

Unimodal 例外型特異点 における 代数的局所コホモロジー類

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1 序

孤立特異点を持つ擬斉次多項式 $f = f(x_1, \dots, x_n)$ に対し, 原点に台を持つ代数的局所コホモロジー類 $[1/f_{x_1} \cdots f_{x_n}] \in \mathcal{H}_{[0]}^n(\mathcal{O}_X)$ を考える. 但し, $f_{x_j} = \partial f / \partial x_j$, $j = 1, \dots, n$ である. コホモロジー類 $[1/f_{x_1} \cdots f_{x_n}]$ の annihilator について, 次が成り立つ.

Fact : $f = f(x_1, \dots, x_n)$ を quasiweight $(\alpha_1, \dots, \alpha_n)$, quasidegree d_w の n 変数擬斉次多項式とする. $f_{x_1} = \partial f / \partial x_1, \dots, f_{x_n} = \partial f / \partial x_n$ に対し, 原点に台を持つ代数的局所コホモロジー類 $[1/f_{x_1} \cdots f_{x_n}]$ を考える. $[1/f_{x_1} \cdots f_{x_n}]$ の微分加群としての annihilating ideal を $\mathcal{Ann}$ とおくと,

$$\mathcal{Ann} = \langle f_{x_1}, \dots, f_{x_n}, P \rangle$$

が成り立つ. 但し, $P = \alpha_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n x_n \frac{\partial}{\partial x_n} + n d_w - (\alpha_1 + \cdots + \alpha_n)$ である.

($\mathcal{Ann}$ のグレブナ基底については [2] を参照されたい.)

この事実を用いると, 微分作用素 $f_{x_1}, \dots, f_{x_n}, P$ を用いて代数的局所コホモロジー類 $[1/f_{x_1} \cdots f_{x_n}]$ を特徴付けることができる ([4], [5] 参照). つまり, f が擬斉次多項式の場合, 代数的局所コホモロジー類 $[1/f_{x_1} \cdots f_{x_n}]$ を特徴付ける微分作用素は euler 型で与えることができる. それに比べ, f が半擬斉次多項式の場合, 一般に, 1 階の微分作用素では代数的局所コホモロジー類を特徴付けることはできない.

我々は, V.I. Arnol'd による分類 ([1]) に従い, Unimodal 例外型孤立特異点 ($E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}, Q_{10}, Q_{11}, Q_{12}, S_{11}, S_{12}, U_{12}$) に付随する代数的局所コホモロジー類の annihilator を計算した. その結果, これらの場合, 代数的局所コホモロジー類を特徴付けるには 2 階の微分作用素が必要となることが明らかになった.

本稿では, まず初めに, §1 で E_{12} 型特異点に対してコホモロジー類の計算法を述べ, §2 で主結果を与える. §3 で Unimodal 特異点に関する計算結果を与え, 最後に §4 で Bimodal 特異点である E_{18} 型の場合について述べる.

2 代数的局所コホモロジー類の計算例 (E_{12} 型)

$X = \mathbb{C}^2$ 上の半擬斉次多項式 $f(x, y) = x^3 + xy^5 + y^7$ で与えられる E_{12} 型 Unimodal singularity を考える. $f_x = \partial f / \partial x = 3x^2 + y^5$, $f_y = \partial f / \partial y = 5xy^4 + 7y^6$ とおく. f_x, f_y で生成される $\mathcal{O}_X$ 上のイデアル $I = \langle f_x, f_y \rangle$ に対し, 全次数辞書式順序 $x \succ y$ でのグレブナ基底は $\text{Gb} = \{5x^3 + 7y^2x^2, 125x^4 - 1029yx^3, 3x^2 + y^5, -21yx^2 + 5y^4x\}$ で与えられる. I の準素イデアル分解は $I = I_1 \cap I_2$, $I_1 = \langle 25y + 147, 3125x + 151263 \rangle$, $I_2 = \langle y^8, 5y^4x + 7y^6, 3x^2 + y^5 \rangle$ で与えられる. $Y = \{(x, y) \in X \mid f_x = f_y = 0\}$ とおくと, $Y = \{3125x + 151263 = 25y + 147 = 0\} \cup \{x = y = 0\}$ であり, 原点の重複度は 12 である.

Y に台を持つ代数的局所コホモロジー類 $[1/f_x f_y]$ に対し, $[1/f_x f_y]$ を annihilate する高々 j 階の微分作用素の生成する左イデアルを $\mathcal{Ann}^{(j)}$ とおく. $\mathcal{Ann}^{(1)}$ を求めると, $\mathcal{Ann}^{(1)} = \langle f_x, f_y, P_1, P_2 \rangle$ を得る. 但し, $P_1 = (5xy + 7y^3)\partial_y + 20x + 42y^2$, $P_2 = (3500y^2 + 2058y)x\partial_x + ((-1000y - 735)x + 7203y^2)\partial_y - 4000x + 7700y^2 + 84378y$ ($\partial_x = \partial / \partial x$, $\partial_y = \partial / \partial y$) である. これらの作用素を用いて原点に台を持つ代数的局所コホモロジー類 $\sigma = [1/f_x f_y]|_{(0,0)}$ の表現を計算する ([4] 参照). 方程式 $f_x \sigma = f_y \sigma = P_1 \sigma = P_2 \sigma = 0$, $|\partial(f_x, f_y) / \partial(x, y)| \sigma = 12[1/xy]$ を解くことにより, σ の表現

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$$\sigma = \left[\begin{array}{l} \frac{1}{xy} - \frac{1220703125}{1483273860320763} \frac{1}{xy^2} + \frac{48828125}{10090298369529} \frac{1}{xy^3} - \frac{1953125}{68641485507} \frac{1}{xy^4} \\ + \frac{466948881}{390625} \frac{xy^5}{1} - \frac{3176523}{15625} \frac{xy^6}{1} + \frac{21609}{625} \frac{xy^7}{1} - \frac{147}{25} \frac{xy^8}{1} - \frac{1441471195647}{9765625} \frac{x^2y}{1} \\ + \frac{9805926501}{3125} \frac{x^2y^2}{1} - \frac{66706983}{125} \frac{x^2y^3}{5} + \frac{453789}{5} \frac{x^2y^4}{1} - \frac{3087}{3087} \frac{x^2y^5}{1} + \frac{1}{21} \frac{x^2y^6}{1} \\ + \frac{9529569}{9529569} \frac{x^3y}{1} - \frac{64827}{64827} \frac{xy^2}{1} + \frac{441}{441} \frac{x^3y^3}{1} - \frac{1}{63} \frac{1}{x^4y} \end{array} \right]$$

を得る。つまり, $\mathcal{Ann}^{(1)}$ を用いた計算では, デルタ関数に相当する部分 $[1/xy]$ の係数が決まらないことが分かる。また, $\mathcal{Ann}^{(1)}$ を原点に局所化したものは, $\langle y^8, 5y^4x + 7y^6, 3x^2 + y^5, P_{0,1}, P_{0,2} \rangle$ となる。但し,

$$\begin{aligned} P_{0,1} &= (5yx + 7y^3)\partial_y + 20x + 42y^2 \\ P_{0,2} &= 470880590578020yx\partial_x + (-16817163949215x + 164808206702307y^2)\partial_y \\ &\quad - 9765625000x^3 + (-80390625000y + 236348437500)x^2 \\ &\quad + (56273437500y^3 - 330887812500y^2 + 1945620337500y - 11440247584500)x \\ &\quad + 463242937500y^4 - 2723868472500y^3 + 16016346618300y^2 + 1930610421369882y \end{aligned}$$

である。 $P_{0,1}, P_{0,2}$ の形から, ホロノミック系 $\mathcal{D}_X/\mathcal{Ann}^{(1)}$ の原点における重複度が 2 であることが分かる。

次に, $\mathcal{Ann}^{(2)}$ を求めると, $\mathcal{Ann}^{(2)} = \langle f_x, f_y, P \rangle$ を得る。但し,

$$\begin{aligned} P &= ((68250000y - 24863426875)x^2 + (-37182906425y^2 - 10656175824y)x)\partial_x^2 \\ &\quad + (-8437500x^2 + (637980000y + 694297170)x - 3708061182y^2)\partial_y\partial_x \\ &\quad + ((-885281250y - 128629930625)x - 179156250y^3 - 111958930125y^2 - 54216894564y)\partial_x \\ &\quad + (-18742500x + 154288260y)\partial_y^2 \\ &\quad + (-25312500x - 511875000y^2 - 2118711000y + 2082891510)\partial_y - 5545968750y - 102242340375 \end{aligned}$$

