

Gap Function Approach to Duality

— basic-model —

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Abstract

This paper discusses a class of pairs of quadratic optimization problem (primal) and its dual. The primal has a fixed initial state. We analyze both the problems through gap function method. A complete solution is given through characteristic equation. The model is based upon a complementary identity, which generates simultaneously a triplet of primal function, dual function and an equality condition.

1 Introduction

Recently, in [12–29], S.Iwamoto, Y.Kimura, T.Fujita and A.Kira show that a duality for paired optimization problems through several methods such as (i) extended Lagrangean, (ii) plus-minus, (iii) inequality, (iv) identity, (v) complementary and others. As a historical background, see Bellman and others [1–7, 30], [9, 11, 32, 33] for dynamic optimization.

In this paper, we propose a method through gap function to show a duality between a primal problem and its dual problem. Section 2 considers a basic pair of n -variable minimization (primal) problem (P_n) and maximization (dual) problem (D_n). Then we define a gap function and discuss duality. In section 3, we give the optimal solution (point and value) of (P_n) and (D_n). Section 4 presents a pair of minimization problem and maximization problem for 4-variable.

2 Basic-model

In this section we assume that n is a natural number and $c, (\in R^1)$ is a constant. c denotes an *initial state* at time 0 of a dynamic system.

As a basic pair of *primal*¹ and *dual*, we take n -variable optimization problems

$$\begin{array}{ll} P_n & \begin{array}{l} \text{minimize} \quad \sum_{k=1}^{n-1} [(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + x_n^2 \\ \text{subject to} \quad \text{(i) } x \in R^n, \quad \text{(ii) } x_0 = c \end{array} \end{array}$$

¹Two nouns *primal* and *dual* mean *primal problem* and *dual problem*, respectively.

$$\begin{array}{ll}
\text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \mu_n^2 \\
D_n & \text{subject to (i) } \mu \in R^n.
\end{array}$$

Let $f, g : R^n \rightarrow R^1$ be the respective objective functions of P_n, D_n :

$$\begin{aligned}
f(x) &= \sum_{k=1}^n [(x_{k-1} - x_k)^2 + x_k^2] \\
g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - 2\mu_n^2.
\end{aligned}$$

Note that $f(x)$ is convex and $g(\mu)$ is concave. Then it holds that

$$f(x) \geq g(\mu) \quad (x, \mu) \in R^n \times R^n. \quad (1)$$

The sign of equality holds iff a linear system of $2n$ -equation on $2n$ -variable

$$\begin{array}{llll}
c - x_1 = \mu_1 & x_1 = \mu_1 - \mu_2 & & \\
(EC_1) & x_{k-1} - x_k = \mu_k & x_k = \mu_k - \mu_{k+1} & 2 \leq k \leq n-1 \\
& x_{n-1} - x_n = \mu_n & x_n = \mu_n &
\end{array}$$

holds. (EC_1) is called an *equality condition* between P_n and D_n . Thus both problems are called *dual* of each other.

Lemma 1 *The equality condition (EC_1) yields a pair of linear systems of n -equation on n -variable:*

$$\begin{array}{llll}
& 3x_1 - x_2 = c & & \\
(EQ_x) & -x_{k-1} + 3x_k - x_{k+1} = 0 & 2 \leq k \leq n-1 & \\
& -x_{n-1} + 2x_n = 0, & & \\
& 2\mu_1 - \mu_2 = c & & \\
(EQ_\mu) & -\mu_{k-1} + 3\mu_k - \mu_{k+1} = 0 & 2 \leq k \leq n-1 & \\
& -\mu_{n-1} + 3\mu_n = 0. & &
\end{array}$$

2.1 Gap function for basic-model

First we present an identity, which takes a fundamental role in analyzing respective pairs of primal and dual. Let $x = \{x_k\}_0^n, \mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$(C_1) \quad \sum_{k=1}^{n-1} [(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n = c\mu_1$$

holds true. This identity is called *complementary*.

Now we derive both P_n and D_n through gap function. Let us define a *gap function* $h = h(x, \mu)$ between $x \in R^n$ and $\mu \in R^n$ by

$$\begin{aligned} h(x, \mu) = & \sum_{k=1}^{n-1} [(x_{k-1} - x_k - \mu_k)^2 + \{x_k - (\mu_k - \mu_{k+1})\}^2] \\ & + [(x_{n-1} - x_n - \mu_n)^2 + (x_n - \mu_n)^2]. \end{aligned} \quad (2)$$

Thus $h(x, \mu)$ denotes a *total difference* between x and μ . It turns out that the quadratic function $h = h(x, \mu)$ is convex in (x, μ) .

