

# COMPOSITE ITERATIVE METHODS FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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**ABSTRACT.** In this paper, we introduce two composite iterative methods (one implicit method and one explicit method) for finding a common element of the solution set of a general system of variational inequalities for continuous monotone mappings and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. First, this system of variational inequalities is proven to be equivalent to a fixed point problem of nonexpansive mapping. Then we establish strong convergence of the sequence generated by the proposed iterative methods to a common element of the solution set and the fixed point set, which is the unique solution of a certain variational inequality.

## 1. INTRODUCTION

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\| \cdot \|$ . Let  $C$  be a nonempty closed convex subset of  $H$  and let  $S : C \rightarrow C$  be a self-mapping on  $C$ . We denote by  $Fix(S)$  the set of fixed points of  $S$ .

A mapping  $F : C \rightarrow H$  is called *monotone* if

$$\langle x - y, Fx - Fy \rangle \geq 0, \quad \forall x, y \in C.$$

and  $F$  is called  $\alpha$ -*inverse-strongly monotone* (see [5, 11]) if there exists a positive real number  $\alpha$  such that

$$\langle x - y, Fx - Fy \rangle \geq \alpha \|Fx - Fy\|^2, \quad \forall x, y \in C.$$

The class of monotone mappings includes the class of  $\alpha$ -inverse-strongly monotone mappings.

A mapping  $T : C \rightarrow H$  is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

and  $T$  is said to be  $k$ -*strictly pseudocontractive* if there exists a constant  $k \in [0, 1)$  such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

where  $I$  is the identity mapping. Note that the class of  $k$ -strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is,  $T$  is nonexpansive (i.e.,  $\|Tx - Ty\| \leq \|x - y\|$ ,  $\forall x, y \in C$ ) if and only if  $T$  is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings as a subclass.

Let  $F$  be a nonlinear mapping of  $C$  into  $H$ . The variational inequality problem (VIP) is to find a  $x^* \in C$  such that

$$(1.1) \quad \langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

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The results presented in this lecture are collected mainly from the work [8] by the author of this report.

We denote the set of solutions of VIP(1.1) by  $VI(C, F)$ . The variational inequality problem has been extensively studied in the literature; see [3, 5, 7, 10, 11, 14, 15, 17, 19] and the references therein.

In 2008, Ceng *et al.* [2] considered the following general system of variational inequalities:

$$(1.2) \quad \begin{cases} \langle \lambda F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \\ \langle \nu F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

where  $F_1$  and  $F_2$  are an  $\alpha$ -inverse-strongly monotone mapping and a  $\beta$ -inverse-strongly monotone mapping, respectively; and  $\lambda \in (0, 2\alpha)$  and  $\nu \in (0, 2\beta)$  are two constants. For finding an element  $Fix(S) \cap \Gamma$ , where  $S : C \rightarrow C$  is a nonexpansive mapping and  $\Gamma$  is the solution set of the problem (1.2), they introduced a relaxed extragradient method ([9]) and proved strong convergence to a common element of  $Fix(S) \cap \Gamma$ .

In 2016, Alofi *et al.* [1] also considered the problem (1.2) coupled with the fixed point problem, and introduced two composite iterative algorithms (one implicit algorithm and one explicit algorithm) based on Jung's composite iterative method [6] to find an element  $Fix(T) \cap \Gamma$ , where  $T : C \rightarrow C$  is a  $k$ -strictly pseudocontractive mapping and  $\Gamma$  is the solution set of the problem (1.2), and showed strong convergence to a common element of  $Fix(T) \cap \Gamma$ . The following problems arise:

Question 1. Can we extend the class of inverse-strongly monotone mappings in [1, 2] to the more general class of continuous monotone mappings ?

Question 2. Can we extend the class of nonexpansive mappings in [2] or the class of strictly pseudocontractive mappings in [1] to the more general class of pseudocontractive mappings ?