である。これらの作用素を用いて σ の表現を計算すると,

$$\sigma = \left[\begin{array}{l} \frac{30517578125}{218041257467152161} \frac{1}{xy} - \frac{1220703125}{1483273860320763} \frac{1}{xy^2} + \frac{48828125}{10090298369529} \frac{1}{xy^3} - \frac{1953125}{68641485507} \frac{1}{xy^4} \\ + \frac{466948881}{390625} \frac{xy^5}{1} - \frac{3176523}{15625} \frac{xy^6}{1} + \frac{21609}{625} \frac{xy^7}{1} - \frac{147}{25} \frac{xy^8}{1} - \frac{1441471195647}{9765625} \frac{x^2y}{1} \\ + \frac{9805926501}{3125} \frac{x^2y^2}{1} - \frac{66706983}{125} \frac{x^2y^3}{5} + \frac{453789}{5} \frac{x^2y^4}{1} - \frac{3087}{3087} \frac{x^2y^5}{1} + \frac{1}{21} \frac{x^2y^6}{1} \\ + \frac{9529569}{9529569} \frac{x^3y}{1} - \frac{64827}{64827} \frac{xy^2}{1} + \frac{441}{441} \frac{x^3y^3}{1} - \frac{1}{63} \frac{1}{x^4y} \end{array} \right]$$

を得る。

また, $\mathcal{Ann}^{(2)}$ を原点に局所化したものは, $\langle y^8, 5y^4x + 7y^6, 3x^2 + y^5, P_0 \rangle$ となる。但し,

$$\begin{aligned} P_0 &= (620690779589826484980x + 42736086463561823556y^2)\partial_x\partial_y \\ &\quad + (32879237237110246500x + 16068800153474932500y^3 - 94484544902432603100y^2 \\ &\quad + 256416518781370941336y)\partial_x \\ &\quad + 305257760454013025400y\partial_y^2 \\ &\quad + ((-1315628472217500000y + 1439969663151534375)x \\ &\quad - 8814296054882311875y^2 + 103829170222453410000y + 3368010623675943713580)\partial_y \\ &\quad - 1165771484375000x^3 + (-2060009765625000y + 116844758789062500)x^2 \\ &\quad + (-35487436523437500y^3 + 144556613085937500y^2 \\ &\quad - 473028944554687500y - 4697651664385312500)x \\ &\quad + 52404357304687500y^4 + 730337234189062500y^3 - 4874975501945062500y^2 \\ &\quad - 20807036096166258750y + 500110503238150591500 \end{aligned}$$

である。イデアル $\langle y^8, 5y^4x + 7y^6, 3x^2 + y^5, P_0 \rangle$ のグレブナ基底を計算することにより, ホロノミック系 $\mathcal{D}_X/\mathcal{Ann}^{(2)}$ の原点における重複度が 1 であることが分かる。従って, E_{12} の場合, $\mathcal{Ann} = \mathcal{Ann}^{(2)}$ が成り立つ。

なお, 微分作用素 P_1, P_2, P 等の構成法については, 文献 [3] を参照されたい。

3 主結果

Unimodal 例外型特異点に対し, 代数的局所コホモロジー類を特徴付ける微分作用素について調べる. 本稿では, *quasihomogeneous* とならない以下の標準形について計算する.

$$\begin{aligned}
 \text{2 変数} \quad E_{12} &: f(x, y) = x^3 + y^7 + axy^5 \\
 E_{13} &: f(x, y) = x^3 + xy^5 + ay^8 \\
 E_{14} &: f(x, y) = x^3 + y^8 + axy^6 \\
 Z_{11} &: f(x, y) = x^3y + y^5 + axy^4 \\
 Z_{12} &: f(x, y) = x^3y + xy^4 + ax^2y^3 \\
 Z_{13} &: f(x, y) = x^3y + y^6 + axy^5 \\
 W_{12} &: f(x, y) = x^4 + y^5 + ax^2y^3 \\
 W_{13} &: f(x, y) = x^4 + xy^4 + ay^6 \\
 \text{3 変数} \quad Q_{10} &: f(x, y, z) = x^3 + y^4 + yz^2 + axy^3 \\
 Q_{11} &: f(x, y, z) = x^3 + y^2z + xz^3 + az^5 \\
 Q_{12} &: f(x, y, z) = x^3 + y^5 + yz^2 + axy^4 \\
 S_{11} &: f(x, y, z) = x^4 + y^2z + xz^2 + ax^3z \\
 S_{12} &: f(x, y, z) = x^2y + y^2z + xz^3 + az^5 \\
 U_{12} &: f(x, y, z) = x^3 + y^3 + z^4 + axyz^2
 \end{aligned}$$

$(x, y, z) \in X = \mathbb{C}^3$ に対し, $f_x = \partial f(x, y, z)/\partial x$, $f_y = \partial f(x, y, z)/\partial y$, $f_z = \partial f(x, y, z)/\partial z$ とおく. f_x, f_y, f_z で生成されるイデアル $I = \langle f_x, f_y, f_z \rangle$ に対し, $Y = V(I)$ に台を持つ代数的局所コホモロジー類 $[1/f_x f_y f_z]$ の $\mathcal{D}_X$ 上の *annihilating ideal* を $\mathcal{A}nn$ とおく. 但し, $\mathcal{D}_X$ は X 上の線形偏微分作用素の層とする. また, $[1/f_x f_y f_z]$ を *annihilate* する高々 j 階の線形偏微分作用素の生成する左イデアルを $\mathcal{A}nn^{(j)}$ とおく. σ を, 原点に台を持つ代数的局所コホモロジー類 $[1/f_x f_y f_z]|_{(0,0,0)}$ とする. $X = \mathbb{C}^2 \ni (x, y)$ の場合も同様の記号を用いる (§2 参照).

次の結果を得た.

Proposition 1

- (i) ホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(1)}$ の原点における重複度 = 2
- (ii) ホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(2)}$ の原点における重複度 = 1

Theorem 2 $\mathcal{A}nn^{(2)} = \mathcal{A}nn$

Theorem 3

- (i) $\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{A}nn^{(1)}, \mathcal{H}_{[0]}^{\dim X}(\mathcal{O}_X)) = \text{Span}\{\delta_0, \sigma\}$,
ここで, δ_0 は原点に台を持つデルタ関数 ($\dim X = 3$ の場合は $[1/xyz]$, $\dim X = 2$ の場合は $[1/xy]$) である.
- (ii) $\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{A}nn^{(2)}, \mathcal{H}_{[0]}^{\dim X}(\mathcal{O}_X)) = \text{Span}\{\sigma\}$.

