

Univalence and starlikeness of a function defined by convolution of analytic function and hypergeometric function ${}_3F_2$

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Abstract

We consider functions defined by a condition of functions in the subclass $\mathcal{U}(\lambda)$ of analytic functions with generalized Gauss hypergeometric functions. In this paper, we give a condition of the parameter λ for which the function to be univalent and starlike.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ that are univalent in $\mathbb{U}$.

Obradović and Ponnusamy define in [4] the class $\mathcal{U}(\lambda)$ of $f(z) \in \mathcal{A}$ satisfying the condition

$$(1.2) \quad \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| \leq \lambda \quad (z \in \mathbb{U})$$

for some real $\lambda > 0$, where f' denotes the derivative of f with respect to the variable z . We set $\mathcal{U}(1) = \mathcal{U}$. It is easy to see that the condition (1.2) is equivalent to

$$\left| z^2 \left(\frac{1}{f(z)} - \frac{1}{z} \right)' \right| \leq \lambda \quad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{S}$ maps $\mathbb{U}$ onto a starlike domain (with respect to the origin), i.e. if $tw \in f(\mathbb{U})$ whenever $t \in [0, 1]$ and $w \in f(\mathbb{U})$, then we say that f is a starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$. A necessary and sufficient condition for $f(z) \in \mathcal{A}$ to be starlike is that the inequality

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$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbf{U})$$

holds.

For these facts, the following lemmas hold.

Lemma 1 ([3]) If $f(z) \in \mathcal{U}(\lambda)$, $a := \frac{|f''(0)|}{2} \leq 1$ and $0 \leq \lambda \leq \frac{\sqrt{2-a^2}-a}{2}$, then $f(z) \in \mathcal{S}^*$.

Lemma 2 ([7]) If $f(z) = z + a_{n+1}z^{n+1} + \dots$ ($n \geq 2$) belongs to $\mathcal{U}(\lambda)$ and

$$0 \leq \lambda \leq \frac{n-1}{\sqrt{(n-1)^2+1}},$$

then $f(z) \in \mathcal{S}^*$.

For analytic functions $f(z)$ and $g(z)$ on $\mathbf{U}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ is said the convolution of $f(z)$ and $g(z)$, denoted by $f * g$ (cf ([5])).

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{A}$, we have a natural convolution operator defined by

$$zF(a, b; c; z) * f(z) := \sum_{n=1}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n, \quad c \in \{-1, -2, -3, \dots\}, z \in \mathbf{U},$$

where $(a)_n$ denotes the Pochhammer symbol $(a)_0 = 1$, $(a)_n = a(a+1) \cdots (a+n-1)$ for $n \in \mathbf{N}$. Here $F(a, b; c; z)$ denotes the Gauss hypergeometric function which is analytic in $\mathbf{U}$. As a special case of the Euler integral representation for the hypergeometric function, one has

$$F(1, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{1}{1-tz} t^{b-1} (1-t)^{c-b-1} dt, \quad z \in \mathbf{U}, \operatorname{Re} c > \operatorname{Re} b > 0.$$

Using this representation, we have, for $f(z) \in \mathcal{A}$,

$$zF(1, c; c+1; z) * f(z) = z \left(F(1, c; c+1; z) * \frac{f(z)}{z} \right) = zc \int_0^1 \frac{f(tz)}{tz} t^{c-1} dt, \quad z \in \mathbf{U}, \operatorname{Re} c > 0.$$

Obradović and Ponnusamy have shown the following result.

Theorem A ([5])

Let $f \in \mathcal{U}(\lambda)$ and $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$ such that

$$\left(\frac{z}{f(z)} \right) * F(1, c; c+1; z) \neq 0 \quad \text{in } z \in \mathbf{U},$$

and $G(z) = G_f^c(z)$ be the transformed function defined by

$$G(z) = \frac{z}{\left(\frac{z}{f(z)}\right) * F(1, c; c+1; z)} \quad (z \in \mathbf{U}).$$

Then we have the following;

- (1) $G \in \mathcal{U}\left(\frac{\lambda|c|}{|c+2|}\right)$. The result is sharp especially when $\left|\frac{f''(0)}{2}\right| \leq 1 - \lambda$. In particular, $G \in \mathcal{U}$ whenever $0 < \lambda \leq \left|\frac{c+2}{c}\right|$.
- (2) $G \in \mathcal{S}^*$ whenever $0 < \lambda \leq \frac{|c+2|}{2|c|}(\sqrt{2-A^2} - A)$ with $A = \left|\frac{c}{c+1} \frac{f''(0)}{2}\right| \leq 1$.