In this paper, in order to give the affirmative answers to the above two questions, we consider a general system of variational inequalities slightly different from the problem (1.2). More precisely, we introduce the following general system of variational inequalities (GSVI) for two continuous monotone mappings  $F_1$  and  $F_2$  of finding  $(x^*, y^*) \in C \times C$  such that

$$(1.3) \quad \begin{cases} \langle \lambda F_1 x^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \\ \langle \nu F_2 y^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

where  $\lambda > 0$  and  $\nu$  are two constants. The solution set of GSVI(1.3) is denoted by  $\Omega$ . First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping. Second, by using Jung's composite iterative algorithms [6], we introduce a composite implicit iterative algorithm and a composite explicit iterative algorithm for finding a common element of  $\Omega \cap Fix(T)$ , where  $T$  is a continuous pseudocontractive mapping. Then we establish strong convergence of these two composite iterative algorithms to a common element of  $\Omega \cap Fix(T)$ , which is the unique solution of a certain variational inequality related to a minimization problem. As a direct consequence, we obtain strong convergence to a common element of  $VI(C, F) \cap Fix(T)$ , where  $F$  is a continuous monotone mapping.

## 2. PRELIMINARIES AND LEMMAS

Let  $H$  be a real Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . We write  $x_n \rightharpoonup x$  to indicate that the sequence  $\{x_n\}$  converges weakly to  $x$ .  $x_n \rightarrow x$  implies that  $\{x_n\}$  converges strongly to  $x$ .

For every point  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C(x)$ , such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

$P_C$  is called the *metric projection* of  $H$  onto  $C$ . It is well known that  $P_C(x)$  is characterized by the property:

$$(2.1) \quad u = P_C(x) \iff \langle x - u, u - y \rangle \geq 0, \quad \forall x \in H, y \in C.$$

In a Hilbert space  $H$ , we have

$$(2.2) \quad \|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle, \quad \forall x, y \in H.$$

We recall that

- (1) an operator  $A$  is said to be *strongly positive* on  $H$  if there exists a constant  $\bar{\gamma} > 0$  such that

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H;$$

- (ii) a mapping  $V : C \rightarrow H$  is said to be  *$l$ -Lipschitzian* if there exists a constant  $l \geq 0$  such that

$$\|Vx - Vy\| \leq l\|x - y\|, \quad \forall x, y \in C;$$

- (iii) a mapping  $G : C \rightarrow H$  is said to be  *$\rho$ -strongly monotone* if there exists a constant  $\rho > 0$  such that

$$\langle Gx - Gy, x - y \rangle \geq \rho\|x - y\|^2, \quad \forall x, y \in C.$$

The following lemma is an immediate consequence of an inner product.

**Lemma 2.1.** *In a real Hilbert space  $H$ , there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

We need the following lemmas for the proof of our main results.

**Lemma 2.2** ([16]). *Let  $\{s_n\}$  be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \omega_n)s_n + \omega_n\delta_n + \nu_n, \quad \forall n \geq 1,$$

*where  $\{\omega_n\}$ ,  $\{\delta_n\}$ , and  $\{\nu_n\}$  satisfy the following conditions:*

- (i)  $\{\omega_n\} \subset [0, 1]$  and  $\sum_{n=1}^{\infty} \omega_n = \infty$  or, equivalently,  $\prod_{n=1}^{\infty} (1 - \omega_n) = 0$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \delta_n \leq 0$  or  $\sum_{n=1}^{\infty} \omega_n |\delta_n| < \infty$ ;
- (iii)  $\nu_n \geq 0$  ( $n \geq 1$ ),  $\sum_{n=1}^{\infty} \nu_n < \infty$ .

*Then  $\lim_{n \rightarrow \infty} s_n = 0$ .*

**Lemma 2.3** ([4]). (Demiclosedness principle) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ , and let  $S : C \rightarrow C$  be a nonexpansive mapping. Then, the mapping  $I - S$  is demiclosed. That is, if  $\{x_n\}$  is a sequence in  $C$  such that  $x_n \rightharpoonup x^*$  and  $(I - S)x_n \rightarrow y$ , then  $(I - S)x^* = y$ .*

