

LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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1. INTRODUCTION

The Dirac Hamiltonian with magnetic vector potential $\mathbf{a} = (a_j(x))_{j=1,\dots,d}$ is expressed by the following form

$$H(\mathbf{a}) = \sum_{j=1}^d \gamma_j (P_j - a_j) + m\gamma_{d+1} + V, \quad (1.1)$$

where $P_j = \frac{1}{i}\partial_{x_j}$, V is a multiplication of an Hermitian matrix $V(x)$. m is the mass of electron. The matrices $\{\gamma_j\}$ satisfy the following relations

$$\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk} \mathbf{1} \quad (j, k = 1, \dots, d+1). \quad (1.2)$$

Here δ_{jk} is Kronecker's delta and $\mathbf{1}$ is an identity matrix. We assume that the speed of the light $c = 1$. When $V \equiv 0$, the square of $H(\mathbf{a})$ has the form

$$H(\mathbf{a})^2 = \sum_{j=1}^d (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \leq j < k \leq d} b_{jk}(x) \gamma_j \gamma_k, \quad (1.3)$$

where

$$b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x). \quad (1.4)$$

It is called Pauli's Hamiltonian. The skew symmetric matrix $(b_{jk}(x))$ is the magnetic field associated with $\mathbf{a}$. We say the magnetic field is asymptotically constant if it satisfies the following conditions as $|x| \rightarrow \infty$:

$$b_{jk}(x) \rightarrow \exists \Lambda_{jk} \quad (1 \leq j, k \leq d), \quad (1.5)$$

where $(\Lambda_{jk})_{j,k}$ is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for $H(\mathbf{a})$ with a constant magnetic field $(b_{jk}(x))$ and a long-range electric potential $V(x)$ when $d = 3$. Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for $d = 2, 3$. As can be inferred from (1.3), the spectrum of $H(\mathbf{a})$ is closely related

with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose $d = 2$ at first. For simplicity we consider the case that the magnetic field $b(x) = \partial_{x_2}a_1(x) - \partial_{x_1}a_2(x) = \lambda > 0$. In this case, the Dirac Hamiltonian $h(\lambda)$ is represented by

$$h(\lambda) = \sigma_1(P_1 + \frac{\lambda}{2}x_2) + \sigma_2(P_2 - \frac{\lambda}{2}x_1) + m\sigma_3, \quad (1.6)$$

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

They are called Pauli's spin matrices. Obviously $\{\sigma_j\}$ satisfy the relation (1.2) and by an elementary calculus we have

$$h(\lambda)^2 = (P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2 + m^2 - \lambda\sigma_3. \quad (1.7)$$

The right hand side is a de-coupled 2 dimensional magnetic Schrödinger operator. So it suggests that the spectrum of $h(\lambda)$ is discrete and

$$\sigma(h(\lambda)) \subset \{\pm\sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \dots\}.$$

In fact we have

$$\sigma(h(\lambda)) = \{\sqrt{2\lambda n + m^2}, -\sqrt{2\lambda(n+1) + m^2} \mid n = 0, 1, 2, \dots\}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [8].) Therefore the spectrum of $h(\lambda)$ is of pure point with infinite multiplicities.

Next we consider the case of $d = 3$. We assume

$$\mathbf{a}_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along x_3 -axis :

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as $H_0(\lambda)$. It is the following operator acting on $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$:

$$H_0(\lambda) = \alpha_1(P_1 + \frac{\lambda x_2}{2}) + \alpha_2(P_2 - \frac{\lambda x_1}{2}) + \alpha_3 P_3 + m\beta, \quad (1.8)$$

where $\{\alpha_j\}$ and β are 4×4 Hermitian matrices such that

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.9)$$

We can easily see that these matrices also satisfy the relation (1.2). It is known that $H_0(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. (See Theorem 4.3 in [8].) Now we consider the spectrum of $H_0(\lambda)$. At first we rewrite $H_0(\lambda)$ as follows.

$$H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \quad (1.10)$$

with $D_0 = \sigma \cdot (P - \mathbf{a}_0)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.

