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Do Rational Animal Spirits Cause Overproduction?”

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Sunspot Equilibria in a Production Economy: Do Rational Animal Spirits Cause Overproduction? *

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Abstract

We study a standard two period economy with one nominal bond and one firm. The input of the firm is done in the first period and financed with the nominal bond, and its profits are distributed to the shareholders in the second period. We show that a sunspot equilibrium exists around each efficient equilibrium. The interest rate is lower than optimal and there is over production in sunspot equilibria, under some conditions. But a sunspot equilibrium does not exist if the profit share can be traded as well as the bond. (JEL classification numbers: D52, D53, D61)

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1 Introduction

We consider a simple two period private ownership economy with production. There is a single perishable good in each period, traded competitively in each period. There is a nominal bond market in the first period. There is one firm which uses the first period good as input and produces the second period good. Since the revenue is earned in the second period, the firm needs to issue the bond for purchasing its input in the good market in the first period. The households are risk averse, and the technology is convex, so no economic agent favors randomness per se. The purpose of this paper is to analyze the properties of sunspot equilibria of this economy.

Recall that sunspots are theoretical device to model random phenomena which do not affect tastes, endowments of goods and resources, and production technology, within the framework of rational expectations.¹ Such phenomena include for instance price uncertainty, fear of inflation, animal spirits of investors, and the psychology of the markets in general. Under the standard convexity assumptions, a sunspot equilibrium is Pareto inefficient. So in other words, we investigate if and how *rational* animal spirits cause inefficiency in a production economy.

The existence and the real indeterminacy of sunspot equilibria in pure exchange economies have been investigated extensively.² But the production economy has not been studied systematically in this context to the best of our knowledge. Introducing production in an incomplete market setting is known to be a challenge, because the objective of the firm is not clearly defined in some cases. In this paper, we do not try to resolve how this issue should be addressed: instead, we postulate that the firm maximizes the expected profits. Although no formal justification is given, the expected profit maximization appears at least very plausible in the simple setup we study.

Given the postulate, many insights from the case of pure exchange are then valid in the production economy as well, and we shall take advantage of them as much as possible in this paper. After providing a formal description of the model and the definition of sunspot equilibria in Section 2, we study the existence problem in Section 3. We shall show that a sunspot equilibrium exists, with only very non-generic exceptions. The idea is very similar to that for the exchange economy with a nominal bond. Since the market clearing of the bond market is enough to establish the general equilibrium of the markets, one can arbitrarily fix the real values of nominal returns depending on the sunspot states, and let the bond price adjust to clear the market. Except for some knife edge cases, the resulting consumption will be random, and we have a sunspot equilibrium.

A sunspot equilibrium is Pareto inefficient, and there are two sources for the inefficiency. The

¹See the seminal paper Cass and Shell (1983).

²See Cass (1992) for a short overview. See also Gotardi and Kajii (1999), where a real asset instead of a nominal bond is considered.

first is distributional inefficiency, which is the focus of the pure exchange models: in effect, households consume according to an extrinsic lottery in the second period, which is welfare worsening. The second is production inefficiency, which cannot be addressed in the pure exchange models, obviously: the firm may be producing too much or too little in a sunspot equilibrium. Section 4 is devoted to the question of production inefficiency, which is the main part of this paper. We establish a characterization result (Proposition 1) which identifies exactly when over/under production occurs in sunspot equilibria around efficient equilibria. As is often the case in general equilibrium analyses, the standard set of assumptions which guarantees the existence of an equilibrium is not strong enough to tell if either over production or under production is to take place. We argue nevertheless that in usual settings, a sunspot equilibrium tends to exhibit over production: in a sunspot equilibrium, the price of bond is too high, i.e., the rate of interest is too low, so that the firm produces too much. That is, rational animal spirits tend to cause overproduction.

In Section 5, we extend the model by introducing a market for the profit share of the firm. Of course, *if* sunspots do not matter, trading profit share is redundant since the price of the share must be determined in such a way that the bond and the share are equivalent as assets. But in a sunspot equilibrium, this equivalence might break down and so introduction of the profit share market could generate a different kind of sunspot equilibrium. Interestingly enough, in this extended set up, we show in Proposition 3 that there is no sunspot equilibrium.³ So if there is a market for profit share, there is no sunspot equilibrium and the profit share market is redundant and unnecessary in any equilibrium, but there are sunspot equilibria without it.⁴

Section 6 contains discussions on a few issues. First we comment on the issue of welfare gains and losses: although a sunspot equilibrium is inefficient, there might be some households who are better off in the sunspot equilibrium than in an efficient equilibrium.⁵ Next, we provide a comparison with models with background income risks, which yield results with a flavor similar to ours. Finally, we give some remarks on extending our model to allow more than one good in each period and multiple firms.

2 The Model

We consider a standard competitive two-period economy with production. There is one perishable consumption good in each period. There is one firm with an increasing and concave production function f with $f(0) = 0$; the firm produces $f(z)$ units of good in the second period (period

³The argument is a slight modification of the ingenious idea of Mas-Colell (1992).

⁴There are other cases where the existence of some markets makes the economy immune to sunspots, but those markets are redundant in equilibria. In a pure exchange setup, Kajii (1997) shows that if there are an enough number of financial options there is no sunspot equilibrium and the options are all redundant.

⁵Goenka, A. and Préchac (2006) and Kajii (2007) investigate this issue for exchange economies.

1) from z units of good in the first period (period 0). There are $H \geq 2$ households, labelled by $h = 1, 2, \dots, H$. Household h is endowed with e_h^0 units of good in period 0 and e_h^1 units in period 1, as well as a profit share θ_h of the firm. We write $e_h = (e_h^0, e_h^1)$.

In the first period, period 0, a nominal bond which pays off one unit in units of account (say, dollar) in the second period is traded in a competitive market. The bond price in units of the period 0 good is denoted by $q > 0$. The firm will finance its input by issuing bond, which will be held by households. Write B for the amount of bond issued by the firm, and z_h for the amount held by household h . So the firm raises qB in units of good in period 0, which is used as input. Consequently, it produces $f(qB)$ units of good in period 1 and is liable for the outstanding bond issued, B dollars.

The real returns of the nominal bond will be determined in the markets, which might be random; the households expect that the price of good in dollars might be random, and then consequently the real value of bond's payoff is expected to be a random variable. In other words, the households expect inflation, and the rate of inflation may vary according to the state of the economy. This idea is formally described using *sunspots* as follows.

At the beginning of the second period, the state of economy is revealed. State s , $s = 1, \dots, S$ occurs with probability $\mu^s > 0$. We assume that these are sunspot states. That is, by assumption, the state is publicly observable, and the fundamentals of the economy described so far are independent of the realization of the sunspot state. It is often convenient to refer to the first period (period 0) as state $s = 0$, and we shall follow this convention throughout the paper.

Write $p^s > 0$ for the price of good in dollars when state s is realized. Then $r^s := \frac{1}{p^s}$ is the real value of one dollar in state s . By construction, the real payoff of the bond per unit is also r^s in state s , so we shall refer to r^s as the (gross) return of the bond in state s . Since only relative prices matters, we will always set $\sum_{s=1}^S \mu^s r^s = 1$ without loss of generality, i.e., we normalize the prices so that the expected real payoffs of the bond in units of the period 1 good is one. We shall write $\tilde{r} = (r^1, \dots, r^S) \in \mathbb{R}_+^S$ for the vector of returns.

