

AN APPLICATION OF THE SECOND MICROLOCALIZATION AT THE BOUNDARY TO THE EXTENSION OF SOLUTIONS OF DIFFERENTIAL SYSTEMS

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ABSTRACT We use the theory of microlocalization of sheaves of [K-S-2], and especially its formulation in [S] for boundary value problems to treat the extension of regular solutions of systems of P.D.E. across an 1-codimensional singular set. Let M be a real analytic manifold, X a complexification of M , N an analytic hypersurface of M , Ω an open component of $M \setminus N$. For a suitable involutive manifold $V \subset T_M^*X$, invariant under the Hamiltonian flow of $N \times T_M^*X$, we introduce a new complex $B_{\Omega|X}^a$ of hyperfunctions in Ω with real analytic parameters and study its applications to non-characteristic boundary value problems. In particular we show that the trace morphism preserves the analytic parameters. The analysis of $B_{\Omega|X}^a$ could be performed from the viewpoint of the 2nd microlocalization at the boundary along V developed in [U-Z]; but we do not need to refer to such a general theory for the purpose of the present paper. We then consider a differential system $\mathcal{M}$ at x , $x \in N$, and a closed set S , $S \subset N$, $x \in \partial S$. We denote by $\tilde{V}$ the union of the leaves of $V^{\mathbb{C}}$ issued from V , we let ρ be the projection $Y \times T^*X \rightarrow T^*Y$, and make the following hypotheses: the conormals to N at x are non-microcharacteristic for $\mathcal{M}$ along $\tilde{V}$ in $\pi^{-1}(x)$; $\text{char } \mathcal{M} \cap \rho^{-1} \rho(\{x\} \times V) \subset V$; $i N_x^*(S) \subset \rho(\{x\} \times V)$. We then prove that $H^0(B_{M|X}^a)$ -solutions of $\mathcal{M}$ on $M \setminus S$ extend to M at x . Under some additional assumptions on "propagation in the interior" we also obtain the extension of A_M -solutions. We refer to [Kan], [$\hat{O}$], and [U-Z] for other results on continuation of (regular) solutions.

§ 1. THE COMPLEXES $B_{\Omega|X}^2$ AND $B_{\Omega|X}^a$

Let $M = M' \times L$ be real analytic manifolds with complexifications $X = X' \times Z$ and

dimensions $n = n_1 + n_2$. For a locally closed set $A = A' \times L$ of M , put $\tilde{A} = A' \times Z$ and define (cf [K-S-2], [S])

$$(1.1) \quad C_{A|X}^h = \mu_{\tilde{A}}^h(O_X) \otimes \omega_{M'/X}[n_1],$$

$$(1.2) \quad B_{A|X}^2 = R \Gamma_{T^*X' \times L}(C_{A|X}^h) \otimes \omega_{L/Z}[n_2].$$

We often consider the case $A = M$ or $A = N$ for an analytic submanifold $N = N' \times L$ of M of codimension 1, or else $A = \Omega$ where $\Omega = \Omega^\pm$ are the components of $M \setminus N$. The following triangle will play an essential role:

$$(1.3) \quad B_{N|X}^2 \rightarrow B_{M|X}^2 \rightarrow B_{\Omega^+|X}^2 \oplus B_{\Omega^-|X}^2 \rightarrow +1.$$

REMARK 1.1. By the results of [U-Z] we could give a canonical definition of the complexes $C_{*|X}^h$ and $B_{*|X}^2$, $* = M, N, \Omega$, associated to a smooth conic regular involutive manifold $V \subset \tilde{T}_M^*X$ such that

$$(1.4) \quad V \text{ and } N \times_M \tilde{T}_M^*X \text{ intersect transversally,}$$

and $N \times_M V$ is regular involutive.

We recall from [K-L] that for $* = M, N$, $B_{*|X}^2$ is concentrated in degree 0 and the natural morphism $C_{*|X|T^*X' \times L} \rightarrow B_{*|X}^2$ is injective, $C_{*|X}$ being the sheaf of usual microfunctions. (As for the case $* = \Omega$ it is proven in [U-Z] that $(B_{\Omega|X}^2)_{T_M^*X' \times L}$ is concentrated in degree 0 but that the corresponding result on injectivity does not hold any more. However this is not needed here.)