By using Foldy-Wouthuysen transform, explained in detail in the following section, $H_0(\lambda)$ can be diagonalized by a unitary operator U_{FW} .

$$U_{FW}H_0(\lambda)U_{FW}^{-1} = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}. \quad (1.11)$$

From the commutation relation (1.2) we have

$$D_0^2 = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 - \lambda\beta. \quad (1.12)$$

We can easily see that $\sigma(D_0^2) = [0, \infty)$. So we have

$$\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).$$

Therefore in the 3 dimensional case, the spectrum of $H_0(\lambda)$ is absolutely continuous.

Let us consider the perturbation of $H_0(\lambda)$: We put

$$H(\lambda) = H_0(\lambda) + V. \quad (1.13)$$

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent $(z - H(\lambda))^{-1}$ on the real axis. As for the Schrödinger operator with constant magnetic field, Iwashita [4] shows the limiting absorption principle for long-range potential by using commutator method. In [4] the following self-adjoint operator is considered.

$$\tilde{H} = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 + V(x). \quad (1.14)$$

The existence of the boundary values

$$\langle x_3 \rangle^{-s} (\tilde{H} - \mu \mp i0)^{-1} \langle x_3 \rangle^{-s}$$

is proved for $s > 1/2$ and $\mu \in \mathbb{R} \setminus (\{\lambda(2n+1) | n = 0, 1, 2, \dots\} \cup \sigma_{pp}(\tilde{H}))$.

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as $|x| \rightarrow \infty$. (See [2].) Hachem [3] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential $V(x)$. Roughly speaking, his assumption means that the absolute value of each components of V is dominated from above by $C\langle x' \rangle^{-1-\epsilon} \langle x \rangle^{-\epsilon}$ ($x' = (x_2, x_3)$) for sufficiently large x . We remark that $\epsilon > 0$ is used as a sufficiently small parameter throughout this paper. To be accurate, $\langle x' \rangle^{1+\epsilon} V(x)$ is required to be a $H_0(\lambda)$ -compact operator.

In this paper we treat directly the following operator

$$H(\lambda) = \alpha_1(P_1 + \frac{\lambda}{2}x_2) + \alpha_2(P_2 - \frac{\lambda}{2}x_1) + \alpha_3P_3 + m\beta + V(x), \quad (1.15)$$

where $V(x)$ is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for $V(x)$. In this case it

seems that an appropriate choice of the conjugate operator is

$$\frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle},$$

which is inspired by [9], when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [9]. Namely we rewrite $H_0(\lambda)$ by a direct integral and the conjugate operator A acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

2. CONJUGATE OPERATOR

Let us recall

$$Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - \mathbf{a}_0), \quad (2.1)$$

with

$$\mathbf{a}_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0). \quad (2.2)$$

The Dirac Hamiltonian $Q_0 + m\beta$ can be diagonalized by sandwiching it between a unitary operator U and $U^* = U^{-1}$. In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator $H_0(\lambda)$. Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre's inequality for original Hamiltonians $H_0(\lambda)$ and $H(\lambda)$.

Let Q_0 be the self-adjoint operator as in (2.1) and $|Q_0| = \sqrt{Q_0^2}$, $|H_0(\lambda)| = \sqrt{H_0(\lambda)^2}$. We define a unitary operator U_{FW} , which diagonalize $H_0(\lambda)$, in the following way.

Definition 2.1. (i). *At first we define a signature function associated with Q_0 by*

$$\text{sgn}Q_0 = \begin{cases} \frac{Q_0}{|Q_0|} & , \quad \text{on } (\ker Q_0)^\perp \\ 0 & , \quad \text{on } (\ker Q_0) \end{cases} \quad (2.3)$$

We note that $\text{sgn}Q_0$ is isometry on $(\ker Q_0)^\perp$.