Sunspots do not influence production, but nevertheless, since the real returns of bond may be random, the level of profits depends on sunspots in general. Specifically, the realized profit is $\Pi^s := f(qB) - r^s B$ in state s , in units of good, which will be distributed to the households according to the profit share θ_h , $h = 1, \dots, H$. We assume that the firm maximizes expected profits; since $\sum_{s=1}^S \mu^s = 1$ and we normalize $\sum_{s=1}^S \mu^s r^s = 1$, this means that the firm takes bond price q as given and solves the following problem:

$$\max_{B \geq 0} f(qB) - B, \tag{1}$$

which is a well defined concave problem under our assumptions.

Notice that the firm's optimal decision as well as the maximized level of profit is independent of $\tilde{r}$. Let $\Pi^*(q)$ be the maximum profit given price q , and $B^*(q)$ be the set of profit maximizers.

That is, the firm will choose $B \in B^*(q)$ given bond price q . Then by construction the expected profit is $\Pi^*(q) = f(qB) - B$ and the level of profits in state s is $\Pi^*(q) + (1 - r^s)B$, $s = 1, \dots, S$. Note that although the expected profit must be non-negative since zero production is feasible, ex post profit Π^s may be negative for some s . Also notice that the choice of $B \in B^*(q)$ is indeterminate as far as the expected profit is concerned, but it does affect the randomness of profits in principle.⁶ When f is strictly concave in the sense of $f'' < 0$ everywhere, $B^*(q)$ is singleton and in such a case we shall abuse notation to denote the single element by $B^*(q)$ as well. Note that $B^*(q)$ is increasing in q .

For household h holding z_h units of the bond at the end of period 0, $r^s z_h$ units of consumption good is delivered at the beginning of period 1 in state s . Also, household h receives $\theta_h \Pi^s$ for profit share, thus the consumption of household h is $e_h^0 - qz_h$ in period 0 and $e_h^1 + r^s z_h + \theta_h \Pi^s$ in state s in period 1. Recall that $\Pi^s < 0$ is possible and so household h may be forced to compensate for firm's loss; i.e., the liability of equity is unlimited. Also z_h may be negative, i.e., household h may choose to borrow.

By assumption, the households take random profits $\tilde{\Pi}$ as given, as well as the other price parameters. Rational expectations then require that random profits are given by an accounting identity $\tilde{\Pi} = \Pi^*(q) + (1 - \tilde{r})B$. Since we focus on rational expectations, we assume that the households take the bond supply B and profit function Π^* as given. Taking this into account, the second period consumption can be written in different ways as follows:

$$e_h^1 + \tilde{r}z_h + \theta_h \tilde{\Pi} = e_h^1 + \theta_h (\Pi^*(q) + B) + \tilde{r}(z_h - \theta_h B) \quad (2)$$

$$= e_h^1 + \theta_h \Pi^*(q) + \tilde{r}z_h + (1 - \tilde{r})\theta_h B. \quad (3)$$

The preferences of household h are represented by a von Neumann Morgenstern utility function $u_h : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$; that is, given bond price q , a vector of rate of returns $\tilde{r} = (r^s)_{s=1}^S \in \mathbb{R}_+^S$ with $\sum_{s=1}^S \mu^s r^s = 1$, and a profile of profits $\tilde{\Pi} := (\Pi^s)_{s=1}^S \in \mathbb{R}^S$, if household h chooses z_h such that consumption is positive, i.e., $e_h^0 - qz_h > 0$ and $e_h^1 + r^s z_h + \theta_h \Pi^s > 0$ for every $s = 1, \dots, S$, the utility is given by:

$$\sum_{s=1}^S \mu^s u_h(e_h^0 - qz_h, e_h^1 + r^s z_h + \theta_h \Pi^s). \quad (4)$$

Using the expectation operator $\mathbf{E}$ with respect to probability measure $(\mu^s)_{s=1}^S$, and denoting with a slight abuse of notation by $\tilde{r}$ and $\tilde{\Pi}$ the random returns and profits, respectively, the utility

⁶For instance, some shareholders may prefer smaller B for less randomness in profits, but the others may not. Then the shareholders might disagree on the choice among B in $B^*(q)$. Moreover, since some shareholders might even prefer lower expected profits if randomness is reduced, the expected profit maximization might not be the shareholders' interests. This is a potentially interesting question, but we do not pursue it in this paper.

function (4) can also be written as

$$\mathbf{E} \left[u_h \left(e_h^0 - qz_h, e_h^1 + \tilde{r}z_h + \theta_h \tilde{\Pi} \right) \right]. \quad (5)$$

Household h 's problem is to choose $z_h \in \mathbb{R}$ to maximize the expected utility (4).

Note that both random returns and random profits contribute to the randomness of income in the second period. But recall the property of the second period consumption (2). We can re-write the utility function (5) further so that the vector $\tilde{r}$ is seen to be the single source of randomness, as follows:

$$\mathbf{E} \left[u_h \left(e_h^0 - qz_h, e_h^1 + \theta_h (\Pi^*(q) + B) + \tilde{r}(z_h - \theta_h B) \right) \right]. \quad (6)$$

From this expression we see that, other things being equal, the utility is sensitive to a small change in random returns $\tilde{r}$, unless $z_h = \theta_h B$.

It is assumed that u_h is C^3 , differentiable strictly increasing (i.e., for any $x_h \in \mathbb{R}_{++}^2$, the gradient $Du_h(x_h)$ is strictly positive), differentiable strictly concave (i.e., for any $x_h \in \mathbb{R}_{++}^2$, the Hessian $D^2u_h(x_h)$ is negative definite), and for each level set, its closure in $\mathbb{R}^2$ is contained in $\mathbb{R}_{++}^2$. The assumption of thrice differentiability is needed since the second derivatives of demand functions are important in our analysis in Section 4. But the reader will see that the differentiability assumption is not essential for the existence problem in Section 3 and for the non-existence result in Section 5.

Under these assumptions, the function (4) is concave in z_h and the optimal choice is characterized by a solution to the first order condition as follows:

$$\sum_{s=1}^S \mu^s \left\{ -q \frac{\partial}{\partial x_0} u_h \left(e_h^0 - qz_h, e_h^1 + r^s z_h + \theta_h \Pi^s \right) + \frac{\partial}{\partial x_1} u_h \left(e_h^0 - qz_h, e_h^1 + r^s z_h + \theta_h \Pi^s \right) r^s \right\} = 0, \quad (7)$$

where $\frac{\partial}{\partial x_0} u_h$ and $\frac{\partial}{\partial x_1} u_h$ are derivatives with respect to the first period consumption and the second period consumption, respectively. Using the expectation operator, and taking the property of the second period consumption (2) into account, (7) can also be written as:

$$\mathbf{E} \left[-q \left(\frac{\partial}{\partial x_0} u_h \right) + \left(\frac{\partial}{\partial x_1} u_h \right) \tilde{r} \right] = 0, \quad (8)$$

where the derivatives are evaluated at $(e_h^0 - qz_h, e_h^1 + \theta_h (\Pi^*(q) + B) + \tilde{r}(z_h - \theta_h B))$.

