

SOME DEGENERATE UNIPOTENT BLOCKS

NAGOYA UNIVERSITY HYOE MIYACHI (名古屋大学 宮地 兵衛)

1. PRELIMINARY

Definition 1. Let G be a finite group of Lie type. Let A be a unipotent block ideal of G with Φ_e -defect torus T and canonical character λ in $M = Z_G(T)$. Let $W(M, \lambda)$ be the inertial group of A . We say that A is a Rouquier block if there exists a Levi subgroup L of G such that

- (i) there exists parabolic subgroup P of G with Levi decomposition $P = LU_P$,
- (ii) L contains M ,
- (iii) $H = L \cdot W(M, \lambda)$ is a proper subgroup of G ,
- (iv) There exists a block B of H with canonical character λ such that A is Morita equivalent to B .

Remark 2. By L. Puig [Pui90], if ℓ does not divide the order of Weyl group W of G and does $q - 1$, then the principal block of G is Morita (Puig) equivalent to the principal block of $T.W$. In the case of type A , there is a generalization of this theorem which was conjectured by R. Rouquier [Rou98] in the context of symmetric groups. This generalization is intensively studied by J. Chuang, R. Kessar, K. Tan, W. Turner, A. Hida and the author [CK02], [CT02a], [Tur02], [HM00], [Miy01]. In this case there is also an interesting interpretation on these Rouquier blocks in terms of Fock space over quantum affine algebra of type A [CT02b], [LM02].

One aim of this note is to report that there are Rouquier blocks in type E_6 and E_8 (see Theorem 4 and Remark 5 below) which are not included in [Pui90].

In through this note, we assume that a prime number ℓ and a prime power q satisfy the following condition:

- (1) (i) $\ell \neq 2, 3$, (ii) ℓ divides $q^2 + 1$, and (iii) q is odd.

Remark 3. (i) and (ii) are essential in this note. (iii) is needed to use Kawanaka's generalized Gelfand-Graev character [GP92]. So, this might be removed once one gets the lower unitriangularity of decomposition matrix.

The main strategy (which I learned from T. Okuyama) is analogous to that in [KM00]. Namely, we construct a stable equivalence between two blocks by Broué's theorem [Bro92, 6.3. Theorem] and checking the assumption, and then we chase the images of simple modules. The most powerful tool in this approach is Linckelmann's theorem [Lin96]. In this note we choose a numbering of simple roots of type E_8 as follows:

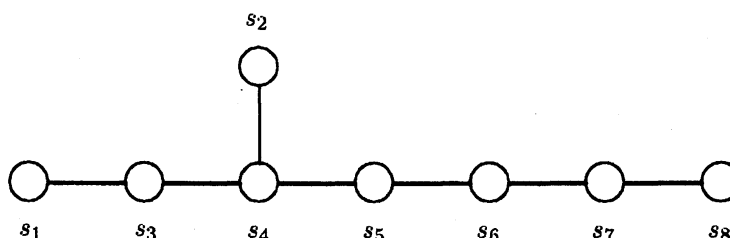


TABLE E_8

2. THE STATEMENT OF MAIN THEOREM

Let G be the Chevalley group of type E_6 with its defining field $\overline{\mathbb{F}_q}$. Let F be its standard Frobenius map. Let L be the Levi subgroup corresponding to $\{s_2, s_3, s_4, s_5\}$. It is of type D_4 .

Let A (resp. B) be the principal block ideal of $\mathbb{k}G^F$ (resp. $\mathbb{k}L^F \cdot \mathfrak{S}_3$). Moreover, $L^F \cdot \mathfrak{S}_3$ contains $\mathcal{N}_G^F(D)$. Hence,

$$(2) \quad \text{there is a Green correspondence } G^F \overset{f}{\underset{g}{\rightleftharpoons}} L^F \cdot \mathfrak{S}_3$$

(see [Alp86].) Let $\Delta(D)$ be $\{(x, x) \mid x \in D\}$. Let X be the Green correspondent of an indecomposable (A, A) -bimodule A in $G \times H$ with vertex $\Delta(D)$. (In other words, X is the Scott (A, B) -bimodule $S_{G^F \times (L^F \cdot \mathfrak{S}_3)}(\Delta(D))$ with vertex $\Delta(D)$.)

Now, we can state the main result of this note as follows:

Theorem 4. *The functor $X \otimes_B -$ induces a Morita equivalence between A and B .*

Remark 5. By [Miy03] we know that the principal block $B_0(\mathbb{k}G^F)$ is Morita equivalent to the unipotent block ideal of $E_8(q)$ with canonical character $\phi_{23,01}$ in the notation [BMM93]. Moreover, thanks to Broué's abelian defect conjecture and our main chart [BMM93], our main result is expected to be useful to settle Broué's abelian defect conjecture for the following unipotent Φ_4 -blocks:

- (i) Group $E_7(q)$: canonical characters ϕ_2^3, ϕ_{11}^3 ,
- (ii) Group $E_8(q)$: canonical characters $\phi_{3,1}, \phi_{123,013}, \phi_{12,03}$.