**Lemma 2.4** ([12]). *Let  $H$  be a real Hilbert space. Let  $A : H \rightarrow H$  be a strongly positive bounded linear operator with a constant  $\bar{\gamma} > 1$ . Then*

$$\langle (A - I)x - (A - I)y, x - y \rangle \geq (\bar{\gamma} - 1)\|x - y\|^2, \quad \forall x, y \in C.$$

*That is,  $A - I$  is strongly monotone with a constant  $\bar{\gamma} - 1$ .*

**Lemma 2.5** ([12]). *Assume that  $A$  is a strongly positive bounded linear operator on  $H$  with a coefficient  $\bar{\gamma} > 0$  and  $0 < \zeta \leq \|A\|^{-1}$ . Then  $\|I - \zeta A\| \leq 1 - \zeta\bar{\gamma}$ .*

**Lemma 2.6** ([17]). *Let  $H$  be a real Hilbert space. Let  $G : H \rightarrow H$  be a  $\rho$ -Lipschitzian and  $\eta$ -strongly monotone mapping with constants  $\rho, \eta > 0$ . Let  $0 < \mu < \frac{2\eta}{\rho^2}$  and  $0 < t < \sigma \leq 1$ . Then  $S := \sigma I - t\mu G : H \rightarrow H$  is a contractive mapping with constant  $\sigma - t\tau$ , where  $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$ .*

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [18], respectively.

**Lemma 2.7** ([18]). *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $F : C \rightarrow H$  be a continuous monotone mapping. Then, for  $r > 0$  and  $x \in H$ , there exists  $z \in C$  such that*

$$\langle y - z, Fz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

For  $r > 0$  and  $x \in H$ , define  $F_r : H \rightarrow C$  by

$$F_r x = \left\{ z \in C : \langle y - z, Fz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

- (i)  $F_r$  is single-valued;
- (ii)  $F_r$  is firmly nonexpansive, that is,
$$\|F_r x - F_r y\|^2 \leq \langle x - y, F_r x - F_r y \rangle, \quad \forall x, y \in H;$$
- (iii)  $\text{Fix}(F_r) = \text{VI}(C, F)$ ;
- (iv)  $\text{VI}(C, F)$  is a closed convex subset of  $C$ .

**Lemma 2.8** ([18]). *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $T : C \rightarrow H$  be a continuous pseudocontractive mapping. Then, for  $r > 0$  and  $x \in H$ , there exists  $z \in C$  such that*

$$\langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C.$$

For  $r > 0$  and  $x \in H$ , define  $T_r : H \rightarrow C$  by

$$T_r x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

- (i)  $T_r$  is single-valued;
- (ii)  $T_r$  is firmly nonexpansive, that is,
$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y \rangle, \quad \forall x, y \in H;$$
- (iii)  $\text{Fix}(T_r) = \text{Fix}(T)$ ;
- (iv)  $\text{Fix}(T)$  is a closed convex subset of  $C$ .

### 3. MAIN RESULTS

Throughout the rest of this paper, we always assume the following:

- $H$  is a real Hilbert space;
- $C$  is a nonempty closed subspace subset of  $H$ ;
- $A : C \rightarrow C$  is a strongly positive linear bounded self-adjoint operator with a constant  $\bar{\gamma} \in (1, 2)$ ;
- $V : C \rightarrow C$  is  $l$ -Lipschitzian with constant  $l \in [0, \infty)$ ;
- $G : C \rightarrow C$  is a  $\rho$ -Lipschitzian and  $\eta$ -strongly monotone mapping with constants  $\rho > 0$  and  $\eta > 0$ ;
- Constants  $\mu$ ,  $l$ ,  $\tau$ , and  $\gamma$  satisfy  $0 < \mu < \frac{2\eta}{\rho^2}$  and  $0 \leq \gamma l < \tau$ , where  $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$ ;
- $F_1$  and  $F_2 : C \rightarrow H$  are continuous monotone mapping;
- $\Omega$  is the solution set of GSVI (1.3) for  $F_1$  and  $F_2$ ;
- $F_{1\lambda} : H \rightarrow C$  is a mapping defined by

$$F_{1\lambda} x = \left\{ z \in C : \langle y - z, F_1 z \rangle + \frac{1}{\lambda} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}$$

for  $\lambda > 0$ ;