(ii). *We can easily see that $m/|H_0(\lambda)| \leq 1$. So we denote the square root of $\frac{1}{2}(1 \pm \frac{m}{|H_0(\lambda)|})$ as $a_\pm$. i.e.*

$$a_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm m/|H_0(\lambda)|}. \quad (2.4)$$

(iii). *Combining these operators we define the operator U_{FW} as*

$$U_{FW} = a_+ + \beta(\text{sgn}Q_0)a_-. \quad (2.5)$$

Lemma 2.2. (i). *U_{FW} is a unitary operator on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$.*

Further,

$$U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(\text{sgn}Q_0)a_-. \quad (2.6)$$

(ii). $H_0(\lambda)$ can be diagonalized by U_{FW} as follows.

$$U_{FW} H_0(\lambda) U_{FW}^{-1} = |H_0(\lambda)|\beta = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}. \quad (2.7)$$

Proof. See 5.6.1 in [8]. □

We denote the diagonalized Dirac Hamiltonian as $\hat{H}_0(\lambda)$. i.e.

$$\hat{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}.$$

We rewrite (1.12) as follows.

$$D_0^2 = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}.$$

Here $D_{\pm}$ are the operators acting on $L^2(\mathbb{R}^3)$ such that

$$D_{\pm} = (P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2 + P_3^2 \pm \lambda.$$

It is well-known that $(P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2$ has eigenvalues

$$\{\lambda(2n+1) | n = 0, 1, 2, \dots\}.$$

We denote the eigenprojection on each eigenspace as Π_n . With these projections, $\sqrt{D_0^2 + m^2}$ can be rewritten as follows.

$$\sqrt{D_0^2 + m^2} = \sum_{n=0}^{\infty} \begin{pmatrix} d_n \otimes \Pi_n & 0 \\ 0 & d_{n+1} \otimes \Pi_n \end{pmatrix}, \quad (2.8)$$

with $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$.

Combining (2.7) and (2.8), we have

$$f(\hat{H}_0(\lambda)) = \sum_{n=0}^{\infty} \begin{pmatrix} f(d_n) \otimes \Pi_n & & & \\ & f(d_{n+1}) \otimes \Pi_n & & \\ & & f(-d_n) \otimes \Pi_n & \\ & & & f(-d_{n+1}) \otimes \Pi_n \end{pmatrix},$$

for any Borel function f .

Now we define the conjugate operator. At first we define

$$\hat{A} = \frac{1}{2} \left\{ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle} \right\}. \quad (2.9)$$

We note that $\hat{A}$ is essentially self-adjoint operator on $D(|x_3|)$. (It is obtained by use of Nelson's commutator theorem [7].) The conjugate operator for the Dirac Hamiltonian associated with constant magnetic field is defined by sandwiching $\hat{A}\beta$ between U_{FW}^{-1} and U_{FW} :

$$A = U_{FW}^{-1} (\hat{A}\beta) U_{FW}. \quad (2.10)$$

Before we show Mourre's inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.

Suppose that $f \in C^\infty(\mathbb{R})$ satisfies the following condition for some $m_0 \in \mathbb{R}$.

$$|f^{(k)}(t)| \leq C_k(1 + |t|)^{m_0 - k}, \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (2.11)$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ having the following properties

$$\begin{aligned} \tilde{f}(t) &= f(t), \quad t \in \mathbb{R}, \\ \text{supp } \tilde{f} &\subset \{z; |Imz| \leq 1 + |Rez|\}, \end{aligned}$$

$$|\partial_{\bar{z}} \tilde{f}(z)| \leq C_N |Imz|^N \langle z \rangle^{m_0 - 1 - N}, \quad \forall N \in \mathbb{N}. \quad (2.12)$$

Then for all f , satisfying (2.11) for $m_0 < 0$ and a self-adjoint operator H , we have

$$f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z) (z - H)^{-1} dz \wedge d\bar{z}. \quad (2.13)$$

3. LIMITING ABSORPTION PRINCIPLE FOR LONG-RANGE POTENTIALS

Now we show the Mourre's inequality for the Dirac Hamiltonian by choosing A defined in the previous section as the conjugate operator.