4 具体的計算結果

前節にあげた定理に関する具体的計算結果 (コホモロジー類の表現や $\mathcal{A}nn^{(1)}, \mathcal{A}nn^{(2)}$ の生成元) を, 下に挙げる標準形で与えられる Unimodal 特異点に対し, それぞれまとめておく. 標準形にはパラメーター a が含まれるが, 本稿では $a = 1$ として計算を行った. グレブナ基底等の計算は, $x \succ y \succ z$ (2 変数の場合は $x \succ y$) として, 全次数辞書式順序を用いて行った. 計算に用いた数式処理システム (*kan/sml*, *risa/asir*) については [6] を参照されたい.

$$\begin{array}{ll}
2 \text{ 变数} & E_{12} : f(x, y) = x^3 + y^7 + xy^5 \\
& E_{13} : f(x, y) = x^3 + xy^5 + y^8 \\
& E_{14} : f(x, y) = x^3 + y^8 + xy^6 \\
& Z_{11} : f(x, y) = x^3y + y^5 + xy^4 \\
& Z_{12} : f(x, y) = x^3y + xy^4 + x^2y^3 \\
& Z_{13} : f(x, y) = x^3y + y^6 + xy^5 \\
& W_{12} : f(x, y) = x^4 + y^5 + x^2y^3 \\
& W_{13} : f(x, y) = x^4 + xy^4 + y^6 \\
3 \text{ 变数} & Q_{10} : f(x, y, z) = x^3 + y^4 + yz^2 + xy^3 \\
& Q_{11} : f(x, y, z) = x^3 + y^2z + xz^3 + z^5 \\
& Q_{12} : f(x, y, z) = x^3 + y^5 + yz^2 + xy^4 \\
& S_{11} : f(x, y, z) = x^4 + y^2z + xz^2 + x^3z \\
& S_{12} : f(x, y, z) = x^2y + y^2z + xz^3 + z^5 \\
& U_{12} : f(x, y, z) = x^3 + y^3 + z^4 + xyz^2
\end{array}$$

4.1 E_{12} 型特異点

$$f = x^3 + y^7 + xy^5,$$

$$f_x = 3x^2 + y^5, f_y = 7y^6 + 5xy^4.$$

$$Gb = \{5x^3 + 7y^2x^2, 125x^4 - 1029yx^3, 3x^2 + y^5, -21yx^2 + 5y^4x\}.$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle 25y + 147, 3125x + 151263 \rangle, \sqrt{I_1} = \langle 25y + 147, 3125x + 151263 \rangle,$$

$$I_2 = \langle y^8, 5y^4x + 7y^6, 3x^2 + y^5 \rangle, \sqrt{I_2} = \langle y, x \rangle.$$

$$Ann^{(1)} = \langle f_x, f_y, P_1, P_2 \rangle,$$

$$P_1 = (5yx + 7y^3)\partial_y + 20x + 42y^2,$$

$$P_2 = (3500y^2 + 2058y)x\partial_x + ((-1000y - 735)x + 7203y^2)\partial_y - 4000x + 7700y^2 + 84378y.$$

$$Ann^{(2)} = \langle f_x, f_y, P \rangle,$$

$$\begin{aligned}
P = & ((68250000y - 24863426875)x^2 + (-37182906425y^2 - 10656175824y)x)\partial_x^2 \\
& + (-8437500x^2 + (637980000y + 694297170)x - 3708061182y^2)\partial_y\partial_x \\
& + ((-885281250y - 128629930625)x - 179156250y^3 - 111958930125y^2 - 54216894564y)\partial_x \\
& + (-18742500x + 154288260y)\partial_y^2 \\
& + (-25312500x - 511875000y^2 - 2118711000y + 2082891510)\partial_y \\
& - 5545968750y - 102242340375.
\end{aligned}$$

$$\begin{aligned}
\sigma = [& \frac{5^{15}}{3^8 7^{16}} \frac{1}{xy} - \frac{5^{13}}{3^7 7^{14}} \frac{1}{xy^2} + \frac{5^{11}}{3^6 7^{12}} \frac{1}{xy^3} - \frac{5^9}{3^5 7^{10}} \frac{1}{xy^4} + \frac{5^7}{3^4 7^8} \frac{1}{xy^5} - \frac{5^5}{3^3 7^6} \frac{1}{xy^6} + \frac{5^3}{3^2 7^4} \frac{1}{xy^7} \\
& - \frac{3 \cdot 7^2}{5^5} \frac{1}{x^2 y^8} - \frac{3^6 7^{11}}{5^3} \frac{1}{x^2 y} + \frac{3^5 7^9}{5} \frac{1}{x^2 y^2} - \frac{3^4 7^7}{5} \frac{1}{x^2 y^3} + \frac{3^3 7^5}{5^2} \frac{1}{x^2 y^4} - \frac{3^2 7^3}{5} \frac{1}{x^2 y^5} + \frac{1}{3 \cdot 7} \frac{1}{x^2 y^6} \\
& + \frac{3^5 7^6}{5^3} \frac{1}{x^3 y} - \frac{3^4 7^4}{5^3} \frac{1}{x^3 y^2} + \frac{3^3 7^2}{5} \frac{1}{x^3 y^3} - \frac{1}{3^3 7} \frac{1}{x^4 y}].
\end{aligned}$$

4.2 E_{13} 型特異点

$$f = x^3 + xy^5 + y^8,$$

$$f_x = 3x^2 + y^5, f_y = 5xy^4 + 8y^7.$$

$$Gb = \{4608x^4 + 125yx^3, 3x^2 + y^5, -24y^2x^2 + 5y^4x, 5x^3 + 8y^3x^2, (-192y^2 - 25y)x^3\}.$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle 192y + 25, 884736x - 3125 \rangle, \sqrt{I_1} = \langle 192y + 25, 884736x - 3125 \rangle,$$

$$I_2 = \langle yx^8, x^4, 3x^2 + y^5, 24y^2x^2 - 5y^4x, 5x^3 + 8y^3x^2 \rangle, \sqrt{I_2} = \langle y, x \rangle.$$

$$Ann^{(1)} = \langle f_x, f_y, P_1, P_2 \rangle,$$

$$P_1 = (5x^2 + 8y^3x)\partial_x + 15x + 16y^3,$$

$$P_2 = ((2688y^2 + 125y)x - 360y^4)\partial_x + (-240x + 50y^2)\partial_y + 5376y^2 + 575y.$$

$$Ann^{(2)} = \langle f_x, f_y, P \rangle,$$

$$\begin{aligned}
P = & ((7077888y - 4492800)x^2 + (-13639680y^3 - 139200y^2 + 71875y)x + 9000y^4)\partial_x^2 \\
& + ((1244160y + 24000)x + 28750y^2)\partial_y\partial_x \\
& + ((155713536y - 3041280)x - 43868160y^3 - 288000y^2 + 402500y)\partial_x \\
& + (42467328y^2 + 6773760y + 24000)\partial_y \\
& + 566231040y + 55710720. \\
\sigma = & \left[-\frac{2^{48}3^8}{5^{17}}\frac{1}{xy} + \frac{2^{42}3^7}{5^{15}}\frac{1}{xy^2} - \frac{2^{36}3^6}{5^{13}}\frac{1}{xy^3} + \frac{2^{30}3^5}{5^{11}}\frac{1}{xy^4} - \frac{2^{24}3^4}{5^9}\frac{1}{xy^5} + \frac{2^{18}3^3}{5^7}\frac{1}{xy^6} - \frac{2^{12}3^2}{5^5}\frac{1}{xy^7} \right. \\
& + \frac{2^63}{5^3}\frac{1}{xy^8} - \frac{1}{5}\frac{1}{xy^9} - \frac{2^{33}3^5}{5^{12}}\frac{1}{x^2y} + \frac{2^{27}3^4}{5^{10}}\frac{1}{x^2y^2} - \frac{2^{21}3^3}{5^8}\frac{1}{x^2y^3} + \frac{2^{15}3^2}{5^6}\frac{1}{x^2y^4} - \frac{2^93}{5^4}\frac{1}{x^2y^5} \\
& \left. + \frac{2^3}{5^2}\frac{1}{x^2y^6} - \frac{2359296}{5^7}\frac{1}{x^3y} + \frac{12288}{5^5}\frac{1}{x^3y^2} - \frac{2^6}{5^3}\frac{1}{x^3y^3} + \frac{1}{3 \cdot 5}\frac{1}{x^3y^4} - \frac{2^3}{3 \cdot 5^2}\frac{1}{x^4y} \right].
\end{aligned}$$

