

Fourier transform for prehomogeneous vector spaces over finite field

Kazuki Ishimoto

Graduate School of Mathematics, Kobe University

Abstract

Let (G, V) be a prehomogeneous vector space over a finite field of odd characteristic. Taniguchi and Thorne [2] developed a method to calculate explicit formulas of the Fourier transforms of any G -invariant functions over V . By means of their method, we calculate the Fourier transform of any G -invariant function for several prehomogeneous vector spaces. .

1 Introduction

Let K be a field and $\overline{K}$ be the algebraic closure. Let V be a finite dimensional representation of a reductive algebraic group G defined over K . When there exists a $G(\overline{K})$ -orbit of $V(\overline{K})$ which is Zariski open, we refer to the pair (G, V) as a prehomogeneous vector space. Taniguchi and Thorne [2] developed a general method to compute the Fourier transform and applied it to obtain explicit formulas for the prehomogeneous vector spaces $1 \otimes \text{Sym}^2(\mathbb{F}_q^2)$, $\text{Sym}^3(\mathbb{F}_q^2)$, $1 \otimes \text{Sym}^2(\mathbb{F}_q^3)$, $2 \otimes \text{Sym}^2(\mathbb{F}_q^2)$, $2 \otimes \text{Sym}^2(\mathbb{F}_q^3)$, where $\mathbb{F}_q$ is the finite field of order a prime power q . There are many prehomogeneous vector spaces for which the explicit formula of the Fourier transform is not yet calculated. The speaker calculated the explicit formula of the Fourier transform for 9 more prehomogeneous vector spaces $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2$, $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^3$, $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^4$, $\mathbb{F}_q^2 \otimes \mathbb{H}_2(\mathbb{F}_{q^2})$, $\mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^4)$, the space of binary tri-Hermitian forms over $\mathbb{F}_{q^3}$, $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$, $\mathbb{F}_q^2 \otimes \mathbb{H}_3(\mathbb{F}_{q^2})$, $\mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^6)$ by using the method developed by Taniguchi and Thorne. In this paper, we see the calculation method and the results for some spaces.

2 Fourier transform

Let p is an odd prime and $\mathbb{F}_p$ is the finite field of order p . Let q is any power of p and $\mathbb{F}_q$ is the finite field of order q . Let V be a finite dimensional vector space over $\mathbb{F}_q$ with a finite group G linearly acting on V . Suppose the pair (G, V) satisfies the following Assumption 2.

Assumption 1. *There exist an automorphism $\iota : G \ni g \mapsto g^\iota \in G$ of order 2 and a non-degenerate bilinear form $\beta : V \times V \rightarrow \mathbb{F}_q$ such that*

$$\beta(gx, g^\iota y) = \beta(x, y) \quad (x, y \in V, g \in G).$$

Then we can identify the dual space V^* with V by the linear isomorphism $V \ni x \mapsto \beta(x, \cdot) \in V^*$ (see [2] for detail). We reformulate the definition of the Fourier transform only in terms of V . For $\phi : V \rightarrow \mathbb{C}$, we define its Fourier transform $\widehat{\phi} : V \rightarrow \mathbb{C}$ as follows:

$$\widehat{\phi}(y) := |V|^{-1} \sum_{x \in V} \phi(x) \exp\left(\frac{2\pi i \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\beta(x, y))}{p}\right). \quad (1)$$

Here $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ is the trace map. Let $\mathcal{F}_V^G$ be the set of all G -invariant maps from V to $\mathbb{C}$, i.e.,

$$\mathcal{F}_V^G := \{\phi : V \rightarrow \mathbb{C} \mid \phi(gx) = \phi(x) \ (g \in G, x \in V)\}.$$

Note that $\mathcal{F}_V^G$ is a finite dimensional vector space over $\mathbb{C}$. We can easily see that if ϕ is a G -invariant function, $\widehat{\phi}$ is also G -invariant. In fact, the Fourier transform map $\mathcal{F}_V^G \ni \phi \mapsto \widehat{\phi} \in \mathcal{F}_V^G$ is a linear isomorphism. Let $\mathcal{O}_i (1 \leq i \leq r)$ be all the distinct G -orbits in V , and for each i let e_i be the indicator function of $\mathcal{O}_i$. The functions $e_1, \dots, e_r$ are clearly G -invariant, and they form a basis of $\mathcal{F}_V^G$. Thus we only have to calculate the Fourier transform of $e_1, \dots, e_r$ to calculate that of all $\phi \in \mathcal{F}_V^G$. We use the following proposition for our calculation of $\widehat{e}_i$.

Proposition 2. [2, Proposition 6] Let W be a subspace of V , and let $W^\perp := \{y \in V \mid \forall x \in W, \beta(x, y) = 0\}$. Then

$$\sum_{i=1}^r \frac{|\mathcal{O}_i \cap W|}{|\mathcal{O}_i|} \cdot \widehat{e}_i = \frac{|W|}{|V|} \sum_{j=1}^r \frac{|\mathcal{O}_j \cap W^\perp|}{|\mathcal{O}_j|} \cdot e_j.$$

In this paper, we call $W^\perp$ the orthogonal complement of W . By Proposition 3, when we choose one subspace of V , we obtain one equation of linear combinations of $\widehat{e}_i$ and e_j . Therefore if we choose r different subspaces and the corresponding equations are linearly independent, we obtain an expression of each $\widehat{e}_i$ in terms of $e_1, \dots, e_r$. In other words, we can determine the following r -by- r matrix M explicitly:

$$(\widehat{e}_1, \dots, \widehat{e}_n) = (e_1, \dots, e_n)M.$$

We calculate the matrix M with this approach.