The solution to (7) is unique if it exists by the strict concavity of the utility function. The existence depends on the returns $\tilde{r}$ among others, but since our analysis will be done locally around a competitive equilibrium where the optimal choice is well defined, we will simply assume that a solution exists in the relevant domain of the analysis. Denote by $Z_h(q, \tilde{r}, B)$ the unique solution to (7); that is, $Z_h(q, \tilde{r}, B)$ is the demand for bond of household h given prices q and $\tilde{r}$ and the bond supply B of the firm. Then $Z(q, \tilde{r}, B) := \sum_{h=1}^H Z_h(q, \tilde{r}, B)$ is the total demand

for the bond of the households. It may first appear unusual that the demand function depends on firm's choice variable B in addition to prices, but as we have explained above, there is no loss as far as rational expectation equilibria are concerned. Note that since the bond supply function $B^*(q)$ is a function of q , and so $Z(q, \tilde{r}, B)$ is effectively just a function of price variables $(q, \tilde{r})$.

The prices endogenously determined in the markets are q and $\tilde{r}$. Thus the rational expectation equilibrium of this economy is defined as follows:

Definition 1 *A bond price q and a vector of returns $\tilde{r} = (\dots, r^s, \dots) \in \mathbb{R}^S$ with $\sum_{s=1}^S \mu^s r^s = 1$ constitute a competitive equilibrium if there exists $B \in B^*(q)$ such that $Z(q, \tilde{r}, B) - B = 0$. A competitive equilibrium is called a sunspot equilibrium if the second period consumption is not constant across the states for some household.*

Remark 2 *Consider the case where the technology is strictly convex and so the supply $B^*(q)$ is a singleton for any q . Denoting the unique element by $B^*(q)$ by convention, the equilibrium condition above can be written as $Z(q, \tilde{r}, B^*(q)) - B^*(q) = 0$.*

Remark 3 *From the property of the second period consumption (2), it readily follows that an equilibrium $(q, \tilde{r})$ is a sunspot equilibrium if and only if there is some h such that $Z_h(q, \tilde{r}, B) \neq \theta_h B$ where B is the corresponding equilibrium bond supply.*

The equilibrium condition above says that the bond market clears. As is usually the case, it can be readily shown that if the bond markets clear, all the good markets clear.

When $S = 1$, our model is a standard two period model of consumption and saving, and so an equilibrium exists and every equilibrium is Pareto efficient. An equilibrium for the case of $S = 1$ is called a *certainty equilibrium*. Under our normalization, the real return of bond is one in any certainty equilibrium.

If $(\bar{q}, 1) \in \mathbb{R}^2$ is a certainty equilibrium, it can be readily seen that $(\bar{q}, \tilde{1})$ is an equilibrium for any $S > 1$, where $\tilde{1} = (1, \dots, 1) \in \mathbb{R}_+^S$. This is an equilibrium where the households think that the sunspot states do not affect the real returns of bond; that is, they expect that the return of bond in units of good is one for sure. Such an equilibrium is called a *non-sunspot equilibrium* when $S > 1$. By the fundamental theorem of welfare economics applied to the certainty equilibrium and the risk aversion of households, a non-sunspot equilibrium is Pareto efficient. From now on, we assume that $S > 1$ to avoid triviality.

Conversely, since there is no uncertainty in production and so the aggregate consumption is independent of sunspots, the risk aversion of the households and the convexity of the technology imply that in any Pareto efficient allocation, the consumption of each household must be independent of sunspots. Hence in particular a sunspot equilibrium is inefficient. But an efficient equilibrium may not be a non-sunspot equilibrium: it is possible that although the equilibrium

returns $\tilde{r}$ are random, the households use the bond to completely offset income risks generated by random profits, as will be seen in Example 4.

3 Existence of sunspot equilibria

We argue that a sunspot equilibrium exists. The intuition for the existence is simple. Basically in this model there are S price variables: bond price q and returns $r^1, \dots, r^S$, but one degree of freedom is lost by normalization. On the other hand, there is one market, the bond market, which needs to be cleared, since the rest of the markets clear automatically if the bond market clears. So even if the returns $\tilde{r}$ are arbitrarily fixed, the bond price q can be adjusted to clear the market. But if $\tilde{r} \neq \tilde{1}$, the income will be random and so will the consumption, except for some coincidental cases. Formally, we have the following existence result.

Lemma 1 *Let $(\bar{q}, 1)$ be a certainty equilibrium, and denote by $\bar{z}_h$ the bond holding of household h and by $\bar{B}$ the bond issued in the equilibrium. Then (1) there exists $\varepsilon > 0$ such that for any normalized returns $\tilde{r}$ with $|\tilde{r} - \mathbf{1}| < \varepsilon$, there is a bond price q such that $(q, \tilde{r})$ is an equilibrium. (2) Moreover, if $\bar{z}_h - \theta_h \bar{B} \neq 0$ for some h , then there exists $\varepsilon > 0$ such that for any normalized returns $\tilde{r}$ with $|\tilde{r} - \mathbf{1}| < \varepsilon$, there is a bond price q such that $(q, \tilde{r})$ is a sunspot equilibrium.*

This result can be shown by a simple continuity argument, so we shall omit a proof. Roughly speaking, if $\tilde{r}$ is close to $\tilde{1}$, the aggregate demand function for bond must look very close to the one for the economy with $S = 1$, and hence in particular there must be an equilibrium $(q, \tilde{r})$. Moreover, if $\bar{z}_h - \theta_h \bar{B} \neq 0$, the continuity implies that $Z_h(q, \tilde{r}, B) \neq \theta_h B$ where B is the corresponding equilibrium bond supply, so it must be a sunspot equilibrium (see Remark 3).

The condition $\bar{z}_h - \theta_h \bar{B} \neq 0$ in (2) of Lemma 1 is indispensable. That is, it is possible that an equilibrium exists for any fixed $\tilde{r}$ arbitrarily close to $\tilde{1}$, but the equilibrium is not a sunspot equilibrium, as the following example shows.

Example 4 *Assume that the households are identical. Then in any equilibrium, $r^s = 1$ for every s . In particular, there is no sunspot equilibrium. Indeed, if the households are identical, their choices must be identical by strict concavity. This means that the consumption cannot be random since there is no aggregate uncertainty.*

This example is effectively a model of a representative agent, where there can be no trade for risk sharing purpose. It is nevertheless instructive since it implies that our results in the following sections will be relying on heterogeneity of households' characteristics.

But a sunspot equilibrium must exist, generically. As long as there is just a slight heterogeneity in the economy (e.g., households have the same preferences but endowed differently in goods and profit share), it is intuitively plausible that $\bar{z}_h - \theta_h \bar{B} = 0$ is unlikely to hold in a certainty

equilibrium. In fact, although we do not elaborate on the details, it can be formally defined and established that $\bar{z}_h - \theta_h \bar{B} = 0$ is a non generic property as long as $H > 1$.⁷ So we contend that except for non-generic cases such as the case of completely homogeneous agents, a sunspot equilibrium exists around a non-sunspot equilibrium.

4 Prudence and Over investment

A sunspot equilibrium is inefficient, and there are two sources for inefficiency. The first is distributional inefficiency: for a given aggregate supply of the good which is independent of sunspots, households' consumption is affected by sunspots. This aspect of inefficiency has been discussed extensively in the literature of exchange economies, so we do not endeavour to clarify further.