We set now:

$$(1.5) \quad B_{*|X}^a = R \Gamma_M(O_X|_{\tilde{*} \times Z}) \otimes \omega_{L/Z}[n_2], \quad * = M, N, \Omega.$$

For $* = M, \Omega$ we have a distinguished triangle

$$(1.6) \quad B_{*|X}^a \rightarrow R \Gamma_*(B_M) \rightarrow R \pi_*(B_{*|X}^2) \rightarrow +1,$$

(B_M being the sheaf of hyperfunctions); for $* = N$ we have to shift

by -1 the first term of (1.6). Using (1.6), the results of [K-L],

(and also the trick of the dummy variable for $* = N$), one easily sees that

$B_{*|X}^a$, $* = M$ or N , are concentrated in degrees 0 and 1 with $H^1(B_{*|X}^a) \neq 0$.

The same should be proven for $* = \Omega$; but this is complicated and needless

here.

The detailed study of the complexes (1.5) is left to [U-Z]; we only treat here their applications to boundary value problems. Thus let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -module on an open set of M . We assume all through this section that Y , the complexification of N , is non-characteristic for $\mathcal{M}$.

PROPOSITION 1.2. The natural morphisms

$$(1.7) \quad H^0(R \operatorname{Hom}(\mathcal{M}, C_{\Omega|X}))|_{T^*_{X',X}L} \rightarrow H^0(R \operatorname{Hom}(\mathcal{M}, B^2_{\Omega|X})),$$

and

$$(1.8) \quad H^0(R \operatorname{Hom}(\mathcal{M}, B^a_{\Omega|X})) \rightarrow \operatorname{Hom}(\mathcal{M}, \Gamma_{\Omega}(B_M)),$$

are injective.

PROOF. By the results of [K-L] it is enough to prove (1.7) and (1.8) in $T^*_{N',X}L$ and N respectively. As for (1.7), set $F = T^*_{M,X} \oplus N^*(\Omega)^a$ ("a"= antipodal) and consider the commuting diagram

$$(1.9) \quad \begin{array}{ccccc} & & C_{\Omega|X} & \rightarrow & B^2_{\Omega|X} \\ & \nwarrow & & & \downarrow \\ R \Gamma_F C_{N|X}[1] & \rightarrow & C_{N|X}[1] & \rightarrow & B^2_{N|X}[1] \end{array}$$

Then the conclusion follows from:

$$(1.10) \quad \operatorname{Hom}(\mathcal{M}, C_{N|X}) = 0, \quad \operatorname{Hom}(\mathcal{M}, B^2_{N|X} / C_{N|X})|_{T^*_{N',X}L} = 0,$$

which are in turn easy consequences of division formulas for $C_{N|X}$ and $B^2_{N|X}$ (cf [K-S-1]).

As for (1.8) we only need to recall (1.6) for $* = \Omega$, and use (1.3), and (1.10). The proof is complete.

Let $\mathcal{M}_Y$ denote the induced system by $\mathcal{M}$ on Y and let $\gamma: \operatorname{Hom}(\mathcal{M}, \Gamma_{\Omega}(B_M)) \rightarrow \operatorname{Hom}(\mathcal{M}_Y, B_N)$ be the trace morphism (cf [S]). By collecting all above results we get:

PROPOSITION 1.3. We have

$$(1.11) \quad H^0(R \operatorname{Hom}(\mathcal{M}, B^a_{\Omega|X}))_X = \{u \in \operatorname{Hom}(\mathcal{M}, \Gamma_{\Omega}(B_M))_X :$$

$$SS(\gamma(u)) \cap (T_{N,Y'}^* \times L) \subset T_Y^* \}, \quad x \in M.$$

PROOF. Let $\mathcal{F} = R \operatorname{Hom}(\mathcal{M}, \mathcal{O}_X)$, put $\mathcal{Q} = \Omega' \times Z$, and note that the natural diagram

$$(1.12) \quad \begin{array}{ccc} \pi^{-1} R \Gamma_{\Omega}(\mathcal{F}) & \rightarrow & R \Gamma_{T_{X'}^* \times L}^* \mu_{\Omega}^*(\mathcal{F}) \\ & \searrow & \nearrow \\ & \mu_{\Omega}(\mathcal{F})|_{T_{X'}^* \times L} & \end{array}$$

is commuting. Thus recalling (1.6) and applying Proposition 1.2, we get

$$(1.13) \quad H^0(R \operatorname{Hom}(\mathcal{M}, \mathcal{B}_{\Omega|X}^a))_x = \{ u \in H^0(\mathcal{M}, \Gamma_{\Omega}(\mathcal{B}_M))_x : \\ SS_{\Omega}^{\mathcal{M}, 0}(u) \cap (T_{X'}^* \times L) \subset T_X^* \},$$

where $SS_{\Omega}^{\mathcal{M}, 0}(u)$ is the support of u identified to a section of $H^0(R \operatorname{Hom}(\mathcal{M}, \mathcal{C}_{\Omega|X}))$ (cf [S]). According to [S] this is in turn equivalent to (1.11).