Lemma 3.1. *Let $\mathbb{R}_{\mathbb{N}}$ be the following discrete subset of $\mathbb{R}$*

$$\mathbb{R}_{\mathbb{N}} = \{\pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \dots\} \subset \mathbb{R}.$$

We take a compact interval $I \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ arbitrarily. Then there exists $\alpha > 0$ such that the following inequality holds for any real valued $f \in C_0^\infty(I)$

$$f(H_0(\lambda)) i[H_0(\lambda), A] f(H_0(\lambda)) \geq \alpha f(H_0(\lambda))^2. \quad (3.1)$$

Proof. By the relations (2.7) and (2.10), it is sufficient to show the inequality

$$f(\hat{H}_0(\lambda)) i[\hat{H}_0(\lambda), \hat{A}\beta] f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \quad (3.2)$$

We rewrite the commutator as follow.

$$i[\hat{H}_0(\lambda), \hat{A}\beta] = \begin{pmatrix} i[\sqrt{D_0^2 + m^2}, \hat{A}] & \\ & i[\sqrt{D_0^2 + m^2}, \hat{A}] \end{pmatrix}. \quad (3.3)$$

We proceed the calculus more precisely to see that

$$i[\sqrt{D_0^2 + m^2}, \hat{A}] = \sum_{n=0}^{\infty} \begin{pmatrix} i[d_n, \hat{A}] \otimes \Pi_n & \\ & i[d_{n+1}, \hat{A}] \otimes \Pi_n \end{pmatrix} \quad (3.4)$$

by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

$$f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) = \begin{pmatrix} I_1 & & & \\ & I_2 & & \\ & & I_3 & \\ & & & I_4 \end{pmatrix} \quad (3.5)$$

where

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty} f(d_n)i[d_n, \hat{A}]f(d_n) \otimes \Pi_n, \\ I_2 &= \sum_{n=0}^{\infty} f(d_{n+1})i[d_{n+1}, \hat{A}]f(d_{n+1}) \otimes \Pi_n, \\ I_3 &= \sum_{n=0}^{\infty} f(-d_n)i[d_n, \hat{A}]f(-d_n) \otimes \Pi_n, \\ I_4 &= \sum_{n=0}^{\infty} f(-d_{n+1})i[d_{n+1}, \hat{A}]f(-d_{n+1}) \otimes \Pi_n. \end{aligned}$$

We note that all the sum in $I_1, \dots, I_4$ are finite since f is a compactly supported function. By an elementary calculus, we have

$$i[d_l, \hat{A}] = \frac{P_3^2}{\sqrt{2\lambda l + P_3^2 + m^2\langle P_3 \rangle}} \quad (l \in \mathbb{N} \cup \{0\}). \quad (3.6)$$

Since $\text{supp} f \subset I \subset \mathbb{R} \setminus \mathbb{R}_N$, P_3 is away from zero when $P_3 \in \text{supp} f(d_l(P_3))$ or $P_3 \in \text{supp} f(-d_l(P_3))$. So there exist $C_l > 0$ such that

$$\begin{aligned} f(d_l)i[d_l, \hat{A}]f(d_l) \otimes \Pi_l &\geq C_l f(d_l)^2 \otimes \Pi_l, \\ f(-d_l)i[d_l, \hat{A}]f(-d_l) \otimes \Pi_l &\geq C_l f(-d_l)^2 \otimes \Pi_l. \end{aligned}$$

Since only a finite number of $l = l_j$ ($j = 1, \dots, N$) is concerned, we have (3.2) with $\alpha = \inf_{j=1, \dots, N} C_{l_j}$. $\square$

Now we give the assumption for the potential, which is necessary to Mourre's inequality associated to $H(\lambda)$. After that we give an example of V satisfying this assumption. It consists of a sum of long-range part and short-range part. In our case short-range potential means $V(x) = O(\langle x \rangle^{-\epsilon} \langle x_3 \rangle^{-1-\epsilon})$ as $|x| \rightarrow \infty$. And long-range part is a multiplication of a real valued function $\varphi(x)$ such that $\varphi(x) = O(\langle x \rangle^{-\epsilon})$ as $|x| \rightarrow \infty$. More precisely we assume that V satisfies the following.