2 Main Result

For the generalized hypergeometric function ${}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z)$, we obtain

Theorem1

Let $f(z) \in \mathcal{U}(\lambda)$. Let $\alpha, \beta \in \mathbb{C}$ satisfying

$$\operatorname{Re} \alpha \geq 0, \operatorname{Re} \beta \geq 0, \frac{1}{|\alpha + \beta|} \left(\frac{|\alpha||\beta|}{|\beta+2|} + \frac{|\beta||\alpha|}{|\alpha+2|} \right) < 1 \text{ and } |\alpha + \beta| > |\alpha\beta|$$

and

$$\frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z) \neq 0, \quad z \in \mathbf{U}.$$

Denote by $G(z) = G_f^{\alpha, \beta}(z)$ the function defined by

$$(2.1) \quad G(z) = \frac{z}{\frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z)}, \quad z \in \mathbf{U},$$

where ${}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z)$ is the generalized hypergeometric function. Then we have the following:

- (1) $G(z) \in \mathcal{U}\left(\frac{\lambda|\alpha + \beta|}{|\alpha + \beta + 4|}\right)$. The result is sharp especially when $\left|\frac{f''(0)}{2}\right| \leq 1 - \lambda$.

In particular, $G(z) \in \mathcal{U}$ whenever $0 < \lambda \leq \frac{|\alpha + \beta + 4|}{|\alpha + \beta|}$.

- (2) $G(z) \in \mathcal{S}^*$

whenever $0 < \lambda \leq \frac{|\alpha + \beta + 4|}{2|\alpha + \beta|}(\sqrt{2-A^2} - A)$ with $A = \left|\frac{\alpha\beta}{(\alpha+1)(\beta+1)} \frac{f''(0)}{2}\right| \leq 1$.

Proof.

Since

$$(2.2) \quad {}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z) = \sum_{n=0}^{\infty} \frac{\alpha\beta}{(\alpha+n)(\beta+n)} z^n = 1 + \sum_{n=1}^{\infty} \frac{\alpha\beta}{(\alpha+n)(\beta+n)} z^n,$$

we have

$$\begin{aligned}
 \frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z) &= 1 - \frac{\alpha\beta a_2}{(\alpha+1)(\beta+1)}z + \frac{\alpha\beta(a_2^2 - a_3)}{(\alpha+2)(\beta+2)}z^2 + \dots \\
 &= \left\{ 1 - \frac{\alpha a_2}{\alpha+1}z + \frac{\alpha(a_2^2 - a_3)}{\alpha+2}z^2 + \dots \right\} * \left\{ 1 - \frac{\beta a_2}{\beta+1}z + \frac{\beta(a_2^2 - a_3)}{\beta+2}z^2 + \dots \right\} \\
 &= \left\{ \frac{z}{f(z)} * F(1, \alpha; \alpha+1; z) \right\} * F(1, \beta; \beta+1; z).
 \end{aligned}$$

Thus $G(z)$ can be written as

$$G(z) = \frac{z}{\left\{ \frac{z}{f(z)} * F(1, \alpha; \alpha+1; z) \right\} * F(1, \beta; \beta+1; z)}.$$

In the same manner, $G(z)$ can be also written as

$$G(z) = \frac{z}{\left\{ \frac{z}{f(z)} * F(1, \beta; \beta+1; z) \right\} * F(1, \alpha; \alpha+1; z)}.$$

Put

$$h_1(z) = \frac{z}{\frac{z}{f(z)} * F(1, \alpha; \alpha+1; z)}, \quad h_2(z) = \frac{z}{\frac{z}{f(z)} * F(1, \beta; \beta+1; z)}.$$

then

$$\frac{z}{f(z)} * F(1, \alpha; \alpha+1; z) = \frac{z}{h_1(z)}, \quad \frac{z}{f(z)} * F(1, \beta; \beta+1; z) = \frac{z}{h_2(z)}$$

By the Theorem A in the introduction, we have

$$h_1(z) \in \mathcal{U} \left(\frac{\lambda|\alpha|}{|\alpha+2|} \right) \quad \text{i.e.} \quad \left| \left(\frac{z}{h_1(z)} \right)^2 h_1'(z) - 1 \right| < \frac{\lambda|\alpha|}{|\alpha+2|}$$

and

$$h_2(z) \in \mathcal{U} \left(\frac{\lambda|\beta|}{|\beta+2|} \right) \quad \text{i.e.} \quad \left| \left(\frac{z}{h_2(z)} \right)^2 h_2'(z) - 1 \right| < \frac{\lambda|\beta|}{|\beta+2|}.$$