3 Main result

The speaker calculated the Fourier transforms for the following prehomogeneous vector spaces over a finite field:

- $V = 2 \otimes 2 \otimes 2$, the space of pairs of 2-by-2matrices; $G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_2$,
- $V = 2 \otimes 2 \otimes 3$, the space of triplets of 2-by-2matrices; $G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_3$,
- $V = 2 \otimes 2 \otimes 4$, the space of quadruples of 2-by-2matrices; $G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_4$,
- $V = 2 \otimes \text{H}_2(\mathbb{F}_{q^2})$, the space of pairs of Hermitian matrices of order 2; $G = \text{GL}_2 \times \text{GL}_2(\mathbb{F}_{q^2})$,
- $V = 2 \otimes \wedge^2(4)$, the space of pairs of alternating matrices of order 4; $G = \text{GL}_2 \times \text{GL}_4$,
- V is the space of binary tri-Hermitian forms over $\mathbb{F}_{q^3}$; $G = \text{GL}_1 \times \text{GL}_2(\mathbb{F}_{q^3})$,
- $V = 2 \otimes 3 \otimes 3$, $G = \text{GL}_2 \times \text{GL}_3 \times \text{GL}_3$,
- $V = 2 \otimes \text{H}_3(\mathbb{F}_{q^2})$, $G = \text{GL}_2 \times \text{GL}_3(\mathbb{F}_{q^2})$,
- $V = 2 \otimes \wedge^2(6)$, $G = \text{GL}_2 \times \text{GL}_6$.

Here, we write about the complicated cases $V = 2 \otimes 3 \otimes 3, 2 \otimes \text{H}_3(\mathbb{F}_{q^2}), 2 \otimes \wedge^2(6)$.

3.1 $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

Let $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ and $G = G_1 \times G_2 \times G_3 = \text{GL}_2 \times \text{GL}_3 \times \text{GL}_3$. We write $x \in V$ as $x = (A, B)$ where A and B are 3-by-3 matrices, and write $g \in G$ as $g = (g_1, g_2, g_3)$ where $g_1 \in \text{GL}_2$ and $g_2, g_3 \in \text{GL}_3$. G acts on V by

$$gx = (g_2 A g_3^T, g_2 B g_3^T) g_1^T.$$

V consists of 21 G -orbits in all. The following elements $x_1, \dots, x_{21}$ are representatives:

$$\begin{aligned} x_1 &= (0, 0), x_2 = \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, 0 \right), x_3 = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, 0 \right), x_4 = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, 0 \right), \\ x_5 &= \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 1 & \\ 0 & & \\ & & 0 \end{bmatrix} \right), x_6 = \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 0 & \\ 1 & & \\ & & 0 \end{bmatrix} \right), x_7 = \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & & \\ & & 0 \end{bmatrix} \right), \\ x_8 &= \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} \right), x_9 = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & -1 & \\ \mu_0 & \mu_1 & \\ & & 0 \end{bmatrix} \right), x_{10} = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 0 & \\ 0 & & \\ & & 1 \end{bmatrix} \right), \end{aligned}$$

$$\begin{aligned}
x_{11} &= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & & 1 \\ & 1 & \\ & & 0 \end{bmatrix} \right), x_{12} = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & & 0 \\ & 0 & \\ 1 & & \end{bmatrix} \right), x_{13} = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 1 & \\ & & 0 \\ 1 & & \end{bmatrix} \right), \\
x_{14} &= \left(\begin{bmatrix} & 1 & \\ 1 & & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix} \right), x_{15} = \left(\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \right), x_{16} = \left(\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix} 0 & & \\ & & 1 \\ & 1 & \end{bmatrix} \right), \\
x_{17} &= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \right), x_{18} = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & & 1 \\ 1 & & \end{bmatrix} \right), x_{19} = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right), \\
x_{20} &= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & -1 & \\ \mu_0 & \mu_1 & \\ & & 1 \end{bmatrix} \right), x_{21} = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} & -1 & \\ \nu_2 & -1 & \\ \nu_0 & \nu_1 & \end{bmatrix} \right).
\end{aligned}$$

Here, $\mu_1, \mu_0, \nu_2, \nu_1, \nu_0 \in \mathbb{F}_q$ are elements such that $X^2 + \mu_1 X + \mu_0, X^3 + \nu_2 X^2 + \nu_1 X + \nu_0 \in \mathbb{F}_q[X]$ is irreducible.

The subspaces we choose to calculate the Fourier transform are as follows:

$$\begin{aligned}
W_1 &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), W_2 = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} \right), W_3 = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \right), \\
W_4 &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \right), W_5 = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} \right), W_6 = \left(\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \right), \\
W_7 &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix} \right), W_8 = \left(\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} \right), W_9 = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right), \\
W_{10} &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \right), W_{11} = \left(\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \right), W_{12} = \left(\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right), \\
W_{13} &= \left(\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right), W_{14} = \left(\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix} \right), W_{15} = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix} \right), \\
W_{16} &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{bmatrix} \right), W_{17} = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \right), W_{18} = \left(\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), \\
W_{19} &= \left(\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} \right), W_{20} = \left(\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \right), W_{21} = \left(\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \right).
\end{aligned}$$

The orthogonal complements of these subspaces are as follows:

$$\begin{aligned}
W_1^\perp &= W_{21}, W_2^\perp = W_{20}, W_3^\perp = W_{17}, W_4^\perp = W_4, W_5^\perp = W_{11}, W_6^\perp = W_{13}, W_7^\perp = W_{18}, W_8^\perp = W_{19}, \\
W_9^\perp &= W_{14}, W_{10}^\perp = W_{10}, W_{12}^\perp = W_{12}, W_{15}^\perp = \left(\begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), W_{16}^\perp = W_{16}. \text{ (See Remark ?? for} \\
&\text{the convention for some of these equalities).}
\end{aligned}$$

Theorem 3. The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbits $\mathcal{O}_i := Gx_i$ and the subspaces W_j are given as follows:

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	W_{11}	W_{12}
$\mathcal{O}_1$	1	1	1	1	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	[1001]	[1011]	[1002]	[1011]	[1011]	[1010] a_1	2[1001]	[1030]	[1001] a_1	[1021]	[1001] a_1
$\mathcal{O}_3$	0	0	[2111]	[2112]	0	0	[2120]	0	[2120]	[2111]	[2121]	[2111]
$\mathcal{O}_4$	0	0	0	[3311]	0	0	0	0	0	0	0	0
$\mathcal{O}_5$	0	0	0	0	[2111]	0	[2110]	0	[2120]	[2111]	[2121]	[2101]
$\mathcal{O}_6$	0	0	0	0	0	[2111]	[2110]	[2001]	[2120]	[2101]	[2111]	[2111]
$\mathcal{O}_7$	0	0	0	0	0	0	[3120]	0	[3130]	[3111]	[3131]	[3111]
$\mathcal{O}_8$	0	0	0	0	0	0	0	[2111]	$\frac{1}{2}$ [2330]	[2311]	$\frac{1}{2}$ [2331]	[2311]
$\mathcal{O}_9$	0	0	0	0	0	0	0	0	$\frac{1}{2}$ [4310]	0	$\frac{1}{2}$ [4311]	0
$\mathcal{O}_{10}$	0	0	0	0	0	0	0	0	0	[3311]	[3331]	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0	0	0	0	[4421]	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0	0	0	0	0	[3311]
$\mathcal{O}_{13}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{15}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{16}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{17}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{18}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{19}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{20}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{21}$	0	0	0	0	0	0	0	0	0	0	0	0

W_{13}	W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	W_{19}	W_{20}	W_{21}	$W_{15}^\perp$
1	1	1	1	1	1	1	1	1	1
[1021]	[1010] b_1	[1000] c_1	[1000] b_2	[1021]	[1010] b_5	[1001] a_2	[1001] b_1	[1012]	[1000] c_1
[2121]	[2130]	[2110] b_1	[2110] a_2	[2112]	[2140]	2[2111]	[2131]	[2122]	[2110] b_1
0	0	[3310]	[3300]	[3311]	[3320]	0	[3311]	[3321]	[3310]
[2111]	[2111]	[2120]	[2100] b_3	[2111]	[2130]	[2111]	[2121]	[2112]	[2120]
[2121]	[2111]	[2120]	[2100] b_3	[2111]	[2130]	[2101] a_3	[2121]	[2112]	[2120]
[3131]	[3130]	[3130]	[3110] a_4	[3121]	[3120] b_3	2[3111]	[3121] a_1	[3132]	[3130]
$\frac{1}{2}$ [2331]	[2330]	[2320]	$\frac{1}{2}$ [2300] b_4	[2321]	$\frac{1}{2}$ [2320] a_5	3[2311]	$\frac{1}{2}$ [2321] a_2	$\frac{1}{2}$ [2332]	[2320]
$\frac{1}{2}$ [4311]	0	0	$\frac{1}{2}$ [4300]	0	$\frac{1}{2}$ [4310]	0	$\frac{1}{2}$ [4311]	$\frac{1}{2}$ [4312]	0
0	[3330]	[3320]	[3300] a_4	[3321]	[3320] a_2	2[3311]	[3321] a_1	[3332]	[3320]
0	0	0	[4400]	0	[4420]	0	[4421]	[4422]	0
[3331]	[3330]	[3320]	[3300] a_4	[3321]	[3320] a_2	[3211] a_2	[3321] a_1	[3332]	[3320]
[4421]	0	0	[4400]	0	[4420]	[4311]	[4421]	[4422]	0
0	[4420]	0	[4400]	0	[4420]	[4311]	[4421]	[4422]	0
0	0	[4320]	[4300] a_1	[4321]	[4320] a_1	0	[4331]	[4332]	[4320]
0	0	0	[5400]	0	[5420]	0	[5421]	[5432]	0
0	0	0	0	[3611]	[3620]	2[3411]	[3611] a_3	[3622]	0
0	0	0	0	0	[4620]	2[4411]	2[4621]	[4632]	0
0	0	0	0	0	0	[4511]	$\frac{1}{2}$ [4721]	$\frac{1}{6}$ [4732]	0
0	0	0	0	0	0	0	$\frac{1}{2}$ [5711]	$\frac{1}{2}$ [5722]	0
0	0	0	0	0	0	0	0	$\frac{1}{3}$ [6731]	0

Here, let $[abcd] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ and

$$\begin{aligned}
 a_1 &= 2q+1 & b_1 &= 2q^2+2q+1 & c_1 &= q^3+4q^2+3q+1 \\
 a_2 &= 3q+1 & b_2 &= 5q^2+3q+1 \\
 a_3 &= q+2 & b_3 &= q^2+3q+1 \\
 a_4 &= 4q+1 & b_4 &= b_7 = q^2+8q+1 \\
 a_5 &= 5q+1 & b_5 &= 3q^2+2q+1
 \end{aligned}$$

By this result and Proposition 3, the representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with

respect to the basis $e_1, \dots, e_{21}$ is given as follows:

$$q^{-18} \begin{bmatrix} 1 & [1012] & [2122] & [3321] & [2112] & [2112] & [3132] & \frac{1}{2}[2332] & \frac{1}{2}[4312] & [3332] & [4422] \\ 1 & e_1 & [1120]d_1 & [2310]c_1 & [1110]d_1 & [1110]d_1 & [2120]d_2 & \frac{1}{2}[1320]d_3 & \frac{1}{2}[3310]c_2 & [2320]d_2 & [3420]c_2 \\ 1 & [0010]d_1 & qg_1 & [1300]e_2 & [1110]c_2 & [1110]c_2 & [1110]e_3 & \frac{1}{2}[1310]d_4 & -\frac{1}{2}[2300]c_1 & [1310]e_3 & -[2410]c_1 \\ 1 & [0001]c_1 & [0101]e_2 & q^3f_1 & -[1111] & -[1111] & -[1111]b_1 & -\frac{1}{2}[1311]a_1 & -\frac{1}{2}[2301] & -[1311]b_1 & [2411] \\ 1 & [0010]d_1 & [1120]c_2 & -[2320] & qf_2 & [1110]c_2 & [1120]e_4 & \frac{1}{2}[1320]d_5 & \frac{1}{2}[2310]d_6 & [1320]e_4 & [2420]d_6 \\ 1 & [1000]d_1 & [1120]c_2 & -[2320] & [1110]c_2 & qf_2 & [1120]e_4 & \frac{1}{2}[1320]d_5 & \frac{1}{2}[2310]d_6 & -[2320]b_2 & -[2410]c_1 \\ 1 & d_2 & qe_3 & -[1300]b_1 & qe_3 & qe_3 & qg_2 & \frac{1}{2}[1300]d_7 & -\frac{1}{2}[1300]d_8 & [1300]e_5 & -[1400]d_9 \\ 1 & d_3 & [1100]d_4 & -[2300]a_1 & [1100]d_5 & [1100]d_5 & [2100]d_7 & -\frac{1}{2}q^3e_6 & -\frac{1}{2}[3320] & [1300]e_7 & -[2400]c_3 \\ 1 & [0010]c_2 & -[0110]c_1 & [1310] & [0110]d_6 & [0110]d_6 & -[0120]d_8 & -\frac{1}{2}[1340] & \frac{1}{2}q^3e_8 & [1330] & -[1410] \\ 1 & d_2 & qe_3 & -[1300]b_1 & qe_4 & -[1100]b_2 & [1100]e_5 & \frac{1}{2}q^3e_7 & \frac{1}{2}[2310] & q^3f_3 & -[1400]d_9 \\ 1 & [0010]c_2 & -[0110]c_1 & [1310] & [0110]d_6 & -qc_1 & -[0110]d_9 & -\frac{1}{2}[0310]c_3 & -\frac{1}{2}[1300] & -[0310]d_9 & q^4d_8 \\ 1 & d_2 & qe_3 & -[1300]b_1 & -[1100]b_2 & qe_4 & [1100]e_5 & \frac{1}{2}q^3e_7 & \frac{1}{2}[2310] & -[1310]c_4 & [2420] \\ 1 & [0010]c_2 & -[0110]c_1 & [1310] & -qc_1 & [0110]d_6 & -[0110]d_9 & -\frac{1}{2}[0310]c_3 & -\frac{1}{2}[1300] & [1330] & -[1410] \\ 1 & [0010]c_2 & -[0110]c_1 & [1310] & -qc_1 & -qc_1 & [1110]d_{10} & \frac{1}{2}[0310]d_{11} & \frac{1}{2}[2310]b_3 & -[0310]d_9 & -[1410] \\ 1 & [0010]c_2 & qe_9 & -q^3c_5 & -qc_1 & -qc_1 & qf_4 & -\frac{1}{2}[0310]c_3 & -\frac{1}{2}[1300] & q^3c_3 & -[1410] \\ 1 & -b_2 & [0120] & -q^3 & [0110] & [0110] & qb_1c_6 & -\frac{1}{2}[0310]b_4 & -\frac{1}{2}[1300]b_3 & -q^3a_1 & q^4 \\ 1 & d_5 & qb_1c_1 & -[2310] & -[1120] & -[1120] & -[1110]c_7 & -\frac{1}{2}[0310]b_5 & \frac{1}{2}[2300] & -[1310]b_4 & [2410] \\ 1 & c_8 & -[0110]b_1 & [1300] & -qb_1 & -qb_1 & qe_{10} & -\frac{1}{2}q^3c_9 & -\frac{1}{2}[1300]b_6 & q^3b_7 & -[1400] \\ 1 & c_{10} & -[1110]a_1 & q^3a_2 & -[1100]a_1 & -[1100]a_1 & -[2110]a_3 & \frac{1}{2}q^3c_{11} & -\frac{1}{2}[3300] & -[1300]b_8 & [2400] \\ 1 & -b_2 & [0120] & -q^3 & [0110] & [0110] & -[0110]c_{12} & -\frac{1}{2}[0310]b_4 & \frac{1}{2}q^2c_{13} & [0310]b_1 & -[1410] \\ 1 & -[0011] & [0111] & -[0310] & [0101] & [0101] & -[0111] & \frac{1}{2}[0311] & -\frac{1}{2}[1301] & -[0311] & [0401] \end{bmatrix}$$