Assumption 3.2. $V = V(x)$ is a multiplicative operator of a 4×4 Hermitian matrix satisfying the following properties.

- (i). V is a $H_0(\lambda)$ -compact operator.
- (ii). The form $[V, A]$ can be extended to a $H_0(\lambda)$ -compact operator.

For example a 4×4 matrix $V(x)$ satisfying the following inequality is $H_0(\lambda)$ -compact.

$$|V(x)| \leq C\langle x \rangle^{-\epsilon} \quad (x \in \mathbb{R}^3). \quad (3.7)$$

It is owing to the fact that $V(x)(-\Delta_x + 1)^{-1}$ is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre's inequality for $H(\lambda)$.

Lemma 3.3. *Suppose V satisfies Assumption 3.2.*

- (i). *We take $\mu \in \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ and $\delta > 0$ so that the closed interval $I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$. There exist $\alpha > 0$ and a compact operator K such that the following inequality holds for all $f \in C_0^\infty(I)$.*

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha f(H(\lambda))^2 + K. \quad (3.8)$$

- (ii). *There is no accumulation point of $\sigma_{pp}(H(\lambda))$ in $\mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$. For $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that the following inequality holds for all $f \in C_0^\infty([\mu - \delta_0, \mu + \delta_0])$.*

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha_0 f(H(\lambda))^2. \quad (3.9)$$

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

Theorem 3.4. *Suppose V satisfies Assumption 3.2. Then for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$, the following limits*

$$R^\pm(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s} \quad (3.10)$$

exist and $R^\pm(\mu)$ are continuous with respect to $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$.

Sketch of proof

From (3.9) and Theorem 2.2 in [5], we can see that the boundary value $\langle A \rangle^{-s} (H(\lambda) - \mu \mp i0)^{-1} \langle A \rangle^{-s}$ exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$. To see the existence of (3.10), it is sufficient if we show the boundness of $\langle A \rangle^s \langle x_3 \rangle^{-s}$. Since $\langle \hat{A} \rangle^s \langle x_3 \rangle^{-s}$ is bounded, it is sufficient to show $\langle x_3 \rangle^s U_{FW} \langle x_3 \rangle^{-s}$ is bounded. We prove it in the following Lemma. Before that we introduce smooth functions.

Let $\chi(t) \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1/\sqrt{2} & (t > -m^2/3) \\ 0 & (t < -2m^2/3). \end{cases} \quad (3.11)$$

With this function we define $F_\pm(t)$ and $F_{\chi,\pm}$ as follows.

$$F_+(t) = \chi(t) \sqrt{1 + \frac{m}{\sqrt{t + m^2}}}$$

$$F_-(t) = \chi(t) \left(\sqrt{1 + \frac{m}{\sqrt{t + m^2}}} \right)^{-1} \frac{1}{\sqrt{t + m^2}}$$

$$F_{\chi,+}(t) = F_+(t) - \chi(t)$$

$$F_{\chi,-}(t) = \sqrt{t + m^2} F_-(t) - \chi(t)$$

Then we can easily verify that

$$\begin{aligned} a_+ &= F_+(Q_0^2), \\ a_- \operatorname{sgn} Q_0 &= F_-(Q_0^2)Q_0 = Q_0 F_-(Q_0^2). \end{aligned}$$

As for the proof of $[Q_0, F_-(Q_0^2)] \equiv 0$, see 5.2.4 in [8]. By the construction of these functions, we can also see that $F_{\chi, \pm}(t)$ satisfy (2.11) with $m_0 < 0$. So we apply the functional calculus in section 2 to $F_{\chi, \pm}(t)$ and see the following properties hold.