Since

$$\frac{z}{G(z)} = \frac{z}{h_1(z)} * F(1, \beta; \beta+1; z) \quad (z \in \mathbf{U}),$$

we have

$$(2.3) \quad (\beta+1) \frac{z}{G(z)} - \left(\frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{G(z)} + z \left(\frac{z}{G(z)} \right)'.$$

On the other hand, $\frac{z}{G(z)}$ can be also written as

$$\frac{z}{G(z)} = \frac{z}{h_2(z)} * F(1, \alpha; \alpha+1; z) \quad (z \in \mathbf{U}),$$

we have

$$(2.4) \quad (\beta + 1) \frac{z}{G(z)} - \left(\frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{G(z)} + z \left(\frac{z}{G(z)} \right)'.$$

Then we have

$$(2.5) \quad (\alpha + 1) \frac{z}{G(z)} - \left(\frac{z}{G(z)} \right)^2 G'(z) = \alpha \frac{z}{h_2(z)} \quad (z \in \mathbb{U})$$

and

$$(2.6) \quad (\beta + 1) \frac{z}{G(z)} - \left(\frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{h_1(z)} \quad (z \in \mathbb{U}).$$

Set

$$p(z) = \left(\frac{z}{G(z)} \right)^2 G'(z).$$

Then $p(z)$ is analytic on $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$, and

$$(2.7) \quad p(z) = (\alpha + 1) \frac{z}{G(z)} - \alpha \frac{z}{h_2(z)}$$

and

$$(2.8) \quad p(z) = (\beta + 1) \frac{z}{G(z)} - \beta \frac{z}{h_1(z)}.$$

From (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8) one then obtain that

$$\begin{aligned} \alpha p(z) + zp'(z) &= (\alpha + 1) \alpha \frac{z}{G(z)} + (\alpha + 1) z \left(\frac{z}{G(z)} \right)' - \alpha^2 \frac{z}{h_2(z)} - \alpha z \left(\frac{z}{h_2(z)} \right)' \\ &= \alpha \left[(\alpha + 1) \frac{z}{h_2(z)} - \alpha \frac{z}{h_2(z)} - z \left(\frac{z}{h_2(z)} \right)' \right] \\ &= \alpha \left[\frac{z}{h_2(z)} - z \left(\frac{z}{h_2(z)} \right)' \right] \\ &= \alpha \left(\frac{z}{h_2(z)} \right)^2 h_2'(z) \end{aligned}$$

and

$$\begin{aligned} \beta p(z) + zp'(z) &= (\beta + 1) \beta \frac{z}{G(z)} + (\beta + 1) z \left(\frac{z}{G(z)} \right)' - \beta^2 \frac{z}{h_1(z)} - \beta z \left(\frac{z}{h_1(z)} \right)' \\ &= \beta \left[(\beta + 1) \frac{z}{h_1(z)} - \beta \frac{z}{h_1(z)} - z \left(\frac{z}{h_1(z)} \right)' \right] \\ &= \beta \left[\frac{z}{h_1(z)} - z \left(\frac{z}{h_1(z)} \right)' \right] \\ &= \beta \left(\frac{z}{h_1(z)} \right)^2 h_1'(z). \end{aligned}$$

Since

$$(\alpha + \beta)p(z) + 2zp'(z) = \alpha \left(\frac{z}{h_2(z)} \right)^2 h_2'(z) + \beta \left(\frac{z}{h_1(z)} \right)^2 h_1'(z),$$

we have

$$p(z) + \frac{2}{\alpha + \beta} zp'(z) = \frac{\alpha}{\alpha + \beta} \left(\frac{z}{h_2(z)} \right)^2 h_2'(z) + \frac{\beta}{\alpha + \beta} \left(\frac{z}{h_1(z)} \right)^2 h_1'(z).$$

Now, as $h_1(z) \in \mathcal{U} \left(\frac{\lambda|\alpha|}{|\alpha + 2|} \right)$ and $h_2(z) \in \mathcal{U} \left(\frac{\lambda|\beta|}{|\beta + 2|} \right)$, it follows that