REMARK 1.4. When considering $\mathcal{B}_M^a|_X$ one can use the injectivity of $\mathcal{C}_M|_X|_{T_M^* \times L}^* \rightarrow \mathcal{B}_M^2|_X$ and $H^0(\mathcal{B}_M^a|_X) \rightarrow \mathcal{B}_M$ as a substitute of Proposition 1.2. (Note that the latter injectivity follows from (1.6) and the (conical) flabbiness of $\mathcal{B}_M^2|_X$ (cf [K-L]).) Then using (1.12) one easily gets

$$(1.14) \quad H^0(\mathcal{B}_M^a|_X)_x = \{ u \in (\mathcal{B}_M)_x : SS(u) \cap (T_M^* \times L) \subset T_X^* \}, \quad x \in M.$$

REMARK 1.5. For a regular involutive manifold V defined on the whole $T_M^* X$ and satisfying (1.4), we can intrinsically define $\mathcal{B}_*^a|_X$, $*$ = M, N, Ω , by replacing in (1.5) $\bar{*} \times Z$ by $\pi(\tilde{V}_*)$ (and $\omega_{L/Z}$ by $\omega_{V/\tilde{V}_M}$), where $\tilde{V}_*$ is the union of the leaves of V^C issued from $\bar{*} \times T_M^* X$; (we also write $\tilde{V} = \tilde{V}_M$). One can also intrinsically define the right hand sides of (1.11), (1.14) just by replacing $T_{N,Y'}^* \times L$ and $T_{M,X'}^* \times L$ by $\rho \varpi^{-1}(V)$ and V respectively (ρ and ϖ being the natural mappings from $Y \times T_X^* X$ to T_Y^* and T_X^* resp.).

It is then clear that if for some coordinates on M we can write

$$V = T_{M,X'}^* \times L, \quad N = N' \times L,$$

then (1.11) and (1.14) still hold. More generally owing to the invariance of

$\mathcal{B}_*^2|_X$ under contact transformation preserving V , $N \times V$, and $\omega_{N/M}$ (cf [U-Z]),

one could prove that (1.6) is fulfilled. But this refined argument is not

needed here.

§ 2. EXTENSION OF SOLUTIONS WITH REAL ANALYTIC PARAMETERS

Let M be a real analytic manifold with complexification X , N an analytic hypersurface of M with complexification Y , $\Omega = \Omega^\pm$ the two components of $M \setminus N$, ρ and ϖ the canonical mappings from $Y \times T^*_X X$ to T^*Y and T^*X respectively. Let $x \in M$, let $U \subset M$ be a neighborhood of x , and let V be a manifold in $U \times T^*_M X$. We assume that, in suitable coordinates on U :

$$(2.1) \quad M = M' \times L, \quad X = X' \times Z, \quad N = N' \times L \\ V = T^*_{M'} X' \times L, \quad \hat{V} = T^*_{M'} X' \times Z.$$

Recall the complexes $\mathcal{B}^a_{M|X}$, $\mathcal{B}^a_{\Omega|X}$ (intrinsically associated to V) and remember (1.11), (1.14). For any $p \in \pi^{-1}(x)$ recall the identification $T^*_x M \hookrightarrow T_p T^*X$ obtained through the embedding $T^*X \times T^*X \hookrightarrow T^*T^*X$ and the Hamiltonian isomorphism, and observe that $(T^*_N M)_x / R^+$ is just a pair of vectors $\pm \theta$.

