \llbracket B \rrbracket \in H^* : \Box^{H^*} \llbracket B \rrbracket \leq \llbracket A \rrbracket \}.$$

Step 3: The construction of a cHag

We now construct a cHag from H^* .

Lemma 20 (Rasiowa & Sikorski's Embedding Lemma): Let H^* be a Ha. Then there exist a cHa H^{**} and an isomorphism from H^* into H^{**} , preserving all infinite meets and joins.

By this lemma, we can obtain a cHa H^{**} from the Ha H^* in Step 2 and an isomorphism $h: H^* \longrightarrow H^{**}$ such that for each indexed set $\{a_i\}_i \subseteq H^*$,

$$h(\bigwedge^{H^*}_i a_i) = \bigwedge^{H^{**}}_i h(a_i) \quad \text{and} \quad h(\bigvee^{H^*}_i a_i) = \bigvee^{H^{**}}_i h(a_i).$$

We denote this cHa $\langle H^{**}, \wedge^{H^{**}}, \vee^{H^{**}}, \rightarrow^{H^{**}}, \neg^{H^{**}}, 0^{H^{**}}, 1^{H^{**}}, \bigwedge^{H^{**}}, \bigvee^{H^{**}} \rangle$ by " H^{**} ."

Definition 21: Define a globalization $\Box^{H^{**}}$ as follows: for each $a \in H^{**}$,

$$\Box^{H^{**}} a = \bigvee^{H^{**}} \{ h(\Box \llbracket A \rrbracket) \in H^{**} : h(\Box \llbracket A \rrbracket) \leq a \},$$

where $\leq$ is the partial order on H^{**} .

Lemma 22: For each $a \in H^{**}$, $\Box^{H^{**}} a = \begin{matrix} 1^{H^{**}} & \text{if } a = 1^{H^{**}}, \\ 0^{H^{**}} & \text{if } a \neq 1^{H^{**}}. \end{matrix}$

Proof: $h(0^{H^*}) = 0^{H^{**}}$ and $h(1^{H^*}) = 1^{H^{**}}$. By Lemma 14, each $\Box \llbracket A \rrbracket \in H^*$ is either 0^{H^*} or 1^{H^*} . So for each $h(\Box \llbracket A \rrbracket) \in H^{**}$,

$$h(\Box \llbracket A \rrbracket) = \begin{matrix} 1^{H^{**}} & \text{if } \Box \llbracket A \rrbracket = 1^{H^*}, \\ 0^{H^{**}} & \text{if } \Box \llbracket A \rrbracket \neq 1^{H^*}. \end{matrix}$$

$$\begin{aligned} \text{Then } \Box^{H^{**}} a &= \bigvee^{H^{**}} \{ h(\Box \llbracket A \rrbracket) \in H^{**} : h(\Box \llbracket A \rrbracket) \leq a \} \\ &= \begin{matrix} 1^{H^{**}} & \text{if } a = 1^{H^{**}}, \\ 0^{H^{**}} & \text{if } a \neq 1^{H^{**}}. \end{matrix} \end{aligned}$$

Lemma 23: For each $a, b \in H^{**}$ and each indexed set $\{a_i\}_i \subseteq H^{**}$,

$$G1_{H^{**}}: \Box^{H^{**}} a \leq a$$

$$G2_{H^{**}}: \Box^{H^{**}} a \rightarrow^{H^{**}} \Box^{H^{**}} b \leq \Box^{H^{**}} (\Box^{H^{**}} a \rightarrow^{H^{**}} b)$$

$$G3_{H^{**}}: \bigwedge^{H^{**}}_i \Box^{H^{**}} a_i \leq \Box^{H^{**}} \bigwedge^{H^{**}}_i a_i$$

$$G4_{H^{**}}: \text{If } \Box^{H^{**}} a \leq b, \text{ then } \Box^{H^{**}} a \leq \Box^{H^{**}} b$$

$$G5_{H^{**}}: \Box^{H^{**}} a \vee^{H^{**}} \neg^{H^{**}} \Box^{H^{**}} a = 1^{H^{**}}.$$

Thus $\langle H^{**}, \wedge^{H^{**}}, \vee^{H^{**}}, \rightarrow^{H^{**}}, \neg^{H^{**}}, \Box^{H^{**}}, 0^{H^{**}}, 1^{H^{**}}, \bigwedge^{H^{**}}, \bigvee^{H^{**}} \rangle$ is a cHag and denoted by “ H^{**} .”

Proof: Using Lemma 22, the proof is straightforward.