The second is production inefficiency, which we focus in this section: the firm may be producing too much or too little. More specifically, starting with a certainty equilibrium where production is done at an efficient level, we study the level of production in nearby sunspot equilibria, whose existence has been established in Lemma 1.

Before proceeding to a formal analysis, let us build up some intuition first. Fix a certainty equilibrium and fix any $\tilde{r}$ close enough to $\tilde{1}$ in the sense of Lemma 1. Since the firm's problem (1) is independent of $\tilde{r}$, the supply curve of the bond is unchanged, and hence we only need to examine households' demand for bond at the sunspot equilibrium. Then the key question will be how the demand curve shifts; that is, we need to see if the demand for bond gets larger or smaller under $\tilde{r}$, other things being equal. If the demand gets larger, then the price of bond must go up to clear the bond market, i.e., the (average) interest rate will go down, which then should induce over production. The case of under production can be understood analogously.

The bond is a risky asset in a sunspot equilibrium so at first sight it might appear that the risk aversion implies the demand for the bond should decrease. It is well known however that the risk aversion alone does not determines the sign in a partial equilibrium setting where the level of income is fixed: in fact, it is the magnitude of the *relative prudence* which plays an important role.⁸

Notice there is another general equilibrium effect through profits, since the households' income depend on the profit level, which is random. Even if the firm's activity does not change so that the *average* profit remains the same, ex post profits will be more random which will make the second period income more random. Therefore, in principle this is potentially a complex problem of increasing risks in asset returns as well as background income risk.

On the other hand, risks in returns and profits are perfectly correlated in equilibrium, and hence the problem turns out to be manageable to some extent. Note that in equilibrium the

⁷See Mas-Colell (1985), section 6.

⁸See Section 4.5 of Gollier (2001), for instance.

second period income is given by (2): as far as the decision problem in equilibrium is concerned, the household effectively take the average profit $\Pi^*(q)$ as given. Moreover, its share of outstanding bond $\theta_h B$ is also taken as given, and the household solves a simple investment problem, controlling the *net investment* $z_h - \theta_h B$ whose returns are $\tilde{r}$.

Now we begin a formal analysis. For expositional simplicity, we shall assume that the utility functions are additively time separable in this section. The analysis can be done without the separability: we show the key result Lemma 2 below without separability assumption in Appendix, and the reader will see that the other results can be readily generalized in a similar manner. So we write $u_h(x^0, x^1) = u_h^0(x^0) + u_h^1(x^1)$ for each h . So the utility maximization in equilibrium (6) can now be written as

$$\max_{z_h} \mathbf{E} [u_h^0(e_h^0 - qz_h) + u_h^1(e_h^1 + \theta_h(\Pi^*(q) + B) + \tilde{r}(z_h - \theta_h B))]. \quad (9)$$

To describe the corresponding first order condition (8), set:

$$F_h(q, \tilde{r}, z_h, B) := -u_h^{0'}(e_h^0 - qz_h)q + \sum_{s=1}^S \mu^s u_h^{1'}(e_h^1 + \theta_h(\Pi^*(q) + B) + r^s(z_h - \theta_h B))r^s. \quad (10)$$

Then the first order condition (8), is now written as $F_h(q, \tilde{r}, z_h, B) = 0$.

Fix a certainty equilibrium $(\bar{q}, 1)$ such that for any $\tilde{r}$ close enough to $\tilde{1}$, there is a sunspot equilibrium $(q, \tilde{r})$. Let $\bar{z}_h$, $h = 1, \dots, H$, and $\bar{B}$ be the corresponding bond demand for household h and bond supply, respectively, in the certainty equilibrium $(\bar{q}, 1)$. To avoid the uninteresting of zero production, assume that $\bar{B} > 0$. Also let $\bar{x}_h^0$ and $\bar{x}_h^1$ be the certainty equilibrium consumption of household h in period 0 and 1, respectively.

Choose any returns $\tilde{r}$ close enough to $\tilde{1}$ so that there is a sunspot equilibrium $(q, \tilde{r})$. We first ask if the demand for bond increase or decrease as returns change from $\tilde{1}$ to $\tilde{r}$, keeping $\bar{q}$ and $\bar{B}$ fixed. That is, we ask how the demand curve shifts around the certainty equilibrium.

First we shall establish some basic results on how individual household's excess demand Z_h changes. We shall calculate changes when returns gets marginally risky. Write $\tilde{r}^{-S}$ for $(r^1, \dots, r^{S-1})$, and we shall set $r^S = (1 - \sum_{s=1}^{S-1} \mu^s r^s) / \mu^S$ to keep the normalization $E[\tilde{r}] = 1$. Using this convention, define $\hat{Z}_h$ by the rule:

$$\hat{Z}_h(q, \tilde{r}^{-S}, B) := Z_h\left(q, \left(\tilde{r}^{-S}, \frac{1 - \sum_{s=1}^{S-1} \mu^s r^s}{\mu^S}\right), B\right), \quad (11)$$

for each h . Then our task is to find the derivatives of $\hat{Z}_h$ with respect to $\tilde{r}^{-S}$, and evaluate them at $\tilde{r}^{-S} = \tilde{1}^{-S}$.

From now on, the derivatives of utility functions are evaluated at the certainty equilibrium: $(\bar{q}, 1)$, $\bar{B}$, and $(\bar{x}_h^0, \bar{x}_h^1)$, $h = 1, \dots, H$, unless specified otherwise. Differentiating (10), set

$$\gamma_h := -\frac{\partial}{\partial z_h} F_h(\bar{q}, \tilde{1}, \bar{z}_h, \bar{B}), \quad (12)$$

$$= -(u_h^{0''} \bar{q}^2 + u_h^{1''}) \quad (13)$$

for each h . Under our assumptions on the utility function, $\gamma_h > 0$. Observe that by symmetry and additive separability, $\frac{1}{\mu^s} \frac{\partial^2}{\partial (r^s)^2} F_h(\bar{q}, \tilde{1}, \bar{z}_h, \bar{B})$ does not depend on s , so set

$$\alpha_h := \frac{1}{\mu^S} \frac{\partial^2}{\partial (r^S)^2} F_h(\bar{q}, \tilde{1}, \bar{z}_h, \bar{B}), \quad (14)$$

$$= u_h^{1'''} \cdot (\bar{z}_h - \theta_h \bar{B})^2 + 2u_h^{1''} \cdot (\bar{z}_h - \theta_h \bar{B}) \quad (15)$$

for each h . We have the following result on the first and the second derivatives of $\hat{Z}_h$:

Lemma 2 *At $(\bar{q}, \tilde{1}, \bar{B})$, $\frac{\partial}{\partial r^s} \hat{Z}_h = 0$ for every $s = 1, \dots, S-1$, and*

$$\left(\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h \right)_{s,s'} = \frac{\alpha_h}{\gamma_h} M, \quad (16)$$

where M is an $S-1$ dimensional positive definite matrix determined by probability μ (thus independent of h).