Lemma 2

- (i) $f(x) - g(\mu) = h(x, \mu) \geq 0 \quad \forall (x, \mu) \in R^n \times R^n$
- (ii) $h(x, \mu) = 0 \implies (x, \mu) \text{ satisfies } (EC_1).$

Theorem 1 (i) *It holds that*

$$f(x) \geq g(\mu) \quad \text{on } R^n \times R^n.$$

(ii) *It holds that*

$$f(x) = g(\mu) \iff (x, \mu) \text{ satisfies } (EC_1).$$

Then P_n attains a minimum $f(x)$, while D_n attains a maximum $g(\mu)$.

Hence a solution (x, μ) to (EC_1) yields a minimum point x for P_n and a maximum point μ for D_n .

Theorem 2 *Let (x, μ) satisfy (EC_1) . Then both sides become a common value with five expressions:*

$$\begin{aligned} (5V_1) \quad & f(x) = c(c - x_1) \\ & = g(\mu) = \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + 2\mu_n^2 = c\mu_1. \end{aligned}$$

The primal P_n has a minimum value

$$m = f(x) = c(c - x_1)$$

at x , while the dual D_n has a maximum value

$$M = g(\mu) = \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + 2\mu_n^2 = c\mu_1$$

at μ .

Hence a solution (x, μ) to (EC_1) yields a minimum value $f(x) = c(c - x_1)$ for P_n and a maximum value $g(\mu) = c\mu_1$ for D_n . Thus *the first argument of optimal point characterizes the common optimum value.*

2.2 Characteristic equation for basic-model

Now let us solve the pair of linear systems (EQ_x) and (EQ_μ). We introduce a second-order linear difference equation

$$x_{n+2} - 3x_{n+1} + x_n = 0, \quad x_1 = 1, \quad x_0 = 0 \quad (3)$$

Lemma 3 *The Eq (3) has a unique solution*

$$x_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} \quad (4)$$

where $\alpha (<) \beta$ are two positive solution

$$\alpha = \frac{3 - \sqrt{5}}{2}, \quad \beta = \frac{3 + \sqrt{5}}{2} \quad (5)$$

to the associated characteristic equation

$$(CE) \quad t^2 - 3t + 1 = 0. \quad (6)$$

We note that

$$\alpha = \bar{\phi}^2, \quad \beta = \phi^2 \quad (7)$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \bar{\phi} = \frac{1 - \sqrt{5}}{2} \quad (8)$$

are positive and negative solutions to a quadratic equation

$$t^2 - t - 1 = 0.$$

Both ϕ and $\bar{\phi}$ are called the *Golden number* and its *conjugate*, respectively. It holds that

$$\phi + \bar{\phi} = 1, \quad \phi\bar{\phi} = -1,$$

$$\alpha + \beta = 3, \quad \alpha\beta = 1.$$

Definition 1 *Let us define the sequence $\{K_n\}$ by*

$$K_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}. \quad (9)$$

We call $\{K_n\}$ a *Kibonacci* sequence. Thus $\{K_n\}$ satisfies a second-order linear difference equation

$$K_{n+1} = 3K_n - K_{n-1}, \quad K_1 = 1, \quad K_0 = 0. \quad (10)$$

This has a unique solution (9). The solution — Kibonacci number — K_n turns out a *two-step Fibonacci* number.

Lemma 4

$$K_n = F_{2n}. \quad (11)$$

Proof. $K_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} = \frac{\phi^{2n} - \bar{\phi}^{2n}}{\phi^2 - \bar{\phi}^2} = \frac{\phi^{2n} - \bar{\phi}^{2n}}{\phi - \bar{\phi}} = F_{2n}. \quad \square$

We remark that *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \quad x_1 = 1, \quad x_0 = 0. \quad (12)$$

Hence

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\phi - \bar{\phi}}. \quad (13)$$

n	$\cdots$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
F_n	$\cdots$	-1	1	0	1	1	2	3	5	8	13	21	34	55	$\cdots$

Table 1 Fibonacci sequence $\{F_n\}$

Lemma 5 *The system (EQ_x) has a unique solution*

$$x_k = c \frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} = c \frac{F_{2n+1-2k}}{F_{2n+1}} \quad 0 \leq k \leq n$$

, while the system (EQ_μ) has a unique solution

$$\mu_k = c \frac{K_{n+1-k}}{2K_n - K_{n-1}} = c \frac{F_{2n+2-2k}}{F_{2n+1}} \quad 1 \leq k \leq n.$$

We see that the x, μ satisfy the equality condition (EC_1) .