4.3 E_{14} 型特異点

$$\begin{aligned}
f &= x^3 + xy^6 + y^8, \\
f_x &= 3x^2 + y^6, \quad f_y = 6xy^5 + 8y^7. \\
\text{Gb} &= \{3x^3 + 4y^2x^2, 9yx^4 - 64yx^3, 9x^5 - 64x^4, 3x^2 + y^6, -4yx^2 + y^5x\}.
\end{aligned}$$

$$I = I_1 \cap I_2,$$

$$\begin{aligned}
I_1 &= \langle 9x - 64, 3y^2 + 16 \rangle, \quad \sqrt{I_1} = \langle 3y^2 + 16, 9x - 64 \rangle, \\
I_2 &= \langle 3x^3 + 4y^2x^2, yx^3, x^4, 3x^2 + y^6, 4yx^2 - y^5x \rangle, \quad \sqrt{I_2} = \langle y, x \rangle.
\end{aligned}$$

$$\text{Ann}^{(1)} = \langle f_x, f_y, P_1, P_2 \rangle,$$

$$\begin{aligned}
P_1 &= (3x^2 + 4y^2x)\partial_x + 9x + 8y^2, \\
P_2 &= ((9y^3 + 32y)x + 4y^5)\partial_x + (3y^4 + 16y^2)\partial_y + 42y^3 + 176y.
\end{aligned}$$

$$\text{Ann}^{(2)} = \langle f_x, f_y, P \rangle,$$

$$\begin{aligned}
P = & ((8064y^2 + 1600512)x^2 + 2076672y^2x)\partial_x^2 \\
& + (-3024yx^2 + 18432yx - 4096y^3)\partial_y\partial_x \\
& + (93312x^3 + (84807y^2 + 554688)x^2 + (-7452y^4 + 756288y^2 + 7117824)x \\
& + 20160y^6 - 29184y^4 + 6201344y^2)\partial_x \\
& + (-6912x - 9216y^2)\partial_y^2 \\
& + (31104yx^2 + (28269y^3 + 268272y)x - 2484y^5 + 380736y^3 - 368640y)\partial_y \\
& + 435456x^2 + (354294y^2 + 3050784)x - 17172y^4 + 4063680y^2 + 746496.
\end{aligned}$$

$$\begin{aligned}
\sigma = & \left[-\frac{3^4}{2^{21}}\frac{1}{xy} + \frac{3^3}{2^{17}}\frac{1}{xy^3} - \frac{3^2}{2^{13}}\frac{1}{xy^5} + \frac{3}{2^9}\frac{1}{xy^7} - \frac{1}{2^5}\frac{1}{xy^9} - \frac{3^2}{2^{15}}\frac{1}{x^2y} + \frac{3}{2^{11}}\frac{1}{x^2y^3} - \frac{1}{2^7}\frac{1}{x^2y^5} \right. \\
& \left. + \frac{1}{2^33}\frac{1}{x^2y^7} - \frac{1}{2^9}\frac{1}{x^3y} + \frac{1}{2^53}\frac{1}{x^3y^3} - \frac{1}{2^33^2}\frac{1}{x^4y} \right].
\end{aligned}$$

4.4 Z_{11} 型特異点

$$\begin{aligned}
f &= x^3y + xy^4 + y^5, \\
f_x &= 3x^2y + y^4, \quad f_y = x^3 + 4xy^3 + 5y^4. \\
\text{Gb} &= \{3yx^2 + y^4, x^3 - 15yx^2 + 4y^3x, -11yx^3 - 15y^2x^2, 58564x^5 - 37125x^4 - 759375y^2x^2\}.
\end{aligned}$$

$$I = I_1 \cap I_2,$$

$$\begin{aligned}
I_1 &= \langle 121y + 675, 1331x - 10125 \rangle, \quad \sqrt{I_1} = \langle 121y + 675, 1331x - 10125 \rangle, \\
I_2 &= \langle 3yx^2 + y^4, x^3 - 15yx^2 + 4y^3x, 11yx^3 + 15y^2x^2, 11x^4 + 225y^2x^2 \rangle, \quad \sqrt{I_2} = \langle y, x \rangle.
\end{aligned}$$

$$\text{Ann}^{(1)} = \langle f_x, f_y, P_1, P_2 \rangle,$$

$$\begin{aligned}
P_1 &= (33x^2 + 45yx)\partial_x + (22yx + 30y^2)\partial_y + 187x + 240y, \\
P_2 &= (270x^2 + (-792y^2 - 3150y)x - 220y^3)\partial_x + (345yx - 528y^3 - 2475y^2)\partial_y + 1770x - 4488y^2 - 18675y.
\end{aligned}$$

$$\text{Ann}^{(2)} = \langle f_x, f_y, P \rangle,$$

$$\begin{aligned}
P = & (1621488000x^2 + 2211120000yx)\partial_x^2 \\
& + (-1507537185x^2 - 463117050yx + 2171748375y^2)\partial_y\partial_x \\
& + (833546736x^2 + (-3104767072y - 8186767725)x - 746016700y^2 + 17492101875y)\partial_x \\
& + (63147480x^2 - 1332723150yx - 1934772750y^2)\partial_y^2 \\
& + ((555697824y - 10098876810)x - 2041779168y^2 - 29844557550y)\partial_y \\
& + 4723431504x - 17355122928y - 112679770725.
\end{aligned}$$

$$\sigma = \left[\frac{11^{11}}{3^{17}5^{12}} \frac{1}{xy} - \frac{11^9}{3^{14}5^{10}} \frac{1}{xy^2} + \frac{11^7}{3^{11}5^8} \frac{1}{xy^3} - \frac{11^5}{3^85^6} \frac{1}{xy^4} + \frac{11^3}{3^55^4} \frac{1}{xy^5} - \frac{11}{3^25^2} \frac{1}{xy^6} + \frac{11^8}{3^{18}5^9} \frac{1}{x^2y} - \frac{11^6}{3^{10}5^7} \frac{1}{x^2y^2} \right. \\ \left. + \frac{11^4}{3^75^5} \frac{1}{x^2y^3} - \frac{11^2}{3^45^3} \frac{1}{x^2y^4} + \frac{1}{3 \cdot 5} \frac{1}{x^2y^5} + \frac{11^5}{3^95^6} \frac{1}{x^3y} - \frac{11^3}{3^65^4} \frac{1}{x^3y^2} + \frac{11}{3^35^2} \frac{1}{x^3y^3} \right. \\ \left. + \frac{11^2}{3^65^3} \frac{1}{x^4y} - \frac{1}{3^25} \frac{1}{x^4y^2} - \frac{1}{3} \frac{1}{x^5y} \right].$$

4.5 Z_{12} 型特異点

$$f = x^3y + xy^4 + x^2y^3,$$

$$f_1 = 3x^2y + y^4 + 2xy^3, f_2 = x^3 + 4xy^3 + 3x^2y^2.$$

$$\text{Gb} = \{3yx^2 + y^4 + 2y^3x, x^3 + (3y^2 - 6y)x^2 - 2y^4, (-14y + 5)x^3 - 30yx^2 - 5y^5 - 10y^4, \\ (105y - 55)x^4 + 484yx^3, 735x^5 + 605x^4 - 5324yx^3\}.$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle 35y - 121, 245x + 1331 \rangle, \sqrt{I_1} = \langle 35y - 121, 245x + 1331 \rangle,$$

$$I_2 = \langle 3yx^2 + y^4 + 2y^3x, x^3 + (3y^2 - 6y)x^2 - 2y^4, 5x^4 - 44yx^3, (14y - 5)x^3 + 30yx^2 + 5y^5 + 10y^4 \rangle,$$

$$\sqrt{I_2} = \langle y, x \rangle.$$

$$\text{Ann}^{(1)} = \langle f_x, f_y, P_1, P_2 \rangle,$$

$$P_1 = (140x^2 + 248yx + 44y^2)\partial_x + ((70y + 33)x + 125y^2)\partial_y + 770x + 1158y,$$

$$P_2 = ((84y + 124)x^2 + (-672y^2 + 2770y)x - 42y^3 - 176y^2)\partial_x \\ + ((42y^2 - 169y - 132)x - 609y^3 + 2008y^2)\partial_y + (462y + 340)x - 4914y^2 + 16686y.$$

$$\text{Ann}^{(2)} = \langle f_x, f_y, P \rangle,$$

$$P = (194685857871120x^2 + 420775133985360yx + 180463194172800y^2)\partial_x^2 \\ + (-82832632888320x^2 + (-9751678231680y - 180463194172800)x + 107193098960640y^2)\partial_y\partial_x \\ + (-133092828502732x^3 + (1373829024443004y - 3548483873971782)x^2 \\ + (-315326509783170y^2 + 2838895228228440y + 76535249452200)x \\ - 696421142828840y^3 - 81448893497400y^2 + 1647347302339200y)\partial_x \\ + ((-92078741076840y + 29963270244600)x - 70175382520320y^2 - 210540393201600y)\partial_y^2 \\ + ((-66546414251366y + 350146787028213)x^2 \\ + (1071890730792481y^2 - 3093414693538428y - 755979214347180)x \\ - 325846150902008y^3 + 2066998836578100y^2 - 1513707145196400y - 1864786339785600)\partial_y \\ - 732010556765026x^2 + (7765120865745538y - 20427450896819658)x \\ - 3303190350044904y^2 + 17240073586133520y - 6930858351546360).$$