THEOREM 2.1. Let N and V be defined, in suitable coordinates by (2.1), and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -module at x which verifies

$$(2.2) \quad \pm \theta \notin C_p(\text{char } \mathcal{M}, \hat{V}) \quad \text{for } \pm \theta \in (T^*_N M)_x / R^+ \text{ and for any } p \in \pi^{-1}(x) \cap V,$$

$$(2.3) \quad \varpi^{-1}(\text{char } \mathcal{M}) \cap \rho^{-1} \rho(\{x\} \times V) \subset T^*_M X.$$

Let S be a closed subset of N with $x \in \partial S$ and

$$(2.4) \quad i^* N^*_x(S) \subset \rho \varpi^{-1}(V)_x,$$

(in the identification $i^* T^* N \simeq T^*_N Y$). We then have, in a neighborhood of x ,

$$(2.5) \quad \text{Hom}(\mathcal{M}, \Gamma_{M \setminus S}(H^0(\mathcal{B}^a_{M|X}))) \simeq \text{Hom}(\mathcal{M}, H^0(\mathcal{B}^a_{M|X})).$$

PROOF. Let $\Omega = \Omega^\pm$ with $\Omega^+ \cup \Omega^- = M \setminus N$; by reasoning as in § 1 and observing that

$$R \pi_* R \Gamma_{(T^*_X X \cup \pi^{-1}(N))}(\mathcal{B}^2_{\Omega|X}) = R \Gamma_{\Omega}(\mathcal{B}^a_{M|X}),$$

we get a distinguished triangle

$$(2.6) \quad \mathcal{B}^a_{\Omega|X} \rightarrow R \Gamma_{\Omega}(\mathcal{B}^a_{M|X}) \rightarrow R \pi_* R \Gamma_{\pi^{-1}(N)}(\mathcal{B}^2_{\Omega|X}) \xrightarrow{+1}$$

Let $\mathcal{F} = R \text{Hom}(\mathcal{M}, C^h_{\Omega|X})|_{M \times T^*_X X}$. We note that (2.2) implies $(p; \pm \theta) \notin \text{SS}(\mathcal{F})$

and thus also $R \Gamma_{\pi^{-1}(N)}(\mathcal{F}) = R \Gamma_{\pi^{-1}(M \setminus \Omega)}(\mathcal{F}) = 0$. By applying

$R \Gamma_{(N \times_M V \cap T_N^* X)}(\cdot) [n_2]$ to the last equality ($N \times_M V$ being defined similarly to $\tilde{V}$ and n_2 being the codimension of V), we then get, for a neighborhood U of x on N ,

$$(2.7) \quad R \Gamma_{\pi^{-1}(N)} R \operatorname{Hom}(\mathcal{M}, B_{\Omega|X}^2) \Big|_{U \times_M V} = 0.$$

Note now that (2.2), (2.3) imply:

$$\omega^{-1}(\operatorname{char} \mathcal{M}) \cap \rho^{-1} \rho(U \times_M V) \subset U \times_M V,$$

which gives, combined with (2.7):

$$(2.8) \quad R \pi_* R \Gamma_{\pi^{-1}(N)} R \operatorname{Hom}(\mathcal{M}, B_{\Omega|X}^2) \Big|_U = 0.$$

By (2.6) this implies:

$$(2.9) \quad R \operatorname{Hom}(\mathcal{M}, B_{\Omega|X}^a) \Big|_U \simeq R \operatorname{Hom}(\mathcal{M}, R \Gamma_{\Omega}(B_M^a|X)) \Big|_U.$$

For $u \in \operatorname{Hom}(\mathcal{M}, \Gamma_{M \setminus S}(H^0(B_M^a|X)))$ let now $u^{\pm} = u|_{\Omega^{\pm}}$. Owing to (2.9) and (1.11) we get

$$SS(\gamma(u^{\pm})) \cap \rho(U \times_M V) \subset T_Y^* Y.$$

We also clearly have

$$\operatorname{supp}(\gamma(u^+) - \gamma(u^-)) \subset S.$$

Therefore the conclusion is an immediate consequence of the following two lemmas.

LEMMA 2.2 (cf $[\hat{O}]$). Let F be a closed set of M and let $u \in (B_M)_x$, $x \in \partial F$.

Then

$$\begin{cases} SS(u) \cap N_x^*(F) \subset \{0\} \\ \operatorname{supp}(u) \subset F \end{cases} \Leftrightarrow u_x = 0$$

PROOF. Easy application of Kashiwara-Holmgren's theorem and of sweeping out procedure by Bony-Schapira.