$$\begin{aligned} \left| p(z) + \frac{2}{\alpha + \beta} zp'(z) - 1 \right| &= \left| \frac{\alpha}{\alpha + \beta} \left\{ \left(\frac{z}{h_2(z)} \right)^2 h_2'(z) - 1 \right\} + \frac{\beta}{\alpha + \beta} \left\{ \left(\frac{z}{h_1(z)} \right)^2 h_1'(z) - 1 \right\} \right| \\ &\leq \left| \frac{\alpha}{\alpha + \beta} \right| \left| \left(\frac{z}{h_2(z)} \right)^2 h_2'(z) - 1 \right| + \left| \frac{\beta}{\alpha + \beta} \right| \left| \left(\frac{z}{h_1(z)} \right)^2 h_1'(z) - 1 \right| \\ &< \frac{|\alpha|}{|\alpha + \beta|} \frac{\lambda|\beta|}{|\beta + 2|} + \frac{|\beta|}{|\alpha + \beta|} \frac{\lambda|\alpha|}{|\alpha + 2|} \\ &= \lambda \left\{ \frac{1}{|\alpha + \beta|} \left(\frac{|\alpha||\beta|}{|\beta + 2|} + \frac{|\beta||\alpha|}{|\alpha + 2|} \right) \right\}. \end{aligned}$$

By the assumption, we have

$$(2.9) \quad \left| p(z) + \frac{2}{\alpha + \beta} zp'(z) - 1 \right| < \lambda.$$

From the work of Hallenbeck and Rusheweyh ([2],[6]), we deduce that

$$(2.10) \quad |p(z) - 1| \leq \frac{\lambda|\alpha + \beta|}{|\alpha + \beta + 4|} \quad (z \in \mathbb{U}).$$

Thus we have $G(z) \in \mathcal{U} \left(\frac{\lambda|\alpha + \beta|}{|\alpha + \beta + 4|} \right)$.

To prove the sharpness, we consider functions $f(z)$ in $\mathcal{U}(\lambda)$ of the form

$$f(z) = \frac{z}{1 - a_2 z + \lambda z^2},$$

where $a_2 = \frac{f''(0)}{2}$ and $|a_2| \leq 1 - \lambda$, so that $1 - a_2 z + \lambda z^2 \neq 0$ for all $z \in \mathbb{U}$. Since $\operatorname{Re} \alpha \geq 0$ and $\operatorname{Re} \beta \geq 0$, it follows that $|\alpha + 2| > |\alpha + 1| > |\alpha|$ and $|\beta + 2| > |\beta + 1| > |\beta|$ and, therefore

$$\left| 1 - a_2 \frac{\alpha\beta}{(\alpha + 1)(\beta + 1)} z + \lambda \frac{\alpha\beta}{(\alpha + 2)(\beta + 2)} z^2 \right| \neq 0$$

for all $z \in \mathbb{U}$, provided $|a_2| \leq 1 - \lambda$. By the series expansion (2.2) of ${}_3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z)$, we have

$$G(z) = \frac{z}{1 - \frac{a_2 \alpha \beta}{(\alpha + 1)(\beta + 1)} z + \frac{\lambda(\alpha\beta)}{(\alpha + 2)(\beta + 2)} z^2}.$$

Obviously, $G(z)$ is analytic on $\mathbf{U}$ and $\frac{z}{G(z)} \neq 0$ on $\mathbf{U}$. Since

$$\left(\frac{z}{G(z)}\right)^2 G'(z) - 1 = -\frac{\lambda\alpha\beta}{(\alpha+2)(\beta+2)}z^2,$$

we have that

$$(2.11) \quad \left|\left(\frac{z}{G(z)}\right)^2 G'(z) - 1\right| \leq \frac{\lambda|\alpha\beta|}{|(\alpha+2)(\beta+2)|}.$$

Now, let us compare the right hand sides of (2.10) and (2.11). Firstly, since $|\alpha + \beta + 4| < |(\alpha + 2)(\beta + 2)|$, then $\frac{1}{|(\alpha + 2)(\beta + 2)|} < \frac{1}{|\alpha + \beta + 4|}$. From the assumption, we see

$$\frac{|\alpha\beta|}{|(\alpha+2)(\beta+2)|} < \frac{|\alpha+\beta|}{|(\alpha+2)(\beta+2)|} < \frac{|\alpha+\beta|}{|\alpha+\beta+4|}.$$

Then, we have that

$$\left|\left(\frac{z}{G(z)}\right)^2 G'(z) - 1\right| \leq \frac{\lambda|\alpha\beta|}{|(\alpha+2)(\beta+2)|} < \frac{|\alpha+\beta|}{|\alpha+\beta+4|}.$$

Thus, we have that the bound $\frac{|\alpha+\beta|}{|\alpha+\beta+4|}$ is sharp. We conclude that the first assertion of Theorem 1.

The second assertion is a direct consequence of Lemma 1. In fact, obviously

$$A = \frac{G''(0)}{2} = \frac{\alpha\beta}{(\alpha+1)(\beta+1)} \frac{f''(0)}{2}$$

is smaller than or equal to 1.