$$\sigma = \left[\frac{5^57^7}{11^{13}} \frac{1}{xy} + \frac{5^47^6}{11^{11}} \frac{1}{xy^2} + \frac{5^37^5}{11^9} \frac{1}{xy^3} + \frac{5^27^4}{11^7} \frac{1}{xy^4} + \frac{5 \cdot 7^3}{11^5} \frac{1}{xy^5} + \frac{7^2}{11^3} \frac{1}{xy^6} - \frac{3}{11} \frac{1}{xy^7} \right. \\ \left. - \frac{5^47^5}{11^{10}} \frac{1}{x^2y} - \frac{5^37^4}{11^8} \frac{1}{x^2y^2} - \frac{5^27^3}{11^6} \frac{1}{x^2y^3} - \frac{5 \cdot 7^2}{11^4} \frac{1}{x^2y^4} - \frac{7}{11^2} \frac{1}{x^2y^5} \right. \\ \left. + \frac{5^37^3}{11^7} \frac{1}{x^3y} + \frac{5^27^2}{11^5} \frac{1}{x^3y^2} + \frac{5 \cdot 7}{11^3} \frac{1}{x^3y^3} + \frac{1}{11} \frac{1}{x^3y^4} - \frac{5^27}{11^4} \frac{1}{x^4y} - \frac{5}{11^2} \frac{1}{x^4y^2} - \frac{2^2}{11} \frac{1}{x^5y} \right].$$

4.6 Z_{13} 型特異点

$$f = x^3y + y^6 + xy^5,$$

$$f_x = 3x^2y + y^5, f_2 = x^3 + 6y^5 + 5xy^4.$$

$$\text{Gb} = \{-7yx^3 - 9y^2x^2, 3yx^2 + y^5, x^3 - 18yx^2 + 5y^4x, -343x^5 - 729y^3x^2\}.$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle 7x + 9y, 49y^2 + 243 \rangle, \sqrt{I_1} = \langle 49y^2 + 243, 7x + 9y \rangle,$$

$$I_2 = \langle 7yx^3 + 9y^2x^2, 7x^4 + 162y^2x^2, 3yx^2 + y^5, x^3 - 18yx^2 + 5y^4x, y^3x^2 \rangle, \sqrt{I_2} = \langle y, x \rangle.$$

$$\text{Ann}^{(1)} = \langle f_x, f_y, P_1, P_2 \rangle,$$

$$P_1 = (14x^2 + 18yx)\partial_x + (7yx + 9y^2)\partial_y + 77x + 90y,$$

$$P_2 = (72x^2 + (-280y^3 - 891y)x - 105y^4)\partial_x + (99yx - 140y^4 - 567y^2)\partial_y + 486x - 1540y^3 - 5184y.$$

$$\text{Ann}^{(2)} = \langle f_x, f_y, P \rangle,$$

$$\begin{aligned}
P = & (3662593200x^2 + 4709048400yx)\partial_x^2 \\
& + (-2683171008x^2 + 1890641088yx + 6866270208y^2)\partial_y\partial_x \\
& + (-42710470250x^3 - 105737060100yx^2 + (-97769239650y^2 - 90813857211)x \\
& - 7368256665y^3 + 55324766448y)\partial_x \\
& + (262919790x^2 - 3426024924yx - 4839511698y^2)\partial_y^2 \\
& + (-21355235125yx^2 + (-52868530050y^2 - 19843900587)x - 48884619825y^3 - 1076714 \\
& - 234907586375x^2 - 554097099675yx - 502429306950y^2 - 667382260608. \\
\sigma = & \left[\frac{7^7}{2 \cdot 3^{19}} \frac{1}{xy} - \frac{7^5}{2 \cdot 3^{14}} \frac{1}{xy^3} + \frac{7^3}{2 \cdot 3^9} \frac{1}{xy^5} - \frac{7}{2 \cdot 3^4} \frac{1}{xy^7} + \frac{7^4}{2 \cdot 3^{12}} \frac{1}{x^2y^2} - \frac{7^2}{2 \cdot 3^7} \frac{1}{x^2y^4} + \frac{1}{2 \cdot 3^2} \frac{1}{x^2y^6} \right. \\
& \left. - \frac{7^3}{2 \cdot 3^{10}} \frac{1}{x^3y} + \frac{7}{2 \cdot 3^5} \frac{1}{x^3y^3} - \frac{1}{2 \cdot 3^3} \frac{1}{x^4y^2} - \frac{1}{3} \frac{1}{x^5y} \right].
\end{aligned}$$

4.7 W_{12} 型特異点

$$\begin{aligned}
f &= x^4 + y^5 + x^2y^3, \\
f_x &= 4x^3 + 2xy^4, \quad f_y = 5y^4 + 3x^2y^2. \\
\text{Gb} &= \{3y^2x^2 + 5y^4, 2x^3 + y^4x, -9x^5 - 50x^3, 6x^4 - 5y^6\}. \\
I &= I_1 \cap I_2, \\
I_1 &= \langle 3y - 10, 27x^2 + 500 \rangle, \quad \sqrt{I_1} = \langle 3y - 10, 27x^2 + 500 \rangle, \\
I_2 &= \langle 2x^3 + y^3x, 3y^2x^2 + 5y^4, yx^3, 6x^4 - 5y^5, x^5 \rangle, \quad \sqrt{I_2} = \langle y, x \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{Ann}^{(1)} &= \langle f_x, f_y, P_1, P_2 \rangle, \\
P_1 &= ((12y - 30)x^2 + 5y^3)\partial_x + (9y^2 - 30y)x\partial_y + (78y - 210)x, \\
P_2 &= (12x^3 + (5y^2 + 50y)x)\partial_x + (9yx^2 + 50y^2)\partial_y + 78x^2 + 350y.
\end{aligned}$$

$$\begin{aligned}
\text{Ann}^{(2)} &= \langle f_x, f_y, P \rangle, \\
P &= (12000x^2 + 20000y^2)\partial_x^2 + (144000y - 480000)x\partial_y\partial_x \\
&+ (-291519x^3 + (-81675y^2 - 1009800y - 933000)x)\partial_x \\
&+ (-45000x^2 + 69000y^2 - 480000y)\partial_y^2 \\
&+ ((-210681y + 106110)x^2 - 723600y^2 + 240000y - 3840000)\partial_y \\
&- 1878957x^2 - 5788800y - 5499000.
\end{aligned}$$

$$\begin{aligned}
\sigma = & \left[-\frac{3^6}{10^7} \frac{1}{xy} - \frac{2^5}{10^6} \frac{1}{xy^2} - \frac{3^4}{10^5} \frac{1}{xy^3} - \frac{3^3}{10^4} \frac{1}{xy^4} - \frac{3^2}{10^3} \frac{1}{xy^5} - \frac{3}{10^2} \frac{1}{xy^6} \right. \\
& \left. + \frac{3^3}{2 \cdot 10^4} \frac{1}{x^3y} + \frac{1}{2 \cdot 10^3} \frac{1}{x^3y^2} + \frac{3}{2 \cdot 10^2} \frac{1}{x^3y^3} + \frac{1}{2 \cdot 10} \frac{1}{x^3y^4} - \frac{1}{2^2 10} \frac{1}{x^5y} \right].
\end{aligned}$$