A proof is given in Appendix. Since $\gamma_h > 0$, Lemma 2 says that as a function of $\tilde{r}^{-S}$, $\hat{Z}_h(\bar{q}, \tilde{r}^{-S}, \bar{B})$ is locally minimized at $\tilde{r}^{-S} = \tilde{1}$ if $\alpha_h > 0$, and it is locally maximized if $\alpha_h < 0$. Thus if $\alpha_h > 0$, then for $\tilde{r}^{-S}$ close enough to $\tilde{1}$, $\hat{Z}_h(\bar{q}, \tilde{r}^{-S}, \bar{B}) > \bar{z}_h$. The demand decreases if $\alpha_h < 0$.

It is useful to develop some intuition about Lemma 2 here. The first order effect disappears because of the envelope property. The reason why the second derivative plays a role can be understood as follows. Since we are interested in increasing risks in the sense of the second order stochastic dominance, if the function $r \mapsto u_h^1(e_h^1 + \theta_h(\Pi^*(\bar{q}) + B) + r(\bar{z}_h - \theta_h \bar{B}))$ is convex, then by the definition (10) we have $F_h(\bar{q}, \tilde{r}, \bar{z}_h, \bar{B}) > 0$. In this case, since F_h is decreasing in z_h , it follows that the demand must increase. It can be readily seen from (15), the parameter α_h is nothing but the derivative of this function. Dividing (15) by $-2u_h^{1''} \cdot (\bar{z}_h - \theta_h \bar{B})^2$ we see that $\alpha_h > 0$ obtains if and only if

$$-\frac{u_h^{1'''} }{u_h^{1''}} > \frac{2}{\bar{z}_h - \theta_h \bar{B}}. \quad (17)$$

If household h is *absolutely prudent* in the sense of $u_h^{1'''} > 0$, the inequality (17) holds if $\bar{z}_h - \theta_h \bar{B} < 0$, that is, household h is a *net lender*. So these households will increase the demand for the bond when $\tilde{r}$ gets random. On the other hand, households with $\bar{z}_h - \theta_h \bar{B} > 0$, a *net borrower*, the effect is ambiguous. So condition (17) above can be stringent in some setup.⁹

Next, we shall study the aggregate demand. Set $\hat{Z}(q, \tilde{r}^{-S}, B) := \sum_{h=1}^H \hat{Z}_h(q, \tilde{r}^{-S}, B)$.

Lemma 3 *If $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$, then for any $\tilde{r}$ with $E[\tilde{r}] = 1$ which is close enough to $\tilde{1}$, we have $Z(\bar{q}, \tilde{r}, \bar{B}) > \bar{B}$. If $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} < 0$, for any $\tilde{r}$ with $E[\tilde{r}] = 1$ which is close enough to $\tilde{1}$, we have $Z(\bar{q}, \tilde{r}, \bar{B}) < \bar{B}$.*

⁹ Thus the logic behind the over production result is different from background risk models. See Section 6.

Proof. Applying Lemma 2, at $(\bar{q}, \tilde{1}, \bar{B})$, $\frac{\partial}{\partial r^s} \hat{Z} = \sum_{h=1}^H \frac{\partial}{\partial r^s} \hat{Z}_h = 0$ for every $s = 1, \dots, S-1$, and $\left(\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z} \right)_{s,s'} = \left(\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} \right) M$, where M is positive definite. So if $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$, $\hat{Z}(\bar{q}, \tilde{r}^{-S}, \bar{B})$ is locally minimized at $\tilde{r}^{-S} = \tilde{1}^{-S}$. If $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} < 0$, $\hat{Z}(\bar{q}, \tilde{r}^{-S}, \bar{B})$ is locally maximized at $\tilde{r}^{-S} = \tilde{1}^{-S}$. Hence the result follows. ■

These results leads us to ask whether or not a natural set of assumptions determines the sign of $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h}$. Assuming absolute prudence, we would like to assert that it tends to be positive. The reason is as follows: as we have seen in (17), assuming absolute prudence, we have $\alpha_h > 0$ for the net lenders. Of course $\alpha_h < 0$ is not ruled out for the net borrowers, and this number could be large enough in absolute value to upset our assertion. But $\alpha_h < 0$ occurs for households whose prudence parameter is low, and/or whose net trade $\bar{z}_h - \theta_h \bar{B}$ is very small. Or to say the least, constructing an example of under production is not simple. Also, we do have $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$ in a special but interesting class of models of “homogeneous” economy.

Lemma 4 *Assume absolute prudence for the second period utility function, $u_h^{1'''} > 0$ for every h . If all the households’ consumption is identical in the certainty equilibrium, i.e., $(\bar{x}_h^0, \bar{x}_h^1)$ is independent of h , then $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$.*

Proof. From (13) and (15), $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} = \sum_{h=1}^H \frac{u_h^{1'''} \cdot (\bar{z}_h - \theta_h \bar{B})^2 + 2u_h^{1''} \cdot (\bar{z}_h - \theta_h \bar{B})}{-(u_h^{0''} \bar{q}^2 + u_h^{1''})} = \sum_{h=1}^H \frac{u_h^{1'''} \cdot (\bar{z}_h - \theta_h \bar{B})^2}{-(u_h^{0''} \bar{q}^2 + u_h^{1''})} + \sum_{h=1}^H \frac{2u_h^{1''} \cdot (\bar{z}_h - \theta_h \bar{B})}{-(u_h^{0''} \bar{q}^2 + u_h^{1''})}$. The first term is positive by the absolute prudence. If $(\bar{x}_h^0, \bar{x}_h^1)$ is independent of h , $u_h^{1''}$ and $u_h^{0''} \bar{q}^2 + u_h^{1''}$ are also independent of h . Hence the second term is zero because of market clearing $\sum_{h=1}^H (\bar{z}_h - \theta_h \bar{B}) = 0$. ■

Remark 5 *The condition in Lemma 4 can be relaxed: the proof only uses the fact that the second derivatives are common.*

Now we are ready to discuss the issue of over/under production. We shall concentrate on two cases: the case of linear technology and the case of strictly convex technology. The analysis for hybrid cases can be done analogously.

Let us first consider the case of linear technology: we assume that $f(z) = kz$ for some constant $k > 0$. Under our normalization, and since $\bar{B} > 0$ by assumption, the no profit condition implies $\bar{q} = k^{-1}$, and of course the zero profit condition $\Pi^*(\bar{q}) = 0$ must hold. Now fix $\tilde{r}$ close enough to $\tilde{1}$ so that a sunspot equilibrium exists. As we mentioned above, the firm’s profit maximization condition is unaffected, so the sunspot equilibrium prices must be $(\bar{q}, \tilde{r})$.

We have the following result on over/under production in sunspot equilibria.

Proposition 1 *Assume linear technology. If $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$, then in any sunspot equilibrium close enough to the certainty equilibrium, the level of production exceeds the efficient level (i.e., over production). Similarly, if $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} < 0$, then there is under production in the nearby sunspot equilibria.*

Proof. Suppose $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$, and fix $\tilde{r}$ close enough to $\tilde{1}$. Note that the function F_h in (10) is decreasing in B , since $\tilde{r} > 0$ and $\theta_h(\tilde{1} - \tilde{r}) > 0$ in (3). Since F_h is decreasing in z_h , this shows that $Z_h(\bar{q}, \tilde{r}, B)$ is decreasing in B for each h , so is $Z(\bar{q}, \tilde{r}, B)$. On the other hand, by Lemma 3, we have $Z(\bar{q}, \tilde{r}, \bar{B}) > \bar{B}$. This means that a sunspot equilibrium obtains, i.e., $Z(\bar{q}, \tilde{r}, B) = B$ only if $B > \bar{B}$.