$$\begin{bmatrix} [3332] & [4422] & [4422] & [4332] & [5432] & [3622] & [4632] & \frac{1}{6}[4732] & -\frac{1}{2}[5722] & \frac{1}{3}[6731] \\ [2320]d_2 & [3420]c_2 & [3420]c_2 & [3330]c_2 & -[4420]b_2 & [2610]d_5 & [3620]c_8 & \frac{1}{6}[3720]c_{10} & -\frac{1}{2}[4710]b_2 & -\frac{1}{3}[5730] \\ [1310]e_3 & -[2410]c_1 & -[2410]c_1 & [2310]e_9 & [3430] & [1600]b_1c_1 & -[2620]b_1 & -\frac{1}{6}[3720]a_1 & \frac{1}{2}[3720] & \frac{1}{3}[4720] \\ -[1311]b_1 & [2411] & [2411] & -[1311]c_5 & -[2411] & -[2611] & [2611] & \frac{1}{6}[1711]a_2 & -\frac{1}{2}[2701] & -\frac{1}{3}[3720] \\ -[2320]b_2 & -[2410]c_1 & -[2410]c_1 & -[2320]c_1 & [3430] & -[2630] & -[2620]b_1 & -\frac{1}{6}[3720]a_1 & \frac{1}{2}[3720] & \frac{1}{3}[4720] \\ [1320]e_4 & [2420]d_6 & -[2410]c_1 & -[2320]c_1 & [3430] & -[2630] & -[2620]b_1 & -\frac{1}{6}[3720]a_1 & \frac{1}{2}[3720] & \frac{1}{3}[4720] \\ [1300]e_5 & -[1400]d_9 & [2400]d_{10} & [1300]f_4 & [2400]b_1c_6 & -[1600]c_7 & [1600]e_{10} & -\frac{1}{6}[3720]a_3 & -\frac{1}{2}[2700]c_{12} & -\frac{1}{3}[3710] \\ [1300]e_7 & -[2400]c_3 & [2400]d_{11} & -[2310]c_3 & -[3410]b_4 & -[1600]b_5 & -[2600]c_9 & \frac{1}{6}[2700]c_{11} & -\frac{1}{2}[3700]b_4 & \frac{1}{3}[4710] \\ [1330] & -[1410] & [2420]b_3 & -[1320] & -[2420]b_3 & [1610] & -[1620]b_6 & -\frac{1}{6}[3720] & \frac{1}{2}[1710]c_{13} & -\frac{1}{3}[3720] \\ -[1310]c_4 & [2420] & -[1400]d_9 & [1300]c_3 & -[2400]a_1 & -[1600]b_4 & [1600]b_7 & -\frac{1}{6}[2700]b_8 & \frac{1}{2}[2700]b_1 & -\frac{1}{3}[3710] \\ [1330] & -[1410] & -[1410] & -[1320] & [1410] & [1610] & -[1610] & \frac{1}{6}[2710] & -\frac{1}{2}[2710] & \frac{1}{3}[2710] \\ q^3f_3 & -[1400]d_9 & -[1400]d_9 & [1300]c_3 & -[2400]a_1 & -[1600]b_4 & [1600]b_7 & -\frac{1}{6}[2700]b_8 & \frac{1}{2}[2700]b_1 & -\frac{1}{3}[3710] \\ -[0310]d_9 & q^4d_8 & -[1410] & -[1320] & [1410] & [1610] & -[1610] & \frac{1}{6}[2710] & -\frac{1}{2}[2710] & \frac{1}{3}[2710] \\ -[0310]d_9 & -[1410] & q^4d_8 & -[1320] & [1410] & [1610] & -[1610] & \frac{1}{6}[2710] & -\frac{1}{2}[2710] & \frac{1}{3}[2710] \\ q^3c_3 & -[1410] & -[1410] & q^3f_5 & -[1400]b_1b_3 & -[1600]b_1 & -[1600]c_{12} & \frac{1}{6}[1700]a_1a_2 & \frac{1}{2}[1700] & \frac{1}{3}[2710] \\ -q^3a_1 & q^4 & q^4 & -q^3b_1b_3 & q^4c_{11} & [1600] & -[1600]b_3 & \frac{1}{6}[1700]a_2 & \frac{1}{2}[1700] & -\frac{1}{3}[1710] \\ -[1310]b_4 & [2410] & [2410] & -[2310]b_1 & [3410] & q^6b_9 & [1610]a_4 & -\frac{1}{2}[1710] & -\frac{1}{2}[2700] & 0 \\ q^3b_7 & -[1400] & -[1400] & -[1300]c_{12} & -[2400]b_3 & q^6a_4 & q^6b_{10} & -\frac{1}{2}[1700] & \frac{1}{2}[1700] & 0 \\ -[1300]b_8 & [2400] & [2400] & [1300]a_1a_2 & [2400]a_2 & -3q^6 & -3[1600] & q^7 & 0 & 0 \\ [0310]b_1 & -[1410] & -[1410] & [0310] & [1410] & -q^6 & [0610] & 0 & -q^7 & 0 \\ -[0311] & [0401] & [0401] & [0311] & -[0411] & 0 & 0 & 0 & 0 & q^7 \end{bmatrix}$$

Here, let $[abcd] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ and

$$\begin{array}{lll}
a_1 = 2q+1 & c_1 = q^3 - q - 1 & e_1 = 2q^5 + 2q^4 - 2q^2 - 2q - 1 \\
a_2 = 2q - 1 & c_2 = q^3 - q^2 - q - 1 & e_2 = q^5 - q^3 - q^2 + q + 1 \\
a_3 = 3q + 1 & c_3 = 2q^3 - 2q - 1 & e_3 = q^5 - 2q^4 - 2q^3 + q^2 + 2q + 1 \\
a_4 = q - 2 & c_4 = 2q^3 - q^2 - 2q - 1 & e_4 = q^5 - 2q^3 + q + 1 \\
b_1 = q^2 - q - 1 & c_5 = q^3 - q^2 + 1 & e_5 = q^5 - 2q^4 - q^3 + 3q^2 + 3q + 1 \\
b_2 = 2q^2 + 2q + 1 & c_6 = q^3 + q + 1 & e_6 = 5q^5 - 7q^4 - 4q^3 + 4q^2 + 3q + 1 \\
b_3 = q^2 + 1 & c_7 = q^3 + q^2 - 2q - 1 & e_7 = 2q^5 - 4q^4 - 3q^3 + 3q^2 + 3q + 1 \\
b_4 = q^2 - 2q - 1 & c_8 = q^3 - 2q^2 - 2q - 1 & e_8 = q^5 + q^4 - q + 1 \\
b_5 = 2q^2 - 3q - 1 & c_9 = q^3 - 4q - 1 & e_9 = q^5 - q^4 - q^3 + q^2 + 2q + 1 \\
b_6 = q^2 - q + 1 & c_{10} = 2q^3 - 2q^2 - 2q - 1 & e_{10} = q^5 - 2q^4 + q^3 + 2q^2 - 2q - 1 \\
b_7 = 2q^2 - 2q - 1 & c_{11} = q^3 - q^2 + 5q + 1 & e_{11} = q^5 - q^4 + q^3 - q^2 - 1 \\
b_8 = q^2 - 3q - 1 & c_{12} = q^3 - q^2 + q + 1 & f_1 = q^6 - q^5 - q^4 + q^2 - 1 \\
b_9 = q^2 - 2 & c_{13} = q^3 + q^2 - q + 1 & f_2 = q^6 + q^5 - q^4 - 2q^3 + q + 1 \\
b_{10} = q^2 - 2q + 2 & d_1 = q^4 + q^3 - q^2 - q - 1 & f_3 = q^6 - 3q^5 + 4q^3 - 2q - 1 \\
& d_2 = 2q^4 - 2q^2 - 2q - 1 & f_4 = q^6 - q^5 + 2q^3 - 2q - 1 \\
& d_3 = 3q^4 - 2q^2 - 2q - 1 & f_5 = q^6 - 2q^5 + q^4 - q^2 + q + 1 \\
& d_4 = 2q^4 - q^3 - 4q^2 - 3q - 1 & g_1 = q^7 + q^6 - 3q^4 - 2q^3 + q^2 + 2q + 1 \\
& d_5 = q^4 + q^3 - 2q^2 - 2q - 1 & g_2 = q^7 - 4q^5 + q^4 + 4q^3 - 2q - 1 \\
& d_6 = q^4 - q^3 + 1 & \\
& d_7 = q^4 - 4q^3 - 7q^2 - 4q - 1 & \\
& d_8 = q^4 - q^2 + 1 & \\
& d_9 = q^4 - q^3 - q^2 + q + 1 & \\
& d_{10} = q^4 + q^2 + 2q + 1 & \\
& d_{11} = q^4 - 2q^3 + 2q + 1 &
\end{array}$$