4.8 W_{13} 型特異点

$$\begin{aligned}
f &= x^4 + xy^4 + y^6, \\
f_x &= 4x^3 + y^4, \quad f_y = 4xy^3 + 6y^5. \\
\text{Gb} &= \{4x^3 + y^4, -6yx^3 + y^3x, -9x^6 - x^5\}. \\
I &= I_1 \cap I_2, \\
I_1 &= \langle 9x + 1, 27y^2 - 2 \rangle, \quad \sqrt{I_1} = \langle 27y^2 - 2, 9x + 1 \rangle, \\
I_2 &= \langle 4x^3 + y^4, 6yx^3 - y^3x, y^3x^2, x^5 \rangle, \quad \sqrt{I_2} = \langle y, x \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{Ann}^{(1)} &= \langle f_x, f_y, P_1, P_2 \rangle, \\
P_1 &= (2x^2 + 3y^2x)\partial_x + 8x + 9y^2, \\
P_2 &= (270yx^2 + 8yx - 33y^3)\partial_x + (-36x^2 + 6y^2)\partial_y + 810yx + 50y.
\end{aligned}$$

$$\begin{aligned}
\text{Ann}^{(2)} &= \langle f_x, f_y, P \rangle, \\
P &= (4032x^2 + (-5778y^2 + 896)x + 2673y^4 - 168y^2)\partial_x^2 + (4512yx - 2304y^3 + 672y)\partial_x\partial_y \\
&+ ((192456y^2 + 34272)x - 31068y^2 + 6496)\partial_x + (336x + 504y^2)\partial_y^2 + (96228y^3 - 600y)\partial_y \\
&+ 1058508y^2 - 22680.
\end{aligned}$$

$$\begin{aligned}
\sigma = & \left[-\frac{3^9}{2^5} \frac{1}{xy} - \frac{3^6}{2^4} \frac{1}{xy^3} - \frac{3^3}{2^3} \frac{1}{xy^5} - \frac{1}{2^2} \frac{1}{xy^7} + \frac{3^7}{2^5} \frac{1}{x^2y} + \frac{3^4}{2^4} \frac{1}{x^2y^3} + \frac{3}{2^3} \frac{1}{x^2y^5} - \frac{3^5}{2^5} \frac{1}{x^3y} \right. \\
& \left. - \frac{3^2}{2^4} \frac{1}{x^3y^3} + \frac{3^3}{2^5} \frac{1}{x^4y} + \frac{1}{2^4} \frac{1}{x^4y^3} - \frac{3}{2^5} \frac{1}{x^5y} \right].
\end{aligned}$$

4.9 Q_{10} 型特異点

$$f = x^3 + y^4 + yz^2 + xy^3,$$

$$f_x = 3x^2 + y^3, f_y = 4y^3 + z^2 + 3xy^2, f_z = 2yz.$$

$$Gb = \{zy, z^3, 3x^2 + y^3, -12x^2 + 3y^2x + z^2, zx^2, -3x^3 - 4yx^2\}.$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle zy, z^3, zx^2, 3x^2 + y^3, 12x^2 - 3y^2x - z^2, 16yx^2 + z^2x, 12x^3 - z^2x \rangle, \sqrt{I_1} = \langle z, y, x \rangle,$$

$$I_2 = \langle z, 3y + 16, 9x - 64 \rangle, \sqrt{I_2} = \langle z, 3y + 16, 9x - 64 \rangle.$$

$$Ann^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3 \rangle,$$

$$P_1 = 2zx\partial_x + 2zy\partial_y + 3z^2\partial_z + 15z,$$

$$P_2 = (18x^2 + 24yx)\partial_x + (12yx + 16y^2)\partial_y + (21zx + 24zy)\partial_z + 111x + 136y,$$

$$P_3 = (162x^3 + (72y - 1152)x^2 - 512yx - 32z^2)\partial_x + (108yx^2 - 768yx)\partial_y + (189zx^2 - 1344zx)\partial_z + 999x^2 + (144y - 7104)x + 192y^2 - 1024y.$$

$$Ann^{(2)} = \langle f_x, f_y, f_z, P \rangle,$$

$$P = (1536x - 384y^2)\partial_x^2 + (768x + 1024y)\partial_y\partial_x + 1536z\partial_z\partial_x + (-1458x - 1080y + 10240)\partial_x + (432x + 486y^2 + 3168y)\partial_y^2 + 4224z\partial_z\partial_y + (3888y + 21696)\partial_y + (1296x^2 - 2304y^2)\partial_z^2 + 729z\partial_z + 2187.$$

$$\sigma = \left[-\frac{3^4}{2^{21}} \frac{1}{xyz} + \frac{3^3}{2^{17}} \frac{1}{xy^2z} - \frac{3^2}{2^{13}} \frac{1}{xy^3z} + \frac{3}{2^9} \frac{1}{xy^4z} - \frac{1}{2^5} \frac{1}{xy^5z} - \frac{3^2}{2^{15}} \frac{1}{x^2yz} - \frac{1}{2 \cdot 3} \frac{1}{x^2yz^3} + \frac{3}{2^{11}} \frac{1}{x^2y^2z} - \frac{1}{2^7} \frac{1}{x^2y^3z} + \frac{1}{2^3} \frac{1}{x^2y^4z} - \frac{1}{2^9} \frac{1}{x^3yz} + \frac{1}{2^5} \frac{1}{x^3y^2z} - \frac{1}{2^3} \frac{1}{x^4yz} \right].$$

4.10 Q_{11} 型特異点

$$f = x^3 + y^2z + xz^3 + z^5$$

$$f_x = 3x^2 + z^3, f_y = 2yz, f_z = 3xz^2 + y^2 + 5z^4.$$

$$Gb = \{zy, 3x^2 + z^3, y^3, -15xz^2 + 3z^2x + y^2, yx^2, 375x^4 + 9x^3 - 5y^2x\}.$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle 25z + 3, y, 125x + 3 \rangle, \sqrt{I_1} = \langle 25z + 3, y, 125x + 3 \rangle,$$

$$I_2 = \langle zy, 3x^2 + z^3, 15xz^2 - 3z^2x - y^2, y^3, yx^2, 9x^3 - 5y^2x \rangle, \sqrt{I_2} = \langle z, y, x \rangle.$$

$$Ann^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3 \rangle,$$

$$P_1 = (120x^2 - 18zx + 10z^3)\partial_x + (120yx - 21zy)\partial_y + (60zx - 12z^2)\partial_z + 660x - 111z,$$

$$P_2 = ((400z^2 + 18z)x - 50z^3)\partial_x + 21zy\partial_y + (-60zx + 12z^2)\partial_z - 60x + 800z^2 + 111z,$$

$$P_3 = ((400z + 18)yx - 50z^2y)\partial_x + (-180zx^2 - 300z^3x + 21y^2)\partial_y + (-60yx + 12yz)\partial_z + 111y.$$

$$Ann^{(2)} = \langle f_x, f_y, f_z, P \rangle,$$

$$P = (162x + 270z^2)\partial_x\partial_x + (-1500x + 840z)\partial_x + (1425z^2 + 171z)x\partial_y^2 + 246y\partial_z\partial_y + 750y\partial_y + (285x + 1375z^2 + 108z)\partial_z^2 + (10000z + 1164)\partial_z + 6500.$$

$$\sigma = \left[\frac{5^{10}}{2 \cdot 3^6} \frac{1}{xyz} - \frac{5^8}{2 \cdot 3^5} \frac{1}{xy^2z} + \frac{5^6}{2 \cdot 3^4} \frac{1}{xyz^3} - \frac{5^4}{2 \cdot 3^3} \frac{1}{xyz^4} + \frac{5^2}{2 \cdot 3^2} \frac{1}{xyz^5} - \frac{1}{2 \cdot 3} \frac{1}{xyz^6} - \frac{1}{2 \cdot 3^5} \frac{1}{x^2yz} + \frac{1}{2 \cdot 3^4} \frac{1}{x^2yz^2} - \frac{1}{2 \cdot 3^3} \frac{1}{x^2yz^3} + \frac{1}{2 \cdot 3^2} \frac{1}{x^2yz^4} - \frac{1}{2 \cdot 3} \frac{1}{x^2yz^5} + \frac{1}{162} \frac{1}{x^3yz} - \frac{1}{2 \cdot 3^3} \frac{1}{x^3yz^2} + \frac{1}{2 \cdot 3^2} \frac{1}{x^3yz^3} - \frac{1}{2 \cdot 3^3} \frac{1}{x^4yz} \right].$$