Lemma 3.5. *Suppose $0 \leq s \leq 2$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

(i). *For $0 < s \leq 1$, there exists $C_s > 0$ such that*

$$\|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C_s (|Imz|^{-1} + |Imz|^{-2} \langle z \rangle). \quad (3.12)$$

(ii). *For $1 < s \leq 2$, there exists $C'_s > 0$ such that*

$$\|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C'_s (|Imz|^{-1} + |Imz|^{-2} \langle z \rangle + |Imz|^{-3} \langle z \rangle^2). \quad (3.13)$$

(iii). *$\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s}$ and $\langle x \rangle^s F_-(Q_0^2) Q_0 \langle x \rangle^{-s}$ are bounded operators.*

Proof. For the proof of (i) and (ii), we use the resolvent equation. Suppose $0 < s \leq 1$. Then

$$\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} = (z - Q_0^2)^{-1} + (z - Q_0^2)^{-1} (Q_0^2 + 1) \quad (3.14)$$

$$\times (Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] (z - Q_0^2)^{-1} \langle x \rangle^{-s}. \quad (3.15)$$

From the boundness of $(Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2]$ and the following estimate

$$\|(z - Q_0^2)^{-1} (Q_0^2 + 1)\|_{\mathbb{H}} \leq C (|Imz|^{-1} \langle z \rangle + 1), \quad (3.16)$$

we obtain (i). As for the case $1 < s \leq 2$, we rewrite the last term $(z - Q_0^2)^{-1} [\langle x \rangle^s, Q_0^2] (z - Q_0^2)^{-1} \langle x \rangle^{-s}$ as

$$(z - Q_0^2)^{-1} (Q_0^2 + 1) (Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] \langle x \rangle^{-s+1} \quad (3.17)$$

$$\times \langle x \rangle^{s-1} (z - Q_0^2)^{-1} \langle x \rangle^{-s+1} \langle x \rangle^{-1}. \quad (3.18)$$

By using the result for $0 < s \leq 1$, we have the inequality for $1 < s \leq 2$.

With these estimates, we prove (iii). Since $\chi(Q_0^2) \equiv 1$, we can easily see that

$$\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s} = \langle x \rangle^s F_{\chi,+}(Q_0^2) \langle x \rangle^{-s} + I. \quad (3.19)$$

Since $F_{\chi,+}(t)$ satisfies (2.12) for $m_0 = -1/2$, $F_{\chi,+}(Q_0^2)$ can be rewritten as follows.

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}. \quad (3.20)$$

From this formula and (i) (ii) we have

$$\begin{aligned} &\|\partial_{\bar{z}} \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \\ &\leq C |\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)| (|Imz|^{-1} + |Imz|^{-2} \langle z \rangle + |Imz|^{-3} \langle z \rangle^2). \end{aligned}$$

From (2.12) we have

$$\|\partial_{\bar{z}} \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C \langle z \rangle^{-5/2}.$$

This implies the boundness of $\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s}$.

In a similar way, we rewrite $\langle x \rangle^s F_-(Q_0^2) Q_0 \langle x \rangle^{-s}$ as

$$\langle x \rangle^s F_{\chi,-}(Q_0^2) \langle x \rangle^{-s} \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s} + \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s}. \quad (3.21)$$

It is sufficient to show the boundness of $\langle x \rangle^s Q_0 / \sqrt{Q_0^2 + m^2} \langle x \rangle^{-s}$. To see this, we denote $\chi(t) / \sqrt{t + m^2} \in C^\infty(\mathbb{R}^3)$ as $S(t)$ and its almost analytic extension as $\tilde{S}(z)$. We can easily see that $S(Q_0^2) \langle x \rangle^s Q_0 \langle x \rangle^{-s}$ is bounded. So we obtain the boundness of $\langle x \rangle^s F_-(Q_0^2) Q_0 \langle x \rangle^{-s}$ if we show that $[\langle x \rangle^s, S(Q_0^2)] Q_0 \langle x \rangle^{-s}$ is bounded. We rewrite it as follows.