- $F_{2\nu} : H \rightarrow C$  is a mapping defined by

$$F_{2\nu}x = \left\{ z \in C : \langle y - z, F_2z \rangle + \frac{1}{\nu} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}$$

for  $\nu > 0$ ;

- $R : H \rightarrow C$  is a mapping defined by  $Rx = F_{1\lambda}F_{2\nu}x$  for each  $x \in H$ ;
- $T : C \rightarrow C$  is a continuous pseudocontractive mapping such that  $Fix(T) \neq \emptyset$ ;
- $T_{r_t} : H \rightarrow C$  is a mapping defined by

$$T_{r_t}x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_t} \langle y - z, (1 + r_t)z - x \rangle \leq 0, \quad \forall y \in C \right\}$$

for  $r_t \in (0, \infty)$ ,  $t \in (0, 1)$ , and  $\liminf_{t \rightarrow 0} r_t > 0$ ;

- $T_{r_n} : H \rightarrow C$  is a mapping defined by

$$T_{r_n}x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n} \langle y - z, (1 + r_n)z - x \rangle \leq 0, \quad \forall y \in C \right\}$$

for  $r_n \in (0, \infty)$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ ;

- $\Omega \cap Fix(T) \neq \emptyset$ .

By Lemma 2.7 and Lemma 2.8, we note that  $F_{1\lambda}$ ,  $F_{2\nu}$ ,  $T_{r_t}$ , and  $T_{r_n}$  are nonexpansive, and  $Fix(T_{r_n}) = Fix(T) = Fix(T_{r_t})$ .

First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping.

**Proposition 3.1.** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . For given  $x^*, y^* \in C$ ,  $(x^*, y^*)$  is a solution of GSVI(1.3) for continuous monotone mappings  $F_1$  and  $F_2$  if and only if  $x^*$  is a fixed point of the mapping  $R : H \rightarrow C$  defined by*

$$Rx = F_{1\lambda}F_{2\nu}x, \quad \forall x \in H,$$

where  $y^* = F_{2\nu}x^*$ .

First, we introduce the following composite algorithm that generates a net  $\{x_t\}$  in an implicit way:

$$(3.1) \quad x_t = (I - \theta_t A)T_{r_t}Rx_t + \theta_t[t\gamma Vx_t + (I - t\mu G)T_{r_t}Rx_t],$$

where  $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\bar{\gamma}l}\})$  and  $\theta_t \in (0, \|A\|^{-1}]$ .

We summarize the basic properties of  $\{x_t\}$ , which can be proved by the same method as in [6].

**Proposition 3.2.** *Let  $\{x_t\}$  be defined via (3.1). Then*

- (i)  $\{x_t\}$  is bounded for  $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\bar{\gamma}l}\})$ ;
- (ii)  $\lim_{t \rightarrow 0} \|x_t - T_{r_t}Rx_t\| = 0$  provided  $\lim_{t \rightarrow 0} \theta_t = 0$ ;
- (iii)  $\lim_{t \rightarrow 0} \|x_t - y_t\| = 0$ , where  $y_t = t\gamma Vx_t + (I - t\mu G)T_{r_t}Rx_t$ ;
- (iv)  $\lim_{t \rightarrow 0} \|x_t - Rx_t\| = 0$ ;
- (v)  $x_t$  defines a continuous path from  $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\bar{\gamma}l}\})$  into  $H$  provided  $\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\bar{\gamma}l}\}) \rightarrow (0, \|A\|^{-1}]$  is continuous, and  $r_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\bar{\gamma}l}\}) \rightarrow (0, \infty)$  is continuous.

We obtain the following theorem for strong convergence of the net  $\{x_t\}$  as  $t \rightarrow 0$ , which guarantees the existence of solutions of the variational inequality (3.2) below.