Theorem 3 *The equality condition (EC_1) has a unique solution (x, μ) ;*

$$x_k = c \frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} = c \frac{F_{2n+1-2k}}{F_{2n+1}} \quad 0 \leq k \leq n$$

$$\mu_k = c \frac{K_{n+1-k}}{2K_n - K_{n-1}} = c \frac{F_{2n+2-2k}}{F_{2n+1}}. \quad 1 \leq k \leq n.$$

where

$$K_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} = F_{2n}.$$

Hence the gap function h attains the zero minimum at (x, μ) .

3 Primal vs dual for basic-model

Let us consider the paired n -variable problem:

$$\begin{array}{ll} P_n & \text{minimize } f(x) \\ & \text{subject to (i) } x \in R^n, \text{ (ii) } x_0 = c \end{array}$$

$$\begin{array}{ll} D_n & \text{Maximize } g(\mu) \\ & \text{subject to (i) } \mu \in R^n \end{array}$$

Lemma 6 (i) *Let x be a minimum point for P_n . Then x satisfies*

Case $n = 1$

$$(EQ_x) \quad 2x_1 = c$$

where

$$f(x_1) = (c - x_1)^2 + x_1^2.$$

Case $n = 2$

$$(EQ_x) \quad \begin{array}{l} 3x_1 - x_2 = c \\ -x_1 + 2x_2 = d \end{array}$$

where

$$f(x_1, x_2) = [(c - x_1)^2 + x_1^2] + [(x_1 - x_2)^2 + x_2^2].$$

Case $n \geq 3$

$$(EQ_x) \quad \begin{array}{l} 3x_1 - x_2 = c \\ -x_{k-1} + 3x_k - x_{k+1} = 0 \quad 2 \leq k \leq n-1 \\ -x_{n-1} + 2x_n = 0, \end{array}$$

The minimum value $f(x)$ is given by $f(x) = c(c - x_1)$.

(ii) *Let μ be a maximum point for D_n . Then μ satisfies*

Case $n = 1$

$$(EQ_\mu) \quad 2\mu_1 = c \tag{14}$$

where

$$g(\mu_1) = 2c\mu_1 - 2\mu_1^2.$$

Case $n = 2$

$$\begin{aligned} (\text{EQ}_\mu) \quad & 2\mu_1 - \mu_2 = c \\ & -\mu_1 + 3\mu_2 = 0 \end{aligned} \tag{15}$$

where

$$g(\mu_1, \mu_2) = 2c\mu_1 - [\mu_1^2 + (\mu_1 - \mu_2)^2] - 2\mu_2^2.$$

Case $n \geq 3$

$$\begin{aligned} (\text{EQ}_\mu) \quad & 2\mu_1 - \mu_2 = c \\ & -\mu_{k-1} + 3\mu_k - \mu_{k+1} = 0 \quad 2 \leq k \leq n-1 \\ & -\mu_{n-1} + 3\mu_n = 0 \end{aligned}$$

The maximum value $g(\mu)$ is given by $g(\mu) = \lambda c\mu_1$.

Theorem 4 The primal P_n attains a minimum

$$m = f(x) = c(c - x_1) = c^2 \left(1 - \frac{K_n - K_{n-1}}{K_{n+1} - K_n} \right) = c^2 \frac{F_{2n}}{F_{2n+1}}$$

at x ;

$$x_k = c \frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} = c \frac{F_{2n+1-2k}}{F_{2n+1}} \quad 0 \leq k \leq n.$$

The dual D_n attains a maximum

$$M = g(\mu) = c\mu_1 = c^2 \frac{K_n}{2K_n - K_{n-1}} = c^2 \frac{F_{2n}}{F_{2n+1}}$$

at μ ;

$$\mu_k = c \frac{K_{n+1-k}}{2K_n - K_{n-1}} = c \frac{F_{2n+2-2k}}{F_{2n+1}} \quad 1 \leq k \leq n.$$

Both the optima are equal.