4.11 Q_{12} 型特異点

$$f = x^3 + y^5 + yz^2 + xy^4,$$

$$f_x = 3x^2 + y^4, f_y = 5y^4 + z^2 + 4xy^3, f_z = 2yz.$$

$$Gb = \{zy, z^3, zx^2, -4x^3 - 5yx^2, 3x^2 + y^4, -15x^2 + 4y^3x + z^2\}$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle zy, z^3, zx^2, 75yx^2 + 4z^2x, 15x^3 - z^2x, 3x^2 + y^4, 15x^2 - 4y^3x - z^2 \rangle, \sqrt{I_1} = \langle z, y, x \rangle,$$

$$I_2 = \langle z, 4x + 5y, 16y^2 + 75 \rangle, \sqrt{I_2} = \langle z, 4x + 5y, 16y^2 + 75 \rangle.$$

$$Ann^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3 \rangle,$$

$$\begin{aligned}
P_1 &= zx\partial_x + zy\partial_y + 2z^2\partial_z + 9z, \\
P_2 &= (8x^2 + 10yx)\partial_x + (4yx + 5y^2)\partial_y + (10zx + 10zy)\partial_z + 50x + 55y, \\
P_3 &= ((192y^2 + 600)x^2 - 375yx - 20z^2)\partial_x + (-120x^2 + 96y^3x - 375y^2)\partial_y \\
&\quad + (240zy^2x - 40zy^3 - 750zy)\partial_z + (1200y^2 + 1200)x - 3375y.
\end{aligned}$$

$$Ann^{(2)} = \langle f_x, f_y, f_z, P \rangle,$$

$$\begin{aligned}
P &= (3750x - 1000y^3)\partial_x^2 + (1500x + 1875y)\partial_y\partial_x + 3750z\partial_z\partial_x + (-4608yx - 3360y^2 + 24375)\partial_x \\
&\quad + (720x + 1152y^3 + 6300y)\partial_y^2 + 11850z\partial_z\partial_y + (12672y^2 + 52950)\partial_y \\
&\quad + (2880yx^2 - 4500y^3)\partial_z^2 + 13824y.
\end{aligned}$$

$$\begin{aligned}
\sigma = [& -\frac{2^9}{3^3 5^6} \frac{1}{xy^2z} + \frac{2^5}{3^2 5^4} \frac{1}{xy^4z} - \frac{2}{3 \cdot 5^2} \frac{1}{xy^6z} + \frac{2^7}{3^3 5^5} \frac{1}{x^2yz} - \frac{1}{2 \cdot 3} \frac{1}{x^2yz^3} - \frac{2^3}{3^2 5^3} \frac{1}{x^2y^3z} \\
& + \frac{1}{2 \cdot 3 \cdot 5} \frac{1}{x^2y^5z} + \frac{2}{3^2 5^2} \frac{1}{x^3y^2z} - \frac{1}{2 \cdot 3^2 5} \frac{1}{x^4yz}].
\end{aligned}$$

4.12 S_{11} 型特異点

$$f = x^4 + y^2z + xz^2 + x^3z,$$

$$f_x = 4x^3 + z^2 + 3x^2z, f_y = 2yz, f_z = y^2 + 2xz + x^3.$$

$$\begin{aligned}
Gb = \{ & zy, y^3, (12y^2 + 15z^2 + 64z)x + 32y^2 - 8z^2, 3zx^2 - 8zx - 4y^2 + z^2, x^3 + 2zx + y^2, \\
& -512z^2x + 75z^4 + 64z^3, (-15z^3 - 64z^2)x + 8z^3 \}.
\end{aligned}$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle 25z + 128, y, 5x - 16 \rangle, \sqrt{I_1} = \langle 25z + 128, y, 5x - 16 \rangle,$$

$$I_2 = \langle zy, 8z^2x - z^3, 3zx^2 - 8zx - 4y^2 + z^2, y^3, (96y^2 + 512z)x + 256y^2 + 15z^3 - 64z^2, x^3 + 2zx + y^2, z^4 \rangle,$$

$$\sqrt{I_2} = \langle z, y, x \rangle.$$

$$Ann^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3 \rangle,$$

$$P_1 = (8yx + 2zy)\partial_x + 4y^2\partial_y + 3zyx\partial_z + 3yx + 36y,$$

$$P_2 = (64x^2 + 56zx + 10z^2)\partial_x + (-64xy + 20zy)\partial_y + (24zx^2 + 15z^2x + 96y^2)\partial_z + 24x^2 + 15zx + 84z,$$

$$P_3 = (16xz + 10z^2)\partial_x + (-96xy - 16yz)\partial_y + (15xz^2 + 96y^2 - 48z^2)\partial_z + (15z - 288)x - 168z.$$

$$Ann^{(2)} = \langle f_x, f_y, f_z, P \rangle,$$

$$\begin{aligned}
P &= (480x + 300z)\partial_x^2 - 288y\partial_x\partial_y + (-2048x - 1280z)\partial_z + (-144x - 90z - 608)\partial_x \\
&\quad + (-360zx + 1152z)\partial_y^2 - 2560y\partial_y\partial_z + 144y\partial_y + (-2304x^2 - 1344zx - 540z^2 - 3072z)\partial_z^2 \\
&\quad + ((-135z - 2112)x - 3168z - 19968)\partial_z - 135x - 2088.
\end{aligned}$$

$$\begin{aligned}
\sigma = [& -\frac{5^5}{2^{23}} \frac{1}{xyz} + \frac{5^3}{2^{16}} \frac{1}{xyz^2} - \frac{5}{2^9} \frac{1}{xyz^3} - \frac{1}{2^2} \frac{1}{xyz^4} - \frac{5^4}{2^{19}} \frac{1}{x^2yz} + \frac{5^2}{2^{12}} \frac{1}{x^2yz^2} - \frac{1}{2^5} \frac{1}{x^2yz^2} - \frac{5^3}{2^{15}} \frac{1}{x^3y^3z} \\
& + \frac{5}{2^8} \frac{1}{x^3yz^2} - \frac{1}{2^3} \frac{1}{x^3y^3z} - \frac{5^2}{2^{11}} \frac{1}{x^4yz} + \frac{1}{2^4} \frac{1}{x^4yz^2} - \frac{5}{2^7} \frac{1}{x^5yz}].
\end{aligned}$$

4.13 S_{12} 型特異点

$$f = x^2y + y^2z + xz^3 + z^5,$$

$$f_x = 2xy + z^3, f_y = x^2 + 2yz, f_z = y^2 + 3xz^2 + 5z^4.$$

$$\begin{aligned}
Gb = \{ & x^2 + 2zy, 2yx + z^3, (-10zy + 3z^2)x + y^2, -13y^2x - 20z^2y^2, -169y^2x - 800zy^3, 2197y^2x + 32000y^4, \\
& -4000y^3x + 130y^3 + 507zy^2 \}.
\end{aligned}$$

$$I = I_1 \cap I_2,$$

$$I_1 = \langle 40y - 13z, 400x - 169, 8000z^2 + 2197 \rangle, \sqrt{I_1} = \langle 8000z^2 + 2197, 40y - 13z, 400x - 169 \rangle,$$

$$I_2 = \langle x^2 + 2zyz, 2yx + z^3, (10zy - 3z^2)x - y^2, 10y^3 + 39zy^2, y^2x, z^2y^2 \rangle, \sqrt{I_2} = \langle z, y, x \rangle.$$

$$Ann^{(1)} = \langle f_x, f_y, f_z, P_1, P_2 \rangle,$$

$$\begin{aligned}
P_1 &= (104zx + 1800z^2y - 425z^3)\partial_x + ((600zy - 295z^2)x + 130zy)\partial_y + (-240zy + 78z^2)\partial_z \\
&\quad - 300zx - 30y + 702z,
\end{aligned}$$

$$\begin{aligned}
P_2 &= ((17000zy - 3000z^2)x - 1100y^2 - 7605zy + 1521z^2)\partial_x \\
&\quad + ((3000y - 3000z)x^2 + (-2535y + 1521z)x - 750z^2y)\partial_y \\
&\quad + (-2400yx + 1014y)\partial_z + 1170x + 11000zy + 1950z^2.
\end{aligned}$$