The case of $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} < 0$ can be shown analogously. ■

Next we consider the case of strictly convex technology: we assume that f is a C^2 function with $f'' < 0$. In this case, the bond supply function is well defined, so denote by $B^*(q)$ the supply of bond when the bond price is q . It can be readily established that $B^*(q)$ is increasing in q : a higher the bond price means a lower interest rate, so the firm will produce more and The idea of analysis is essentially the same as before, except that in this case, shifts of demand function is not enough to identify over or under production, since the excess demand function may be upward sloping around the certainty equilibrium.

Proposition 2 *Assume strictly convex technology, and suppose $Z(q, \tilde{1}, B^*(q)) - B^*(q)$ is decreasing in q at $\bar{q}$. If $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$, then in any sunspot equilibrium close enough to the certainty equilibrium, the level of production exceeds the efficient level (i.e., over production). Similarly, if $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} < 0$, then there is under production in the sunspot equilibria.*

Proof. Suppose $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$, and fix $\tilde{r}$ close enough to $\tilde{1}$. By Lemma 3, we have $Z(\bar{q}, \tilde{r}, \bar{B}) > \bar{B}$. Recall that $B^*(q)$ is increasing and $\bar{B} = B^*(\bar{q})$ by definition. By assumption, $Z(q, \tilde{1}, B^*(q)) - B^*(q)$ is decreasing in q at $\bar{q}$, so is $Z(q, \tilde{r}, B^*(q)) - B^*(q)$ by continuity, if $\tilde{r}$ is close enough to $\tilde{1}$. Therefore, $Z(q, \tilde{r}, B^*(q)) = B^*(q)$ implies $q > \bar{q}$ and $B^*(q) > B^*(\bar{q})$ and so the production level in the sunspot equilibrium is higher than that in the certainty equilibrium. The case of $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} < 0$ can be shown analogously. ■

Remark 6 *If $Z(q, \tilde{1}, B^*(q)) - B^*(q)$ is increasing in q at $\bar{q}$ instead, i.e., the law of demand is violated at the certainty equilibrium, $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$ corresponds to under production and $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} < 0$ corresponds to over production.*

We have argued that $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$ is prevalent. With this assertion, Proposition 1 and Proposition 2 roughly indicate that we tend to see over production in a sunspot equilibrium. In general, one has to check if $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$ actually holds. One can construct various parametric examples, but here we shall give one example of some generality.

Corollary 7 *Assume linear technology $f(z) = z$, and absolute prudence $u_h^{1''''} > 0$ for every h . Suppose for each h , $u_h^0 = u_h^1$, and $e_h^0 + e_h^1 = 2\bar{e}$ for every h , where $\bar{e} > 0$, and $\sum_{h=1}^H (e_h^0 - e_h^1) > 0$. Then there is over production in sunspot equilibria.*

Proof. By assumption $\bar{q} = 1$, and so the income of the households must be the same. Thus, perfect consumption smoothing must take place, so each household consume $(\bar{x}_h^0, \bar{x}_h^1) = (\bar{e}, \bar{e})$. Thus assuming absolute prudence, Lemma 4 implies that $\sum_{h=1}^H \frac{\alpha_h}{\gamma_h} > 0$, and so there is over production in any nearby sunspot equilibrium by Proposition 1. ■

5 The Role of Stock markets

Why do we keep the profit share fixed in a standard textbook general equilibrium model? An answer would be that one could introduce a market for trading shares, but it does not really matter if markets are already complete: the value of firm is determined by the no arbitrage condition so that it is equivalent to the bond. Then the share will be a redundant asset in equilibrium, and the households need not trade the share anyway.

But in our setup, it makes a difference. If the share can be exchanged competitively in addition to the bond, it is still the case that the share is redundant in any equilibrium, but there will be no sunspot equilibrium. Thus the certainty equilibria represent all the equilibria, essentially. We shall see this result below.

Let q_S be the market price of the share. Denote by $\hat{\theta}_h$ the share after trade. Thus the induced utility function of household h is now:

$$\sum_{s=1}^S \mu^s u_h \left(e_h^0 - q z_h - q_S (\hat{\theta}_h - \theta_h), e_h^1 + r^s z_h + \hat{\theta}_h \Pi^s \right).$$

A competitive equilibrium can now be defined analogously: $(q, q_S, \tilde{r})$ constitutes an equilibrium if both the bond market and the stock market clear. An equilibrium is a sunspot equilibrium if the consumption is random for some households in the second period.¹⁰

Clearly, a certainty equilibrium is an equilibrium in this setup: simply set $q_S = f(\bar{q}\bar{B})^{-1}$, and then the bond and the share are equivalent assets, so set $\hat{\theta}_h = \theta_h$ for all h . The next non-existence result is established, by an argument which is roughly the same as that of the standard first fundamental theorem of welfare economics:¹¹ it is always possible to construct a portfolio of the stock and the bond whose payoffs are independent of sunspots. Using this portfolio, every household's utility could be improved by avoiding random income. Then such portfolio must be too expensive for every household, but this is inconsistent with the market clearing conditions.

Proposition 3 *If the profit share can be traded in period 0, there is no sunspot equilibrium.*

Proof. Suppose there is one, and let q and q_S be the bond price and the equity price in equilibrium, respectively. Let B be the bond issued by the firm in this equilibrium.

¹⁰Note that by assumption the firm's behavior is independent of the composition of shareholders.

¹¹The argument is a modification of the ingenious idea in Mas-Colell (1992). The main difference is that the returns of assets are fixed exogeneously in Mas-Colell (1992), whereas they are endogenously determined in this model.

Let x_h^s be the consumption of household h in state $s = 0, 1, \dots, S$ in the sunspot equilibrium. The feasibility implies that in the second period $\sum_{h=1}^H x_h^s = \sum_h e_h^1 + f(qB)$ for every s . Let $\hat{x}_h^1 = \sum_{s=1}^S \mu^s x_h^s$, i.e., the expected consumption, for every h . Note that from the feasibility of the equilibrium consumption x , we have

$$\begin{aligned}
\sum_{h=1}^H \hat{x}_h &= \sum_{h=1}^H \sum_{s=1}^S \mu^s x_h^s, \\
&= \sum_{s=1}^S \mu^s \sum_{h=1}^H x_h^s, \\
&= \sum_{s=1}^S \mu^s \left(\sum_{h=1}^H e_h^1 + f(qB) \right), \\
&= \sum_{h=1}^H e_h^1 + f(qB),
\end{aligned} \tag{18}$$

thus $(\hat{x}_h)_{h=1}^H$ can be attained by reallocating the good available for consumption, $\sum_h e_h^1 + f(qB)$.