3.2 $\mathbb{F}_q^2 \otimes H_3(\mathbb{F}_q)$

For $a \in \mathbb{F}_{q^2}$, let $\bar{a}$ be the conjugate of a over $\mathbb{F}_q$. Define the norm map as follows:

$$N_2 : \mathbb{F}_{q^2} \ni z \mapsto z\bar{z} \in \mathbb{F}_q.$$

N_2 is surjective and $N_2|_{\mathbb{F}_q^\times} : \mathbb{F}_q^\times \rightarrow \mathbb{F}_q^\times$ is a surjective group homomorphism. Let $H_n(\mathbb{F}_{q^2})$ be the set of Hermitian matrices of order n . We consider $H_3(\mathbb{F}_{q^2})$, i.e.,

$$H_3(\mathbb{F}_{q^2}) := \left\{ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \in M_3(\mathbb{F}_{q^2}) \mid a_{ii} \in \mathbb{F}_q, a_{ij} \in \mathbb{F}_{q^2} (1 \leq i < j \leq 3) \right\}.$$

Let $V = \mathbb{F}_q^2 \otimes H_3(\mathbb{F}_{q^2})$ and $G = G_1 \times G_2 = \text{GL}_2(\mathbb{F}_q) \times \text{GL}_3(\mathbb{F}_{q^2})$. We write $x \in V$ as $x = (A, B)$ where $A, B \in H_3(\mathbb{F}_{q^2})$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \text{GL}_2(\mathbb{F}_q)$ and $g_2 \in \text{GL}_3(\mathbb{F}_{q^2})$. The action of G on V is defined by

$$gx = (g_2 \overline{A g_2^T}, g_2 \overline{B g_2^T}) g_1^T.$$

Here, for a matrix h , $\bar{h}$ is the matrix whose (i, j) -entry is the conjugate over $\mathbb{F}_q$ of the (i, j) -entry of h .

V consists of 15 G -orbits in all. The following elements $x_1, \dots, x_{15}$ are representatives:

$$\begin{aligned}
x_1 &= (0, 0), x_2 = \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, 0 \right), x_3 = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, 0 \right), x_4 = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, 0 \right), \\
x_5 &= \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \right), x_6 = \left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} \right), x_7 = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} -1 & & \\ \mu_0 & \mu_1 & \\ & & 0 \end{bmatrix} \right), \\
x_8 &= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right), x_9 = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \right), x_{10} = \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right),
\end{aligned}$$

W_{10}	W_{11}	W_{12}	W_{13}	W_{14}	W_{15}	$W_2^\perp$	$W_5^\perp$
1	1	1	1	1	1	1	1
[100001]	[100001]	[10001]	[10000] d_1	[101010]	[101101]	[101010]	[101010]
[110000] c_2	[110010] c_3	[110000] f_1	[110010] d_2	[111000] c_1	[111111]	[111010] c_3	[111020]
[231000]	[231010]	[231010]	[230011]	[232000]	[231011]	[232010]	[231010]
[211000] b_2	[211010]	[211000] b_2	[211011]	[212000] b_3	[212111]	[212020]	[211011]
$\frac{1}{2}$ [230010]	[230000]	$\frac{1}{2}$ [230010]	$\frac{1}{2}$ [230010] b_2	$\frac{1}{2}$ [231010]	$\frac{1}{2}$ [231111]	$\frac{1}{2}$ [231010]	$\frac{1}{2}$ [231010]
$\frac{1}{2}$ [232000]	0	$\frac{1}{2}$ [232000]	$\frac{1}{2}$ [231010]	$\frac{1}{2}$ [231000] b_4	$\frac{1}{2}$ [231111]	$\frac{1}{2}$ [231010] b_3	$\frac{1}{2}$ [231010]
[242000]	0	[243000]	[242010]	[242010]	[242111]	[242010] b_3	[242010]
[331000]	[331000]	[331000]	[331020]	[332000]	[332111]	[332010]	[331010]
[342000]	0	[343000]	[342010]	[343000]	[343111]	[343020]	[342010]
0	[261000]	[261000]	[260010] b_1	[262000]	[261111]	[262010]	[261010]
0	0	[362000]	2[361010]	[362000]	[362111]	[362011]	[361010]
0	0	0	$\frac{1}{2}$ [470010]	0	$\frac{1}{6}$ [471111]	$\frac{1}{6}$ [473010]	0
0	0	0	$\frac{1}{2}$ [371010]	[372000]	$\frac{1}{2}$ [372111]	$\frac{1}{2}$ [372020]	0
0	0	0	0	0	$\frac{1}{3}$ [473011]	$\frac{1}{3}$ [473010]	0

Here, let $[abcdef] = (q-1)^a q^b (q+1)^c (q^2-q+1)^d (q^2+1)^e (q^2+q+1)^f$ and

$$\begin{aligned}
 b_1 &= q^2 + 2 & c_1 &= q^3 + 2q^2 + 1 & d_1 &= q^4 + q^3 + q^2 + q + 1 \\
 b_2 &= 2q^2 + q + 1 & c_2 &= q^3 + 3q^2 + q + 1 & d_2 &= q^4 + q^2 + q + 1 \\
 b_3 &= 2q^2 + 1 & c_3 &= q^3 + q^2 + 1 & f_1 &= q^5 + q^4 + q^3 + 3q^2 + q + 1 \\
 b_4 &= 3q^2 + 1
 \end{aligned}$$

By this result and Proposition 3, the representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, \dots, e_{21}$ can be calculated.