$$Ann^{(2)} = \langle f_x, f_y, f_z, P \rangle,$$

$$\begin{aligned}
P = & (4732x + 7280z^2)\partial_x^2 + (-4550zx + 5915y)\partial_x\partial_y + (-28700zx + 26390y + 3549z)\partial_x\partial_z \\
& + (904300x + 1500000z^2 + 36673)\partial_x + (-12600zy + 4095z^2)\partial_y\partial_z \\
& + ((-9000000y + 2055000z)x + 1244400y - 37310z)\partial_y + (9555x - 90000zy + 43950z^2)\partial_z^2 \\
& + (-3000000zx + 1020000y + 1012000z)\partial_z - 21000000x + 6916600. \\
\sigma = & \left[-\frac{2^{13}5^6}{13^7} \frac{1}{xy^2z^2} + \frac{2^75^3}{13^4} \frac{1}{xyz^4} - \frac{2}{13} \frac{1}{xyz^6} - \frac{2^{10}5^5}{13^6} \frac{1}{xy^2z} + \frac{2^45^2}{13^3} \frac{1}{xy^2z^3} + \frac{2 \cdot 5}{13^2} \frac{1}{xy^3z^2} - \frac{3}{13} \frac{1}{xy^4z} \right. \\
& \left. - \frac{2^95^4}{13^5} \frac{1}{x^2yz^2} + \frac{2^35}{13^2} \frac{1}{x^2yz^2} - \frac{2^65^3}{13^4} \frac{1}{x^2y^2z} + \frac{1}{13} \frac{1}{x^2y^2z^3} - \frac{2^55^2}{13^3} \frac{1}{x^3yz^2} - \frac{2^25}{13^2} \frac{1}{x^3y^2z} - \frac{2}{13} \frac{1}{x^4yz^2} \right].
\end{aligned}$$

4.14 U_{12} 型特異点

$$\begin{aligned}
f &= x^3 + y^3 + z^4 + xyz^2, \\
f_x &= 3x^2 + yz^2, f_y = 3y^2 + xz^2, f_z = 4z^3 + 2xyz. \\
\text{Gb} &= \{3x^2 + z^2y, z^2x + 3y^2, zyx + 2z^3, -x^3 + y^3, -3y^3 + 2z^4, -6y^2x + y^4, -6yx^2 + y^3x\}. \\
I &= I_1 \cap I_2 \cap I_3, \\
I_1 &= \langle x + y + 6, z^2 + 18, y^2 + 6y + 36 \rangle, \sqrt{I_1} = \langle z^2 + 18, x + y + 6, y^2 + 6y + 36 \rangle, \\
I_2 &= \langle y - 6, x - 6, z^2 + 18 \rangle, \sqrt{I_2} = \langle z^2 + 18, y - 6, x - 6 \rangle, \\
I_3 &= \langle 3x^2 + z^2y, z^2x + 3y^2, zyx + 2z^3, xz^2, y^2x, yx^2, x^3 - y^3, 3y^3 - 2z^4, y^4 \rangle, \sqrt{I_3} = \langle z, y, x \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{Ann}^{(1)} &= \langle f_x, f_y, f_z, P_1, P_2, P_3 \rangle, \\
P_1 &= (48x^2 - 11y^2x - 6z^2y)\partial_x + (60yx - 10y^3)\partial_y + (36zx - 6zy^2)\partial_z + 324x - 69y^2, \\
P_2 &= (4x^3 - 24yx)\partial_x + (5y^2x + 6z^2x - 12y^2)\partial_y + (3zx^2 - 18zy)\partial_z + 30x^2 - 126y, \\
P_3 &= (26zyx^2 - 576zx - 60zy^2)\partial_x + (29zy^2x + (18z^3 - 720z)y)\partial_y + ((16z^2 + 72)yx - 432z^2)\partial_z \\
&\quad - 362z^3 - 3888z.
\end{aligned}$$

$$\begin{aligned}
\text{Ann}^{(2)} &= \langle f_x, f_y, f_z, P \rangle, \\
P &= (864x - 144y^2)\partial_x\partial_y + (-2x^2 - 276y)\partial_x + (-144x^2 + 864y)\partial_y^2 + (24z^3 + 432z)\partial_y\partial_z \\
&\quad + (yx + 218z^2 + 5616)\partial_y + (-36x + 6y^2)\partial_z^2 - 3x. \\
\sigma &= \left[-\frac{1}{2^53^6} \frac{1}{xyz} + \frac{1}{2^43^4} \frac{1}{xyz^3} - \frac{1}{2^33^2} \frac{1}{xyz^5} - \frac{1}{2^23^3} \frac{1}{xy^4z} - \frac{1}{2^33^4} \frac{1}{x^2y^2z} + \frac{1}{2^23^2} \frac{1}{x^2y^2z^3} - \frac{1}{2^23^3} \frac{1}{x^4yz} \right].
\end{aligned}$$

5 Bimodal singularity E_{18} に関する計算

2 変数半擬斉次多項式 $f(x, y) = x^3 + y^{10} + xy^7 + xy^8$ の定める E_{18} 型 Bimodal 特異点に対し, $\text{Ann}^{(1)}$ を求め, §4 と同様の計算を行った.

$\text{Ann}^{(1)}$ を用いて代数的局所コホモロジー類 $[1/f_x f_y]_{(0,0)}$ を計算すると, 次の表現を得る.

$$\begin{aligned}
& \left[a \frac{1}{xy} + b \frac{1}{xy^2} + \frac{52326274982537898625543}{3^{10}10^{20}} \frac{1}{xy^3} + \frac{354285915106436807093}{3^9 10^{18}} \frac{1}{xy^4} \right. \\
& + \frac{272111019228806143}{3^8 10^{16}} \frac{1}{xy^5} - \frac{3787973405502307}{3^7 10^{14}} \frac{1}{xy^6} - \frac{13210502493257}{3^6 10^{12}} \frac{1}{xy^7} \\
& + \frac{14232708293}{3^5 10^{10}} \frac{1}{xy^8} + \frac{188727343}{3^4 10^8} \frac{1}{xy^9} + \frac{438893}{3^3 10^6} \frac{1}{xy^{10}} - \frac{2057}{3^2 10^4} \frac{1}{xy^{11}} \\
& - \frac{7}{3 \cdot 10^2} \frac{1}{xy^{12}} - \frac{24652664985303778499}{3^9 10^{17}} \frac{1}{x^2y} - \frac{75715525087212649}{3^8 10^{15}} \frac{1}{x^2y^2} \\
& + \frac{107457356825701}{3^7 10^{13}} \frac{1}{x^2y^3} + \frac{1264904961551}{3^6 10^{11}} \frac{1}{x^2y^4} + \frac{1815069901}{3^5 10^9} \frac{1}{x^2y^5} \\
& - \frac{11224249}{3^4 10^7} \frac{1}{x^2y^6} - \frac{45899}{3^3 10^5} \frac{1}{x^2y^7} - \frac{49}{3^2 10^3} \frac{1}{x^2y^8} + \frac{1}{30} \frac{1}{x^2y^9} - \frac{70850911193}{3^6 10^{10}} \frac{1}{x^3y} \\
& \left. - \frac{320395243}{3^5 10^8} \frac{1}{x^3y^2} + \frac{178207}{3^4 10^6} \frac{1}{x^3y^3} + \frac{4157}{3^3 10^4} \frac{1}{x^3y^4} + \frac{7}{3^2 10^2} \frac{1}{x^3y^5} - \frac{251}{3^3 10^3} \frac{1}{x^4y} - \frac{1}{3^2 10} \frac{1}{x^4y^2} \right].
\end{aligned}$$

E_{18} の場合, 1 階の微分作用素を用いた計算では $[1/xy]$ と $[1/xy^2]$ の係数 a, b を決めることはできないことが分かる. まとめると, 次の (i), (ii) を得る.

(i) ホロノミック系 $\mathcal{D}_X/\text{Ann}^{(1)}$ の原点における重複度 = 3

(ii) $\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}^{(1)}, \mathcal{H}_{[(0,0)]}^2(\mathcal{O}_X)) = \text{Span}\{[1/xy], [1/xy^2], \sigma\}$

このことは、特異点の”複雑さ”とホロノミック系 $\mathcal{D}_X/\text{Ann}^{(1)}$ の重複度の間には何らかの関係があることを示唆していると思われる。

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