LEMMA 2.3. Let $u \in \operatorname{Hom}(\mathcal{M}, \Gamma_{(M \setminus N)}(B_M))$; then

$$u \in \operatorname{Hom}(\mathcal{M}, H^0(B_M^a|X)) \Leftrightarrow \begin{cases} \gamma(u^{\pm}) \in H^0(B_N^a|Y) \\ \gamma(u^+) - \gamma(u^-) = 0. \end{cases}$$

PROOF. It is enough to recall the triangle

$$C_M|X \rightarrow C_{\Omega^+|X} \oplus C_{\Omega^-|X} \rightarrow C_N|X[1] \rightarrow +1,$$

and the estimation

$$SS(u) \subset \bigcup_{\pm} SS_{\Omega^{\pm}}^{m,0}(u^{\pm}) \subset \rho^{-1}\left(\bigcup_{\pm} SS(\gamma(u^{\pm}))\right),$$

(cf [S]).

COROLLARY 2.4. In the situation of Theorem 2.1 assume in addition:

$$(2.10) \quad \text{Hom}(\mathcal{M}, \Gamma_S(C_M|X))_p = 0 \quad \forall p \in \dot{T}_M^*X \setminus V, \pi(p) = x.$$

Then (for $A_M = O_X|_M$):

$$(2.11) \quad \text{Hom}(\mathcal{M}, \Gamma_{(M \setminus S)}(A_M))_x \simeq \text{Hom}(\mathcal{M}, A_M)_x.$$

By the argument in the proof of (2.7) and by the injectivity of $C_M|X|_V \rightarrow \mathcal{B}_M^2|X$, a sufficient condition for (2.10) is that (2.2) is fulfilled for some V_p and θ_p such that $p \in V_p$, $S \subset \{x \in M : \langle x, \theta_p \rangle \geq 0\}$.

REMARK 2.5. It is clear from Lemma 2.2 that we can even consider in Theorem 2.1 some singular set S such that $N_{x_0}^*(S) = T_{x_0}^*N$. In fact for $M = M' \times L \simeq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni x = (x', x'')$, we only need to assume that $N \setminus S$ contains spheres of the L -plane whose diameters are infinite over the distance to ∂S . For example this is the case of any $S \subset \{\phi \leq 0\}$ for $\phi \in C^0(N)$ with $\phi(x_0) = 0$,

$$\partial_{x_{n_1}} \phi(x_0) \neq 0, \quad \partial_{x''} \phi(x_0) = 0, \quad \partial_{x'} \phi \in C^0, \quad \partial_{x''} \phi \in C^0.$$

REMARK 2.6. Theorem 2.1 extends the results of [Kan], [Ô]. These are obtained by choosing $L \simeq \mathbb{R}^{n-2} \subset M \simeq \mathbb{R}^n$ and by replacing $\hat{V}$ with T_M^*X in (2.2).

EXAMPLE 2.7. Let $M = M' \times L \ni (x', x'')$, $N = N' \times L$, $M' = \mathbb{R} \times N'$, $x' = (x_1, \hat{x})$, $S = S' \times L$, $x_0 = 0 \in \partial S$. Let (z, ζ) , $z = x + iy$, $\zeta = \xi + i\eta$, be coordinates in T^*X , let $V = \{\eta'' = 0\}$ and consider

$$m : \zeta_1^2 - (z_1^r + z^s) \zeta^2 + \zeta''^2, \quad r, s \text{ even}, r \geq 2.$$

Then (2.2)–(2.4) hold with $\pm \theta = \pm dx_1$ (cf [S–Z]) and thus we get (2.5) and (2.11) (as (2.10) is trivial in the present situation).

EXAMPLE 2.8. In the above situation let $M' \simeq \mathbb{R} \times N' \simeq \mathbb{R} \times \mathbb{R}^3$, let $V =$

$= \{ \eta_3 = \eta_4 = \eta'' = 0 \}$, and consider

$$m : (\zeta_1^3 + \zeta_3^3 + \zeta''^3, \zeta_2(\zeta_3^2 + \zeta_4^2)) .$$

For $S = S' \times L$ with $0 \in \partial S$ we have (2.2)–(2.4) and thus also (2.5). Moreover

for any $p \in V$ and for $\pm \theta_p = \pm dx_1$ or $\pm dx_2$ we have (2.2) with $\hat{V}_p = T_M^* X$. Therefore if we let $S = \{x_1 = x_2 = 0\}$, we get (2.10) and (2.11). (This extends Example 1.1 of $[\hat{O}]$.)

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