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{S}(z) \frac{Q_0^2 + 1}{z - Q_0^2} (Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] Q_0 \langle x \rangle^{-s} \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}.$$

By an elementary calculus, we have $(Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] Q_0 \langle x \rangle^{-s}$ bounded. Combining (i) and (ii), we have

$$\begin{aligned} \|[\langle x \rangle^s, S(Q_0^2)] Q_0 \langle x \rangle^{-s}\|_{\mathbb{H}} &\leq C \int_{\mathbb{C}} |\partial_{\bar{z}} \tilde{S}(z)| \{1 + |Imz|^{-1}\} \\ &\times \{|Imz|^{-1} + |Imz|^{-2} \langle z \rangle + |Imz|^{-3} \langle z \rangle^2\} dz \wedge d\bar{z} < \infty. \end{aligned}$$

This implies the boundness of $\langle x \rangle^s F_-(Q_0^2) Q_0 \langle x \rangle^{-s}$. $\square$

Next we give an example of V . It requires smoothness, but allows long-range part in its diagonal components.

Lemma 3.6. *Let V be a 4×4 Hermitian matrix of the form*

$$V(x) = (v_{ij}(x)) + \varphi(x) I_4 \equiv V_s(x) + V_l(x) \quad (3.22)$$

where $V_s(x) = (v_{ij}(x))$ is an Hermitian matrix and I_4 is an identity matrix. Suppose the following conditions hold. Then $V(x)$ satisfies Assumption 3.2.

There exist $\delta > 0$ such that the following inequalities hold for all multi-index α .

$$|\partial_x^\alpha v_{ij}(x)| \leq C_\alpha \langle x \rangle^{-\delta - |\alpha|} \langle x_3 \rangle^{-1} \quad (1 \leq i, j \leq 4). \quad (3.23)$$

$\varphi(x) \in C^\infty(\mathbb{R}^3)$ is real valued and satisfies

$$|\partial_x^\alpha \varphi(x)| \leq C'_\alpha \langle x \rangle^{-\delta - |\alpha|}. \quad (3.24)$$

The relatively compactness of $V(x)$ itself is clear since V satisfies (3.7). So we only have to show the relatively compactness of $[V, A]$. We prove the relatively compactness of $[V_s, A] = [V_s, U_{FW}^{-1} \hat{A} \beta U_{FW}]$ at first. From the boundness of $\langle x_3 \rangle^{-1} \hat{A} \beta$ and the relatively compactness of $V_s \langle x_3 \rangle$, it is sufficient to show that $\langle x_3 \rangle U_{FW} \langle x_3 \rangle^{-1}$ and $\langle x_3 \rangle U_{FW}^{-1} \langle x_3 \rangle^{-1}$ are bounded operators in $\mathbb{H}$. We have already proved it in Lemma 3.5.

Next we treat the long-range term. The conjugate operator A can be decomposed into the sum of $J_1, \dots, J_4$ where

$$\begin{aligned} J_1 &= F_+(Q_0^2) \hat{A} \beta F_+(Q_0^2), \\ J_2 &= F_+(Q_0^2) \hat{A} \beta^2 F_-(Q_0^2) Q_0, \\ J_3 &= \beta F_-(Q_0^2) Q_0 \hat{A} \beta F_+(Q_0^2), \\ J_4 &= \beta F_-(Q_0^2) Q_0 \hat{A} \beta^2 F_-(Q_0^2) Q_0. \end{aligned}$$

We prove that the $H_0(\lambda)$ -compactness holds for each of $[V_l, J_1], \dots, [V_l, J_4]$. To see this we use the functional calculus again and rewrite J_1 as follows.

$$\begin{aligned} F_+(Q_0^2) \hat{A} \beta F_+(Q_0^2) &= \hat{A} \beta F_+(Q_0^2)^2 + [F_{\chi,+}(Q_0^2), \hat{A} \beta] F_+(Q_0^2) \\ &\equiv J'_1 + J''_1 \end{aligned}$$

At first we prove the boundness of J''_1 and consequently the relatively compactness of $[V_l, J''_1]$. By using (2.13), we rewrite $[F_{\chi,+}(Q_0^2), \hat{A} \beta]$ as follows.