By risk aversion $u_h(x_h^0, \hat{x}_h^1) \geq \sum_{s=1}^S u_h(x_h^0, x_h^s)$ and the inequality is strict for at least one h whose consumption is random. Now consider the following portfolio: buy 1 unit of share and B units of bond: then in state s , the share yields the total profit of the firm $\Pi^*(q) + (1 - r^s)B$, and the bond pays off $r^s B$. So the payoff of this portfolio is $\xi := \Pi^*(q) + B$, which is independent of states, and the cost of the portfolio is $q\xi := qB + q_S$. Since $\xi > 0$, $q\xi > 0$ follows by no arbitrage, so

$$qB + q_S > 0. \tag{19}$$

So if household h sells the whole θ_h units of the initially owned share and buys $\frac{1}{\xi}(\bar{x}_h - e_h^1)$ units of this portfolio, then household h 's consumption is exactly $\hat{x}_h$ in every state, which is more desirable. Therefore, if household h follows this activity in the bond market and the stock market, household h 's consumption in period 0 must not increase, and must decrease if h strictly prefers $\hat{x}_h^1$ to x_h in period 1: that is, we have $x_h^0 \geq e_h^0 + q_S \theta_h - q\xi \frac{1}{\xi}(\bar{x}_h - e_h^1)$, and the inequality is strict for some h . Summing up, we have

$$\begin{aligned}
\sum_{h=1}^H (x_h^0 - e_h^0) &> \sum_{h=1}^H \left(q_S \theta_h - q\xi \frac{1}{\xi}(\bar{x}_h - e_h^1) \right) \\
&= q_S - \frac{q\xi}{\xi} \sum_{h=1}^H (\bar{x}_h - e_h^1) \\
&= q_S.
\end{aligned}$$

On the other hand, the bond market clearing condition implies $\sum_{h=1}^H (x_h^0 - (e_h^0 - qB)) = 0$. So the inequality above implies $-qB > q_S$, which is a contradiction to the no-arbitrage condition (19). ■

6 Remarks

6.1 Welfare Gains and Losses

Consider a certainty equilibrium and a sunspot equilibrium close to it. Although the sunspot equilibrium must necessarily be inefficient, some households may nevertheless be better off in the sunspot equilibrium than in the certainty equilibrium. This point is first raised by Goenka - Préchac (2006) in a simple symmetric pure exchange setting, and then it is elaborated in a general exchange economy setup by Kajii (2007). These papers however do not take production into account. Here we shall discuss how the question of welfare gains and losses can be addressed in the model with production.

There are three effects which determines the economic welfare in a sunspot equilibrium, relative to the certainty equilibrium. First, sunspots make the returns of asset more random, which is bad for all households since they lose a perfect saving device.

Secondly, the equilibrium bond price is different from the efficient one. As we have argued, the equilibrium bond price tends to be higher in the non-linear technology case, making the expected real interest rate lower in sunspot equilibria. This is bad news for those who save. Consider a typical setup where households are endowed with good in period 0 only, so all the households save in equilibrium. Then this second effect is also bad for all the households.

The third effect is more delicate. A lower real interest rate is good news for the firm, and the firm tends to be more profitable in the sunspot equilibrium. The additional profits are distributed to the shareholders, so this is welfare improving. Especially for those households with relatively large share, the positive welfare effect from this channel can be large enough to offset the first two negative effects.

To sum up the discussion, we conclude that: (1) a household whose share θ_h is zero must be worse off in the sunspot equilibrium. (2) if the technology exhibits constant returns to scale, then all the households must be worse off in the sunspot equilibrium because expected profit is always zero. A formal analysis including other cases appears to be a very interesting research agenda.

6.2 Comparison with background income risk models

The sunspot model we developed in this paper has some flavor of the so called background income risk model. More specifically, imagine that the second period endowments gets slightly riskier, thus states are no longer sunspots, and the real return of the bond is fixed at one. Then by the precautionary saving argument, the saving of each household will increase assuming that $u_h^{1'''} > 0$ for every household. Therefore, the price of bond must go up and the level of production also goes up, and so this background risk model also explains a higher level of production.

However, a higher level of production in this model does not mean that there is over production. Notice that since the background risks cannot be insured, one cannot hope for full efficiency to begin with. And more importantly, one cannot necessarily say that the higher level of production under background risk is excessive, since there is no benchmark efficient level of production within the model. In our sunspot model, the certainty equilibrium is a benchmark for comparison, and the meaning of over/under production is very clear.

In the background risk model, a relevant exercise close to ours is to check the constrained efficiency of the equilibrium.¹² For instance, suppose the government can control the level of input and output by some criterion different from profit maximization, letting all the other variables be endogenously determined in the markets. Should the government find reducing the level of output beneficial to the economy, one can then argue that there is over production.¹³

6.3 Extensions

To conclude, let us provide a few remarks concerning the single good assumption in our analysis. If there are multiple consumption goods, the set of sunspot equilibria is still parametrized by $\tilde{r}$, and we believe that the existence of sunspot equilibria can be established analogously. A potential complication arises due to changes in equilibrium relative prices of goods within each spot markets. This will make the analysis potentially involved, but it appears to us that the nature of the analysis will not change as far as the existence is concerned.¹⁴ The issue of under/over production will become less clear cut, obviously. Nevertheless, we believe that analogous exercise can be done to see if the real interest rate goes down or not due to sunspots.

In the case of multiple goods, it is natural to think of many firms as well. In the standard complete markets setup, one could regard these firms as one firm which does a joint production because of the equivalence of individual firms' profit maximization and profit maximization of the aggregated firm. Then even in the sunspot set up, as long as we assume expected profit maximization, the same argument would work. However, for the non-existence result (Proposition 3), such aggregation is not neutral. *If* each consumption good is produced by one firm, and *if* all the firms' shares are traded in their respective markets, then the non-existence result will still hold. Then, it means that the aggregation of production side does not work as in the complete markets. There seems to be many interesting directions for further research.

¹²Various sorts of constrained efficiency exercises are possible in the incomplete market models. See Citanna, Kajii, and Villanacci (1998) for a general treatment and an overview of the literature.

¹³Davila *et al* (2007) considered a model of background individual uninsurable risks with no aggregate risk and asked if a competitive equilibrium is constrained efficient. They found that over production tends to occur if the households are relatively homogeneous.

¹⁴See Gottardi and Kajii (1999) for the case of pure exchange.

Appendix A Proof of Lemma 2

We shall give a proof without time additive separability assumption. The reader then will see that the other results reported in the main text can readily be shown without the separability assumption.

Fix a certainty equilibrium $(\bar{q}, 1)$ and denote by $\bar{B}$ and $\bar{z}_h$, $h = 1, \dots, H$, the bond supply and the demand in the equilibrium, respectively. For each h , let

$$F_h(q, \tilde{r}, z_h, B) := \sum_{s=1}^S \mu^s \left\{ -\bar{q} \frac{\partial}{\partial x_0} u_h + \frac{\partial}{\partial x_1} u_h \cdot r^s \right\},$$

where derivatives are evaluated at $(e_h^0 - \bar{q}z_h, (e_h^1 + \theta_h(\Pi^*(q) + B) + r^s(z_h - \theta_h B))_{s=1}^S)$. That is, $F_h(q, \tilde{r}, z_h, B) = 0$ is the first order condition for utility maximization. Thus by construction, $F_h(\bar{q}, \bar{1}, \bar{z}_h, \bar{B}) = 0$ for every h . Note also that by the additive separability and the symmetry across the states, we have for any pair of states s and s' :

$$\frac{1}{\mu^s} \frac{\partial}{\partial r^s} F_h = \frac{1}{\mu^{s'}} \frac{\partial}{\partial r^{s'}} F_h; \quad (20)$$

$$\frac{\partial^2}{\partial r^s \partial r^{s'}} F_h = 0, \text{ if } s \neq s'; \quad (21)$$

$$\frac{1}{\mu^s} \frac{\partial^2}{\partial (r^s)^2} F_h = \frac{1}{\mu^{s'}} \frac{\partial^2}{\partial (r^{s'})^2} F_h, \quad (22)$$

where the derivatives are evaluated at $(\bar{q}, \bar{1}, \bar{z}_h, \bar{B})$.