3.3 $\mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^6)$

Let $\wedge^2(\mathbb{F}_q^6)$ be the set of all alternating matrices of order 6 over $\mathbb{F}_q$. We write $A \in \wedge^2(6)$ as

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}_q.$$

Let $V = \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^6)$ and $G = G_1 \times G_2 = \text{GL}_2 \times \text{GL}_6$. We write $x \in V$ as $x = (A, B)$ where $A, B \in \wedge^2(6)$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \text{GL}_2$ and $g_2 \in \text{GL}_6$. The action of G on V is defined by

$$gx = (g_2 A g_2^T, g_2 B g_2^T) g_1^T.$$

Let $u_{lmn} (1 \leq l \leq 2, 1 \leq n < m \leq 6)$ be the element of V that the (n, m) -entry and (m, n) -entry of l th matrix is 1 and -1 respectively and the rest are all 0. For example,

$$u_{112} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

The set $\{u_{lmn} \mid 1 \leq l \leq 2, 1 \leq n < m \leq 6\}$ is a $\mathbb{F}_q$ -basis of V .

V consists of 18 G -orbits in all. The following elements $x_1, \dots, x_{18}$ are representatives:

$$\begin{aligned}
 x_1 &= 0, \\
 x_2 &= u_{112}, \\
 x_3 &= u_{112} + u_{134}, \\
 x_4 &= u_{112} + u_{134} + u_{156},
 \end{aligned}$$

$$\begin{aligned}
x_5 &= u_{112} + u_{213}, \\
x_6 &= u_{112} + u_{214} + u_{223}, \\
x_7 &= u_{112} + u_{234}, \\
x_8 &= u_{112} + u_{134} + u_{214} + \mu_0 u_{223} + \mu_1 u_{234}, \\
x_9 &= u_{112} + u_{215} + u_{234}, \\
x_{10} &= u_{112} + u_{134} + u_{215} + u_{223}, \\
x_{11} &= u_{114} + u_{123} + u_{216} + u_{225}, \\
x_{12} &= u_{112} + u_{216} + u_{225} + u_{234}, \\
x_{13} &= u_{114} + u_{123} + u_{216} + u_{225} + u_{234}, \\
x_{14} &= u_{112} + u_{236} + u_{245}, \\
x_{15} &= u_{112} + u_{134} + u_{236} + u_{245}, \\
x_{16} &= u_{112} + u_{134} + u_{234} + u_{256}, \\
x_{17} &= u_{112} + u_{134} + u_{214} + \mu_0 u_{223} + \mu_1 u_{234} + u_{256}, \\
x_{18} &= u_{112} + u_{134} + u_{156} + \nu_2 u_{212} + u_{216} + u_{223} + \nu_1 u_{225} + \nu_0 u_{245}.
\end{aligned}$$

Here, $\mu_1, \mu_0, \nu_2, \nu_1, \nu_0 \in \mathbb{F}_q$ are elements such that $X^2 + \mu_1 X + \mu_0, X^3 + \nu_2 X^2 + \nu_1 X + \nu_0 \in \mathbb{F}_q[X]$ is irreducible.

The subspaces we choose to calculate the Fourier transform are as follows:

$$\begin{aligned}
W_1 &= \{0\}, \\
W_2 &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116} \rangle_{\mathbb{F}_q}, \\
W_3 &= \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156} \rangle_{\mathbb{F}_q}, \\
W_4 &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156} \rangle_{\mathbb{F}_q}, \\
W_5 &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216} \rangle_{\mathbb{F}_q}, \\
W_6 &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{212} \rangle_{\mathbb{F}_q}, \\
W_7 &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{223}, u_{224}, u_{225}, u_{226} \rangle_{\mathbb{F}_q}, \\
W_8 &= \langle u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}, \\
W_9 &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226} \rangle_{\mathbb{F}_q}, \\
W_{10} &= \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}, \\
W_{11} &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226} \rangle_{\mathbb{F}_q}, \\
W_{12} &= \langle u_{114}, u_{115}, u_{116}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}, \\
W_{13} &= \langle u_{112}, u_{113}, u_{114}, u_{123}, u_{124}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234} \rangle_{\mathbb{F}_q}, \\
W_{14} &= \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}, \\
W_{15} &= \langle u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}, \\
W_{16} &= \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{213}, u_{214}, u_{215}, u_{216}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}, \\
W_{17} &= \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, \\
&\quad , u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}, \\
W_{18} &= V.
\end{aligned}$$

The orthogonal complements of these subspaces are as follows:

$$\begin{aligned}
W_1^\perp &= W_{18}, W_2^\perp = W_{17}, W_3^\perp = W_{14}, W_4^\perp = W_4, W_5^\perp = W_{10}, W_6^\perp = W_{15}, W_7^\perp = W_{16}, W_8^\perp = W_{11}, \\
W_{12}^\perp &= W_{12}, W_{13}^\perp = W_{13} \text{ and}
\end{aligned}$$

$$W_9^\perp = \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}.$$

Theorem 5. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbits $\mathcal{O}_i := Gx_i$ and the subspaces W_j are given as*

follows:

	W_1	W_2	W_3	W_4	W_5	W_6	W_7
$\mathcal{O}_1$	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	[1, 0, 0, 0, 0, 1, 0]	[1, 0, 0, 0, 1, 1, 0]	[1, 0, 0, 1, 0, 1, 1]	[1, 0, 1, 0, 0, 1, 0]	[1, 0, 1, 0, 0, 1, 0]	[1, 0, 1, 0, 0, 0, 0] c_1
$\mathcal{O}_3$	0	0	[2, 2, 0, 1, 0, 1, 0]	[2, 2, 0, 2, 0, 1, 1]	0	[2, 2, 1, 1, 1, 0, 0]	0
$\mathcal{O}_4$	0	0	0	[3, 6, 0, 1, 0, 1, 0]	0	0	0
$\mathcal{O}_5$	0	0	0	0	[2, 1, 1, 0, 1, 1, 0]	[2, 1, 2, 0, 1, 0, 0]	[2, 1, 1, 0, 1, 0, 0] a_1
$\mathcal{O}_6$	0	0	0	0	0	[3, 2, 1, 1, 1, 0, 0]	0
$\mathcal{O}_7$	0	0	0	0	0	0	[2, 3, 1, 1, 1, 0, 0]
$\mathcal{O}_8$	0	0	0	0	0	0	0
$\mathcal{O}_9$	0	0	0	0	0	0	0
$\mathcal{O}_{10}$	0	0	0	0	0	0	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0
$\mathcal{O}_{13}$	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0
$\mathcal{O}_{16}$	0	0	0	0	0	0	0
$\mathcal{O}_{17}$	0	0	0	0	0	0	0
$\mathcal{O}_{18}$	0	0	0	0	0	0	0