Theorem 2

For a fixed $n \geq 2$, let $f(z) = z + a_{n+1}z^{n+1} + \dots$ belong to $\mathcal{U}(\lambda)$. Let $\alpha, \beta \geq 0$ and

$$\operatorname{Re} \alpha \geq 0, \operatorname{Re} \beta \geq 0, \frac{1}{|\alpha+\beta|} \left(\frac{|\alpha||\beta|}{|\beta+n|} + \frac{|\alpha||\beta|}{|\alpha+n|} \right) < 1,$$

and

$$\frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z) \neq 0, \quad z \in \mathbf{U}.$$

and $G(z) = G_f^{\alpha, \beta}(z)$ be the transform function defined by (2.1). Then we have the following:

- (1) $G(z) \in \mathcal{U} \left(\frac{\lambda|\alpha+\beta|}{|\alpha+\beta+2n|} \right)$. In particular, $G(z) \in \mathcal{U}$ whenever $0 < \lambda \leq \frac{|\alpha+\beta+2n|}{|\alpha+\beta|}$.
- (2) $G(z) \in S^*$ whenever $0 < \lambda \leq \frac{(n-1)|\alpha+\beta+2n|}{|\alpha+\beta|\sqrt{(n-1)^2+1}}$.

Proof. Using the Gaussian hypergeometric function, $G(z)$ can be written as

$$G(z) = \frac{z}{\left\{ \frac{z}{f(z)} * F(1, \alpha; \alpha + 1; z) \right\} * F(1, \beta; \beta + 1; z)}$$

and

$$G(z) = \frac{z}{\left\{ \frac{z}{f(z)} * F(1, \beta; \beta + 1; z) \right\} * F(1, \alpha; \alpha + 1; z)}.$$

Put

$$h_3(z) = \frac{z}{\frac{z}{f(z)} * F(1, \alpha; \alpha + 1; z)}, \quad h_4(z) = \frac{z}{\frac{z}{f(z)} * F(1, \beta; \beta + 1; z)}.$$

Then

$$\frac{z}{f(z)} * F(1, \alpha; \alpha + 1; z) = \frac{z}{h_3(z)}, \quad \frac{z}{f(z)} * F(1, \beta; \beta + 1; z) = \frac{z}{h_4(z)},$$

We see

$$h_3(z) \in \mathcal{U} \left(\frac{\lambda|\alpha|}{|\alpha + n|} \right) \quad \text{i.e.} \quad \left| \left(\frac{z}{h_3(z)} \right)^2 h_3'(z) - 1 \right| < \frac{\lambda|\alpha|}{|\alpha + n|}$$

and

$$h_4(z) \in \mathcal{U} \left(\frac{\lambda|\beta|}{|\beta + n|} \right) \quad \text{i.e.} \quad \left| \left(\frac{z}{h_4(z)} \right)^2 h_4'(z) - 1 \right| < \frac{\lambda|\beta|}{|\beta + n|}.$$

Since

$$\frac{z}{f(z)} = \frac{1}{1 + a_{n+1}z^n + \dots} = 1 - a_{n+1}z^n + \dots,$$

so that

$$\frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z) = 1 - a_{n+1} \left\{ \frac{\alpha\beta}{(\alpha + n)(\beta + n)} \right\} z^n + \dots.$$

Thus, $G(z)$ can be written in the form

$$G(z) = z + a_{n+1} \left\{ \frac{\alpha\beta}{(\alpha + n)(\beta + n)} \right\} z^{n+1} + \dots.$$

Therefore, as in the proof of Theorem 1, the function $p(z)$ defined by

$$p(z) = \left(\frac{z}{G(z)} \right)^2 G'(z) = 1 + (n-1)a_{n+1} \left\{ \frac{\alpha\beta}{(\alpha + n)(\beta + n)} \right\} z^n + \dots$$

is analytic in $\mathbf{U}$ and $p(0) = 1$, $p'(0) = \dots = p^{(n-1)}(0) = 0$. $p(z)$ can be written as

$$p(z) = (\alpha + 1) \frac{z}{G(z)} - \alpha \frac{z}{h_3(z)}$$

and

$$p(z) = (\beta + 1) \frac{z}{G(z)} - \beta \frac{z}{h_4(z)}.$$

By the same argument of proof of Theorem 1 using $h_3(z)$ and $h_4(z)$ instead of $h_1(z)$ and $h_2(z)$, $p(z)$ satisfies (2.9). Consequently, we obtain that

$$|p(z) - 1| \leq \frac{\lambda|\alpha + \beta||z|^n}{|\alpha + \beta + 2n|} \quad (z \in \mathbf{U}),$$

and the proof of part(1) is complete. The second part is a direct consequence of Lemma 2.

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