**Theorem 3.3.** *Let the net  $\{x_t\}$  be defined via (3.1). If  $\lim_{t \rightarrow 0} \theta_t = 0$ , then  $x_t$  converges strongly to  $\tilde{x}$  in  $\Omega \cap \text{Fix}(T)$  as  $t \rightarrow 0$ , which solves the variational inequality*

$$(3.2) \quad \langle (A - I)\tilde{x}, \tilde{x} - p \rangle \leq 0, \quad \forall p \in \Omega \cap \text{Fix}(T),$$

*Equivalently, we have*

$$P_{\Omega \cap \text{Fix}(T)}(2I - A)\tilde{x} = \tilde{x}.$$

Now, we propose the following composite algorithm which generates a sequence in an explicit way:

$$(3.3) \quad \begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) T_{r_n} R x_n, \\ x_{n+1} = (I - \beta_n A) T_{r_n} R x_n + \beta_n y_n, \quad \forall n \geq 0, \end{cases}$$

where  $\{\alpha_n\} \in [0, 1]$ ;  $\{\beta_n\} \subset (0, 1]$ ;  $\{r_n\} \subset (0, \infty)$ ; and  $x_0 \in C$  is an arbitrary initial guess, and establish strong convergence of this sequence to  $\tilde{x} \in \Omega \cap \text{Fix}(T)$ , which is the unique solution of the variational inequality (3.2).

**Theorem 3.4.** *Let  $\{x_n\}$  be the sequence generated by the explicit algorithm (3.3). Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{r_n\}$  satisfy the following conditions:*

- (C1)  $\{\alpha_n\} \subset [0, 1]$  and  $\{\beta_n\} \subset (0, 1]$ ,  $\alpha_n \rightarrow 0$  and  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ ;
- (C2)  $\sum_{n=0}^{\infty} \beta_n = \infty$ ;
- (C3)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ , and  $|\beta_{n+1} - \beta_n| \leq o(\beta_{n+1}) + \sigma_n$ ,  $\sum_{n=0}^{\infty} \sigma_n < \infty$  (the perturbed control condition);
- (C4)  $\{r_n\} \subset (0, \infty)$ ,  $\liminf_{n \rightarrow \infty} r_n > 0$ , and  $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ .

*Then  $\{x_n\}$  converges strongly to  $\tilde{x} \in \Omega \cap \text{Fix}(T)$ , which is the unique solution of the variational inequality (3.2).*

Taking  $G \equiv I$ ,  $\mu = 1$ , and  $\gamma = 1$  in Theorem 3.5, we obtain the following corollary.

**Corollary 3.5.** *Let  $\{x_n\}$  be generated by the following iterative algorithm:*

$$\begin{cases} y_n = \alpha_n V x_n + (1 - \alpha_n) T_{r_n} R x_n, \\ x_{n+1} = (I - \beta_n A) T_{r_n} R x_n + \beta_n y_n, \quad \forall n \geq 0. \end{cases}$$

*Assume that the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{r_n\}$  satisfy the conditions (C1) – (C4) in Theorem 3.5. Then  $\{x_n\}$  converges strongly to  $\tilde{x} \in \Omega \cap \text{Fix}(T)$ , which is the unique solution of the variational inequality (3.2).*

**Remark 3.6.** 1) The  $\tilde{x} \in \Omega \cap \text{Fix}(T)$  in our results is the unique solution of minimization problem

$$(3.4) \quad \min_{x \in D} \frac{1}{2} \langle (A - I)x, x \rangle,$$

where the constraint set  $D$  is  $\Omega \cap \text{Fix}(T)$ . In fact, the variational inequality (3.2) is the optimality condition for the minimization problem (3.4). Thus, for finding an element of  $\Omega \cap \text{Fix}(T)$ , where  $T$  is a continuous pseudocontractive mapping, and  $F_1$  and  $F_2$  are continuous monotone mappings, Theorem 3.4, Theorem 3.5 and Corollary 3.6 are new ones different from previous those introduced by some authors (for example, see [1, 2]).

2) Taking  $F_1 = F_2 = F$ ,  $\lambda = \nu$  and  $x^* = y^*$  in GSVI(1.3) and replacing  $F_\lambda$  by  $F_{r_n}$  along with the condition (C4) on  $\{r_n\}$ , we can obtain a new result, which improves, supplements and develops the corresponding results of [3, 5, 7, 14, 15, 19].

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