We remark that

$$\begin{aligned} & K_{n+1} - 3K_n + K_{n-1} = 0 \\ \text{i.e.} \quad & K_{n+1} - K_n - (K_n - K_{n-1}) = K_n \end{aligned}$$

yields

$$1 - \frac{K_n - K_{n-1}}{K_{n+1} - K_n} = \frac{K_n}{2K_n - K_{n-1}} = \frac{F_{2n}}{F_{2n+1}}.$$

4 Four-variable pair for basic-model

The primal problem

$$\begin{aligned}
 & \text{minimize} \quad (c - x_1)^2 + x_1^2 + [(x_1 - x_2)^2 + x_2^2] \\
 P_4 \quad & + [(x_2 - x_3)^2 + x_3^2] + [(x_3 - x_4)^2 + x_4^2] \\
 & \text{subject to} \quad (i) \ (x_1, x_2, x_3, x_4) \in R^4, \quad (ii) \ x_0 = c.
 \end{aligned}$$

has a minimum value

$$m_4 = c(c - \hat{x}_1) = \frac{21}{34} c^2$$

at a point $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) :$

$$\hat{x}_1 = \frac{13}{34} c, \quad \hat{x}_2 = \frac{5}{34} c, \quad \hat{x}_3 = \frac{2}{34} c, \quad \hat{x}_4 = \frac{1}{34} c.$$

The dual problem

$$\begin{aligned}
 & \text{Maximize} \quad 2c\mu_1 - [\mu_1^2 + (\mu_1 - \mu_2)^2] - [\mu_2^2 + (\mu_2 - \mu_3)^2] \\
 D_4 \quad & - [\mu_3^2 + (\mu_3 - \mu_4)^2] - 2\mu_4^2 \\
 & \text{subject to} \quad (i) \ (\mu_1, \mu_2, \mu_3, \mu_4) \in R^4
 \end{aligned}$$

has a maximum value

$$M_4 = \mu_1^* c = \frac{\rho^3 + 4\rho^2 + 8\rho + 8}{34} c^2$$

at a point $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*) :$

$$\mu_1^* = \frac{21}{34} c, \quad \mu_2^* = \frac{8}{34} c, \quad \mu_3^* = \frac{3}{34} c, \quad \mu_4^* = \frac{1}{34} c$$

Note that the equality condition EC₁ for 4-variable pair P₄, D₄ has a unique solution $(x, \mu) ;$

$$x_k = c \frac{K_{5-k} - K_{4-k}}{2K_4 - K_3} \quad 1 \leq k \leq 4$$

$$\mu_k = c \frac{K_{5-k}}{2K_4 - K_3} \quad 1 \leq k \leq 4.$$

We show how both the minimum point $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$ and the maximum point $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$ are obtained. First we have

$$\begin{aligned}
 2K_4 - K_3 &= 2 \cdot 21 - 8 \\
 &= 34
 \end{aligned}$$

that is

$$2K_4 - K_3 = 34. \quad (16)$$

Further

$$K_4 - K_3 = 13$$

$$K_3 - K_2 = 5$$

$$K_2 - K_1 = 2$$

$$K_1 - K_0 = 1.$$

Here

$$K_2 = 3, \quad K_3 = 8, \quad K_4 = 21.$$

Thus we obtain the desired minimum point :

$$\hat{x}_1 = c \frac{K_4 - K_3}{2K_4 - K_3} = \frac{13}{34} c$$

$$\hat{x}_2 = \frac{c}{\rho} \cdot \frac{K_3 - K_2}{2K_4 - K_3} = \frac{5}{34} c$$

$$\hat{x}_3 = \frac{c}{\rho^2} \cdot \frac{K_2 - K_1}{2K_4 - K_3} = \frac{2}{34} c$$

$$\hat{x}_4 = \frac{c}{\rho^3} \cdot \frac{K_1 - K_0}{2K_4 - K_3} = \frac{1}{34} c.$$

In a similar way, the desired maximum point is obtained as follows.

$$\mu_1^* = c \frac{K_4}{2K_4 - K_3} = \frac{21}{34} c$$

$$\mu_2^* = \frac{c}{\rho} \cdot \frac{K_3}{2K_4 - K_3} = \frac{8}{34} c$$

$$\mu_3^* = \frac{c}{\rho^2} \cdot \frac{K_2}{2K_4 - K_3} = \frac{3}{34} c$$

$$\mu_4^* = \frac{c}{\rho^3} \cdot \frac{K_1}{2K_4 - K_3} = \frac{1}{34} c.$$

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