These will be discussed in elsewhere.

3. THE DECOMPOSITION MATRIX

Now, we recall what is known for the decomposition matrix of A without Theorem 4. Using [Gec93], [GH97], [GP92], etc, we can approximate the decomposition matrix as follows:

Lemma 6. *The decomposition matrix of the unipotent characters lying in the principal block of G^F has the following shape:*

	a	name	ps	ps	ps	ps	D_4	ps	ps	ps	D_4	ps	A_3	D_4	D_4	D_4
E_6	0	$\phi_{1,0}$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$E_6(a_1)$	1	$\phi_{6,1}$	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$E_6(a_3)$	3	$\phi_{15,5}$	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A_5	3	$\phi_{15,4}$	1	1	.	1	.	.	.	.	.	.	.	.	.	.
A_5	3	$D_{4,1}$	.	.	.	.	1	.	.	.	.	.	.	.	.	.
A_4	6	$\phi_{81,6}$	.	1	1	.	*	1	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{80,7}$	.	1	.	1	*	1	1	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{90,8}$	.	.	1	.	*	1	.	1	.	.	.	.	.	.
$D_4(a_1)$	7	$D_{4,r}$	.	.	.	.	*	.	.	.	1	.	.	.	.	.
$2A_2 + A_1$	7	$\phi_{10,9}$	1	.	.	1	*	.	.	.	*	1	.	.	.	.
A_3	10	$\phi_{81,10}$	.	.	.	.	*	1	1	1	*	.	1	.	.	.
A_2	15	$\phi_{15,17}$	.	.	.	.	*	.	.	1	*	.	*	1	.	.
$3A_1$	15	$\phi_{15,16}$	.	.	.	1	*	.	1	.	*	1	*	*	1	.
$3A_1$	15	$D_{4,\epsilon}$	.	.	.	.	*	.	.	.	*	.	*	*	*	1
A_1	25	$\phi_{6,25}$	.	.	.	.	*	.	1	.	*	.	*	*	*	1
1	36	$\phi_{1,36}$	.	.	.	.	*	.	.	*	1	*	*	*	*	1

Here, * means an unknown non-negative integer.

I would like to remove these unknown parameters *'s in Lemma 6 as possible. Using Theorem 4, Lemma 6 and the precise correspondence on the simple modules over A and B , we get the following new result:

Theorem 7. *The decomposition matrix of the unipotent characters lying in the principal block of G^F is given as follows:*

	a	name	ps	ps	ps	ps	D_4	ps	ps	ps	D_4	ps	A_3	D_4	A_3	D_4	D_4	D_4
E_6	0	$\phi_{1,0}$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$E_6(a_1)$	1	$\phi_{6,1}$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$E_6(a_3)$	3	$\phi_{15,5}$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A_5	3	$\phi_{15,4}$	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
A_5	3	$D_{4,1}$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A_4	6	$\phi_{81,6}$	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{80,7}$	.	1	.	1	.	1	1	.	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$\phi_{90,8}$	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.
$D_4(a_1)$	7	$D_{4,r}$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$2A_2 + A_1$	7	$\phi_{10,9}$	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.
A_3	10	$\phi_{81,10}$	.	.	.	.	.	1	1	1	.	.	1	.	.	.	.	.
A_2	15	$\phi_{15,17}$	.	.	.	.	*	.	.	1	.	.	1	1	.	.	.	.
$3A_1$	15	$\phi_{15,16}$	.	.	.	1	.	.	1	.	.	1	.	.	1	.	.	.
$3A_1$	15	$D_{4,\epsilon}$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
A_1	25	$\phi_{6,25}$	.	.	.	.	.	.	1	.	*	.	1	.	1	.	1	.
1	36	$\phi_{1,36}$	.	.	.	.	.	.	.	.	.	1	.	.	1	*	.	1

Here, $*$ is equal to the decomposition number $d_{(1111),\varphi}$ in type D_4 .

4. REMARKS ON HECKE ALGEBRAS

By Theorem 4, we can know that

Theorem 8. *Let F be a field with an invertible element q . We assume that*

- (i) *The characteristic of F is not 2, 3.*
- (ii) *$q^4 = 1, q^2 \neq 1$.*
- (iii) *F contains $q^{\frac{1}{2}}$.*
- (iv) *If the characteristic of F is positive, then, q lie in the prime field of F .*

Then, $B_0(\mathcal{H}_{F,q}(E_6))$ and $B_0(\mathcal{H}_{F,q}(D_4)).\mathfrak{S}_3$ are Morita equivalent.