Lemma 24: The isomorphism $h: H^* \longrightarrow H^{**}$ preserves $\Box$, i.e.

$$h(\Box^{H^*} |\llbracket A \rrbracket|) = \Box^{H^{**}} h(|\llbracket A \rrbracket|).$$

Proof: $h(\Box^{H^*} |\llbracket A \rrbracket|) = h(\bigvee^{H^*} \{ \Box^{H^*} |\llbracket B \rrbracket| \in H^* : \Box^{H^*} |\llbracket B \rrbracket| \leq |\llbracket A \rrbracket| \})$,

by Lemma 19

$$= \bigvee^{H^{**}} \{ h(\Box^{H^*} |\llbracket B \rrbracket|) \in H^{**} : h(\Box^{H^*} |\llbracket B \rrbracket|) \leq h(|\llbracket A \rrbracket|) \},$$

since h preserves infinite joins

$$= \Box^{H^{**}} h(|\llbracket A \rrbracket|) \text{ by the definition of } \Box^{H^{**}} \text{ in } H^{**}.$$

Therefore the map $h \circ g: H \longrightarrow H^{**}$ is a homomorphism and preserves not only infinite meets and joins but also $\Box$. The definition of a map $\llbracket \cdot \rrbracket^{**}$ in H^{**} goes as follows:

Definition 25:

(1) For the constants of $L(\mathfrak{D})$, set

$$\llbracket c \rrbracket^{**} \triangleq c \in \mathfrak{D} \text{ for each individual constant } c \text{ of } \mathfrak{D}, \text{ and}$$

$$\llbracket R^n \rrbracket^{**}: \mathfrak{D}^n \longrightarrow H^{**} \text{ is defined by : for each } c_{i_1}, \dots, c_{i_n} \in \mathfrak{D},$$

$$\llbracket R^n \rrbracket^{**} (\llbracket c_{i_1} \rrbracket^{**}, \dots, \llbracket c_{i_n} \rrbracket^{**}) \triangleq h \circ g(\llbracket R^n(c_{i_1}, \dots, c_{i_n}) \rrbracket) \in H^{**}.$$

(2) For the sentences of $L(\mathfrak{D})$, set $\llbracket A \rrbracket^{**} \triangleq h \circ g(\llbracket A \rrbracket) \in H^{**}$.

Lemma 26: For the map $\llbracket \cdot \rrbracket^{**}$, we have

$$(1) \llbracket A \wedge B \rrbracket^{**} = \llbracket A \rrbracket^{**} \wedge^{H^{**}} \llbracket B \rrbracket^{**}$$

$$(2) \llbracket A \vee B \rrbracket^{**} = \llbracket A \rrbracket^{**} \vee^{H^{**}} \llbracket B \rrbracket^{**}$$

$$(3) \llbracket A \rightarrow B \rrbracket^{**} = \llbracket A \rrbracket^{**} \rightarrow^{H^{**}} \llbracket B \rrbracket^{**}$$

$$(4) \llbracket \neg A \rrbracket^{**} = \neg^{H^{**}} \llbracket A \rrbracket^{**}$$

$$(5) \llbracket \forall x A(x) \rrbracket^{**} = \bigwedge^{H^{**}}_{c \in \mathcal{D}} \llbracket A(c) \rrbracket^{**}$$

$$(6) \llbracket \exists x A(x) \rrbracket^{**} = \bigvee^{H^{**}}_{c \in \mathcal{D}} \llbracket A(c) \rrbracket^{**}$$

$$(7) \llbracket \Box A \rrbracket^{**} = \Box^{H^{**}} \llbracket A \rrbracket^{**}$$

Proof: For (5) we have $\llbracket \forall x A(x) \rrbracket^{**} = \text{h} \circ \text{g}(\llbracket \forall x A(x) \rrbracket) = \text{h} \circ \text{g}(\bigwedge_{c \in \mathcal{D}} \llbracket A(c) \rrbracket) = \bigwedge^{H^{**}}_{c \in \mathcal{D}} \text{h} \circ \text{g}(\llbracket A(c) \rrbracket) = \bigwedge^{H^{**}}_{c \in \mathcal{D}} \llbracket A(c) \rrbracket^{**}$. The rest are similar.

Therefore $\langle \mathcal{D}, H^{**}, \llbracket \cdot \rrbracket^{**} \rangle$ is a cHag interpretation in which $\llbracket \bar{P} \rrbracket^{**} = \text{h} \circ \text{g}(\llbracket \bar{P} \rrbracket) = 1^{H^{**}}$ and $\llbracket \bar{Q} \rrbracket^{**} = \text{h} \circ \text{g}(\llbracket \bar{Q} \rrbracket) = 0^{H^{**}}$. Thus $\not\models \bar{P} \Rightarrow \bar{Q}$.

This completes the proof of the completeness theorem.

References

- [1] H. Ono, *On some intuitionistic modal logic*, Publications of Research Institute for Mathematical Sciences, Kyoto University 13 (1977), pp. 687—722.
- [2] G. Takeuti and S. Titani, *Globalization of intuitionistic set theory*, Annals of Pure and Applied Logic 33 (1987), pp. 195—211.
- [3] S. Titani, *Completeness of global intuitionistic set theory*, Preprint (1993).