To keep the normalization $E[\tilde{r}] = 1$, as in the main text write $\tilde{r}^{-S}$ for $(r^1, \dots, r^{S-1})$, and define $\Phi_h(\tilde{r}^{-S}, z_h)$ for each h by the rule:

$$\Phi_h(\tilde{r}^{-S}, z_h) := F_h\left(\bar{q}, \left(\tilde{r}^{-S}, \frac{1}{\mu^S} \left(1 - \sum_{s=1}^{S-1} \mu^s r^s\right)\right), z_h, \bar{B}\right).$$

Under our maintained assumptions on the utility function, the change in the modified demand $\hat{Z}_h(\bar{q}, \cdot, \bar{B})$ (see (11)) is given by the implicit function theorem applied to the identity $\Phi_h(\tilde{r}^{-S}, z_h) = 0$.

First, we shall show that $\frac{\partial}{\partial r^s} \hat{Z}_h = 0$ at $\tilde{r}^{-S} = \bar{1}^{-S}$. Indeed, evaluated at $\tilde{r}^{-S} = \bar{1}^{-S}$ and $z_h = \bar{z}_h$, we have $\frac{\partial}{\partial r^s} \Phi_h = \frac{\partial}{\partial r^s} F_h + \frac{\partial}{\partial r^S} F_h \cdot \left(-\frac{\mu^s}{\mu^S}\right)$ for each $s = 1, \dots, S-1$, and $\frac{\partial}{\partial z_h} \Phi_h = \frac{\partial}{\partial z_h} F_h$. Therefore, by differentiating the identity $\Phi_h(\tilde{r}^{-S}, z_h) = 0$, we have $\frac{\partial}{\partial r^s} \Phi_h + \frac{\partial}{\partial z_h} \Phi_h \frac{\partial}{\partial r^s} \hat{Z}_h = \frac{\partial}{\partial r^s} F_h - \left(\frac{\mu^s}{\mu^S}\right) \frac{\partial}{\partial r^S} F_h + \frac{\partial}{\partial z_h} F_h \frac{\partial}{\partial r^s} \hat{Z}_h = 0$, for each $s = 1, \dots, S-1$. The symmetry relation (20) then implies that this equation is reduced to $\frac{\partial}{\partial z_h} F_h \frac{\partial}{\partial r^s} \hat{Z}_h = 0$, so $\frac{\partial}{\partial r^s} \hat{Z}_h = 0$ must hold for each $s = 1, \dots, S-1$, since $\frac{\partial}{\partial z_h} F_h$ is not zero.

Next, we calculate the second order effect, $\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h$. Set $\gamma_h := -\frac{\partial}{\partial z_h} F_h (= -\frac{\partial}{\partial z_h} \Phi_h)$, and set α_h to be the common constant in (22), so $\frac{\partial^2}{\partial (r^s)^2} F_h = \mu^s \alpha_h$, $s = 1, \dots, S-1$. We shall show that $\left(\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h\right)_{s,s'} = \frac{\alpha_h}{\gamma_h} M$, where M is an $S-1$ dimensional positive definite matrix determined by probability μ (thus in particular independent of h).

Differentiate the identity $\frac{\partial}{\partial r^s} \Phi_h + \frac{\partial}{\partial z_h} \Phi_h \frac{\partial}{\partial r^s} \hat{Z}_h = 0$ with respect to $r^{s'}$, and evaluate the result at $\tilde{r}^{-S} = \tilde{1}^{-S}$ and $z_h = \bar{z}_h$. Both $\frac{\partial}{\partial r^s} \Phi_h$ and $\frac{\partial}{\partial z_h} \Phi_h$ are functions of z_h , but since $\frac{\partial}{\partial r^s} \hat{Z}_h = 0$ when $\tilde{r}^{-S} = \tilde{1}^{-S}$ and $z_h = \bar{z}_h$, these indirect effects vanish. Also the cross effect $\frac{\partial^2}{\partial z_h \partial s'} \Phi_h$ is multiplied by $\frac{\partial}{\partial r^s} \hat{Z}_h$ and so this also vanishes. Thus the resulting equation is simplified as follows:

$$\frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h + \frac{\partial}{\partial z_h} \Phi_h \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h = 0. \quad (23)$$

So solving (23) we have $\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h = - \left(\frac{\partial}{\partial z_h} \Phi_h \right)^{-1} \frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h = \frac{1}{\gamma_h} \frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h$. Now by construction, $\frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h = \frac{\partial}{\partial r^{s'}} \left(\frac{\partial}{\partial r^s} F_h + \frac{\partial}{\partial r^s} F_h \cdot \left(-\frac{\mu^s}{\mu^S} \right) \right) = \left(\frac{\partial^2}{\partial r^s \partial r^{s'}} F_h + \frac{\partial^2}{\partial r^s \partial r^s} F_h \cdot \left(-\frac{\mu^{s'}}{\mu^S} \right) \right) + \left(-\frac{\mu^s}{\mu^S} \right) \left(\frac{\partial^2}{\partial r^s \partial r^{s'}} F_h + \frac{\partial^2}{(\partial r^s)^2} F_h \left(-\frac{\mu^{s'}}{\mu^S} \right) \right) = \frac{\partial^2}{\partial r^s \partial r^{s'}} F_h + \alpha_h \left(\frac{\mu^s \mu^{s'}}{\mu^S} \right)$, where the last equality holds by (21) and $\frac{\partial^2}{\partial (r^s)^2} F_h = \mu^S \alpha_h$. Thus if $s \neq s'$, $\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h = \frac{\alpha_h \mu^s \mu^{s'}}{\gamma_h \mu^S}$ by (21). For $s = s'$, $\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h = \frac{1}{\gamma_h} \left(\mu^s \alpha_h + \alpha_h \left(\frac{(\mu^s)^2}{\mu^S} \right) \right) = \frac{\alpha_h}{\gamma_h} \left(\mu^s + \left(\frac{(\mu^s)^2}{\mu^S} \right) \right)$. Writing these in a matrix form, we obtain the following:

$$\left(\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h \right)_{s,s'} = \frac{\alpha_h}{\gamma_h} \left(\begin{bmatrix} \mu^1 & 0 & \cdots & 0 \\ 0 & \mu^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mu^{S-1} \end{bmatrix} + \frac{1}{\mu^S} \begin{bmatrix} \mu^1 \\ \vdots \\ \mu^{S-1} \end{bmatrix} [\mu^1, \dots, \mu^{S-1}] \right). \quad (24)$$

The two matrices consisting of $(\mu^1, \dots, \mu^S)$ in (24) are both positive definite, and they are determined by probabilities only. Thus we have established the desired property of $\left(\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h \right)_{s,s'}$.

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