We can construct $B_0(\mathcal{H}_{F,q}(D_4)).\mathfrak{S}_3$ as a block ideal of a well-known Iwahori-Hecke algebra in the following way. $\mathcal{H}_{F,q}(D_4)$ is a q -deformation of the group algebra of Weyl group $W(D_4)$ of type D_4 . And, in our situation, $W(D_4).\mathfrak{S}_3$ is nothing but the Weyl group $W(F_4)$ of type F_4 . Moreover, $W(D_4)$ is realized as a reflection subgroup of $W(F_4)$ generated by all the reflections of $W(F_4)$ whose roots are long. So, let us recall the definition of Iwahori-Hecke algebra of type F_4 .

Definition 9. *Let R be an integral domain with invertible elements $u^{\frac{1}{2}}, v^{\frac{1}{2}}$. The Iwahori-Hecke algebra $\mathcal{H}_{R,u,v}(F_4)$ over R with parameter u, v is an associative algebra with generators T_1, T_2, T_3, T_4 and relations*

$$\begin{aligned}
 (T_i - u)(T_i + 1) &= 0 \text{ for } i = 1, 2, (T_j - v)(T_j + 1) = 0 \text{ for } j = 3, 4, \\
 T_i T_j T_i &= T_{i+1} T_i T_{i+1} \text{ for } i = 1, 3, \\
 T_2 T_3 T_2 T_3 &= T_3 T_2 T_3 T_2, \\
 T_i T_j &= T_j T_i \text{ for } 1 \leq i < j - 1 \leq 3.
 \end{aligned}$$

From now on, we consider the Iwahori-Hecke algebra $\mathcal{H}_{k,q,1}(F_4)$ of type F_4 with parameter q and 1. Put a_1, a_2, a_3, a_4 respectively to be

$$a_1 = T_2, a_2 = T_1, a_3 = T_3 T_2 T_3, a_4 = T_4 T_3 T_2 T_3 T_4.$$

$\mathcal{H}_{k,q}(D_4)$ is isomorphic to the subalgebra $\mathcal{H}'$ of $\mathcal{H}_{k,q,1}(F_4)$ generated by a_1, a_2, a_3, a_4 . One can easily check that a_i 's satisfy the quadratic relations and braid relations. Moreover, clearly, $\mathbb{k}\langle T_3, T_4 \rangle$ is isomorphic to the group algebra $\mathbb{k}\mathfrak{S}_3$ since $T_3^2 = 1 = T_4^2$. By definition, the action of T_3 and T_4 on $\mathcal{H}'$ is also clear. Since the principal block idempotent of $\mathcal{H}_{k,q}(D_4)$ is normalized by $\mathbb{k}\langle T_3, T_4 \rangle$, it is lifted to the idempotent

of the whole algebra $\mathcal{H}_{k,q,1}(F_4)$. The decomposition matrix of $\mathcal{H}_{k,q,1}(F_4)$ is first calculated by Bremke [Bre94, p.342]. So, it is worth saying the correspondences among characters, simple modules, PIM's, etc over $B_0(\mathcal{H}_{k,q}(E_6))$ and $B_0(\mathcal{H}_{k,q,1}(F_4))$. The correspondence is given as follows:

E_6										F_4									
$\phi_{1,0}$	1	.	.	.	.	.	.	.	.	$\phi_{1,0}$	1	.	.	.	.	.	.	.	.
$\phi_{6,1}$	.	1	.	.	.	.	.	.	.	$\phi_{2,4}'$	.	1	.	.	.	.	.	.	.
$\phi_{15,5}$	.	.	1	.	.	.	.	.	.	$\phi_{1,12}'$	.	.	1	.	.	.	.	.	.
$\phi_{15,4}$	1	1	.	1	.	.	.	.	.	$\phi_{9,2}$	1	1	.	1	.	.	.	.	.
$\phi_{81,6}$	.	1	1	.	1	.	.	.	.	$\phi_{9,6}''$	.	1	1	.	1	.	.	.	.
$\phi_{90,8}$	.	.	1	.	1	1	.	.	.	$\phi_{8,9}''$	.	.	1	.	1	1	.	.	.
$\phi_{80,7}$	.	1	.	1	1	.	1	.	.	$\phi_{16,5}$	.	1	.	1	1	.	1	.	.
$\phi_{10,9}$	1	.	.	1	.	.	.	1	.	$\phi_{8,3}'$	1	.	.	1	.	.	.	1	.
$\phi_{81,10}$	.	.	.	.	1	1	1	.	.	$\phi_{9,10}$	.	.	.	.	1	1	1	.	.
$\phi_{15,16}$	.	.	.	1	.	.	1	1	.	$\phi_{9,6}'$	.	.	.	1	.	.	1	1	.
$\phi_{15,17}$	.	.	.	.	.	1	.	.	.	$\phi_{1,24}$	.	.	.	.	.	1	.	.	.
$\phi_{6,25}$	.	.	.	.	.	.	1	.	.	$\phi_{2,16}$	.	.	.	.	.	.	1	.	.
$\phi_{1,36}$	.	.	.	.	.	.	.	1	.	$\phi_{1,12}'$	.	.	.	.	.	.	.	1	.

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