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{F}_{\chi,+}(z) (z - Q_0^2)^{-1} [Q_0^2, \hat{A} \beta] (z - Q_0^2)^{-1} dz \wedge d\bar{z}. \quad (3.25)$$

From (3.16) we have $[Q_0^2, \hat{A} \beta] (z - Q_0^2)^{-1}$ is dominated from above by $C\{1 + |\operatorname{Im} z|\}$. So we have

$$\|[F_{\chi,+}(Q_0^2), \hat{A} \beta]\| \leq C \int_{\mathbb{C}} |\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)| \{|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle\} dz \wedge d\bar{z}. \quad (3.26)$$

Since the almost analytic extension $\tilde{F}_{\chi,+}(z)$ satisfies

$$|\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)| \leq C_N |\operatorname{Im} z|^N \langle z \rangle^{-3/2-N} \quad (\forall N \in \mathbb{N}), \quad (3.27)$$

we have $[F_{\chi,+}(Q_0^2), \hat{A} \beta]$ is bounded and inductively $[V_l, J''_1]$ is $H_0(\lambda)$ -compact. So we only have to show the relatively compactness of $[V_l, J'_1]$.

$$[V_l, J'_1] = [V_l, \hat{A} \beta] F_+(Q_0^2)^2 + \hat{A} \beta [V_l, F_+(Q_0^2)^2]. \quad (3.28)$$

Clearly $[V_l, \hat{A} \beta] F_+(Q_0^2)$ is $H_0(\lambda)$ -compact. Again we rewrite the commutator in the second term, by use of (2.13). Then we have $\langle x \rangle^{1+\delta} [V_l, F_+(Q_0^2)^2]$ is bounded. Combining these facts, we have the relatively compactness of $[V_l, J_1]$.

As for the commutator $[V_l, J_2], \dots, [V_l, J_4]$ we also replace $F_{\pm}$ by $F_{\chi,\pm}$ and use the functional calculus. The proof of relatively compactness of $[V_l, J_2]$ and $[V_l, J_3]$ are almost the same. We only give the proof for J_2 . We also estimate the 'principle' part before we compute the commutator with V_l .

$$J_2 = \hat{A} F_+(Q_0^2) F_-(Q_0^2) Q_0 + [F_+(Q_0^2), \hat{A}] F_-(Q_0^2) Q_0 \quad (3.29)$$

It is sufficient to show that $[V_l, \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0]$ is a $H_0(\lambda)$ -compact operator. We decompose it into the following sum.

$$\begin{aligned} & [V_l, \hat{A}]F_+(Q_0^2)F_-(Q_0^2)Q_0 \\ & + \hat{A}[V_l, F_+(Q_0^2)F_-(Q_0^2)]Q_0 \\ & + \hat{A}F_+(Q_0^2)F_-(Q_0^2)[V_l, Q_0]. \end{aligned}$$

We can easily see that the first and the third term is relatively compact since $\langle x \rangle^{1+\delta}[V_l, Q_0]$ is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for J_4 , the proof is similar. We rewrite it as

$$\hat{A}F_-(Q_0^2)^2Q_0^2 + [F_-(Q_0^2)Q_0, \hat{A}]F_-(Q_0^2)Q_0 \quad (3.30)$$

We can also obtain the relatively compactness by estimating the term $[V, \hat{A}F_-(Q_0^2)^2Q_0^2]$.

Corollary 3.7. *Let V be a 4×4 Hermitian matrix. Suppose V satisfies the condition in Lemma 3.6. Then the following limits*

$$R^\pm(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s} \quad (3.31)$$

exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_N \cup \sigma_{pp}(H(\lambda)))$ and $R^\pm(\mu)$ are continuous with respect to μ .

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