W_8	W_9	W_{10}	W_{11}	W_{12}	W_{13}
1	1	1	1	1	1
[1, 0, 1, 1, 1, 0, 0]	[1, 0, 1, 0, 0, 0, 0] d_1	[1, 1, 0, 0, 1, 1, 0]	[1, 0, 1, 1, 0, 0, 0] c_2	[1, 0, 0, 1, 0, 1, 0]	[1, 0, 0, 0, 0, 0, 0] e_1
[2, 2, 1, 1, 0, 0, 0]	[2, 2, 1, 1, 1, 0, 0]	[2, 2, 1, 1, 0, 1, 0]	[2, 2, 2, 1, 1, 0, 0]	[2, 2, 0, 2, 0, 0, 0] c_2	[2, 2, 0, 0, 0, 0, 0] e_2
0	0	0	0	[3, 6, 1, 1, 0, 0, 0]	[3, 6, 1, 0, 0, 0, 0]
[2, 1, 2, 1, 1, 0, 0]	[2, 1, 1, 0, 1, 0, 0] d_2	[2, 1, 1, 1, 1, 1, 0]	[2, 1, 2, 1, 2, 0, 0]	[2, 1, 2, 1, 1, 0, 0]	[2, 1, 2, 0, 0, 0, 0] d_3
[3, 2, 2, 1, 1, 0, 0]	[3, 2, 2, 1, 1, 0, 0]	[3, 2, 2, 1, 1, 1, 0]	[3, 2, 2, 2, 1, 0, 0]	[3, 2, 1, 2, 1, 0, 0]	[3, 2, 1, 0, 0, 0, 0] e_3
$\frac{1}{2}$ [2, 5, 1, 1, 1, 0, 0]	[2, 5, 1, 1, 1, 0, 0]	$\frac{1}{2}$ [2, 5, 1, 1, 1, 1, 0]	$\frac{1}{2}$ [2, 5, 3, 1, 1, 0, 0]	[2, 5, 0, 2, 0, 0, 0]	$\frac{1}{2}$ [2, 5, 0, 0, 0, 0, 0] d_4
$\frac{1}{2}$ [4, 5, 1, 1, 0, 0, 0]	0	$\frac{1}{2}$ [4, 5, 1, 1, 0, 1, 0]	$\frac{1}{2}$ [4, 5, 1, 1, 1, 0, 0]	0	$\frac{1}{2}$ [4, 5, 2, 0, 0, 0, 0]
0	[3, 5, 2, 1, 1, 0, 0]	[3, 5, 2, 1, 1, 1, 0]	[3, 5, 4, 1, 1, 0, 0]	[3, 5, 2, 2, 0, 0, 0]	[3, 5, 2, 0, 0, 0, 0] c_3
0	0	[4, 6, 2, 1, 1, 1, 0]	[4, 6, 3, 1, 1, 0, 0]	0	[4, 6, 4, 0, 0, 0, 0]
0	0	0	[4, 8, 2, 1, 1, 0, 0]	0	[4, 8, 2, 0, 0, 0, 0]
0	0	0	0	[4, 6, 1, 2, 0, 0, 0]	[4, 6, 1, 0, 0, 0, 0] c_4
0	0	0	0	0	[5, 8, 2, 0, 0, 0, 0]
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	$W_9^\perp$
1	1	1	1	1	1
[1, 0, 0, 0, 0, 1, 0] d_5	[1, 0, 1, 0, 1, 1, 0]	[1, 0, 1, 0, 1, 0, 0] c_5	[1, 0, 0, 0, 0, 2, 0]	[1, 0, 1, 1, 0, 1, 1]	[1, 0, 2, 0, 2, 0, 0]
[2, 2, 0, 2, 0, 1, 1]	[2, 2, 1, 1, 0, 0, 1] d_2	[2, 2, 1, 1, 0, 0, 0] c_1	[2, 2, 0, 0, 1, 1, 0] d_5	[2, 2, 1, 2, 0, 1, 1]	[2, 2, 1, 1, 0, 0, 0] c_6
[3, 6, 0, 1, 0, 1, 0]	[3, 6, 1, 1, 1, 0, 0]	0	[3, 6, 0, 1, 0, 1, 0]	[3, 6, 1, 1, 0, 1, 0]	0
[2, 1, 2, 0, 1, 1, 0]	[2, 1, 2, 1, 2, 0, 0]	[2, 1, 1, 0, 1, 0, 0] e_4	[2, 1, 2, 0, 2, 1, 0]	[2, 1, 2, 1, 1, 1, 1]	[2, 1, 1, 2, 1, 0, 0]
[3, 2, 1, 1, 1, 1, 0]	[3, 2, 1, 1, 1, 0, 0] d_2	[3, 2, 2, 1, 1, 0, 0] a_2	[3, 2, 1, 2, 1, 1, 0]	[3, 2, 2, 2, 1, 1, 1]	[3, 2, 3, 1, 1, 0, 0]
[2, 5, 0, 1, 1, 1, 0]	$\frac{1}{2}$ [2, 5, 1, 1, 1, 0, 0] b_1	$\frac{1}{2}$ [2, 5, 1, 1, 1, 0, 0] b_2	$\frac{1}{2}$ [2, 5, 0, 1, 1, 1, 0] b_4	$\frac{1}{2}$ [2, 5, 1, 2, 1, 1, 1]	$\frac{1}{2}$ [2, 5, 1, 1, 1, 0, 0] a_2
0	$\frac{1}{2}$ [4, 5, 1, 1, 0, 0, 0]	$\frac{1}{2}$ [4, 5, 1, 1, 0, 0, 0]	$\frac{1}{2}$ [4, 5, 1, 1, 0, 1, 0]	$\frac{1}{2}$ [4, 5, 1, 2, 0, 1, 1]	$\frac{1}{2}$ [4, 5, 1, 1, 0, 0, 0]
[3, 5, 2, 1, 1, 1, 0]	[3, 5, 3, 2, 1, 0, 0]	[3, 5, 2, 1, 1, 0, 0] b_3	[3, 5, 2, 2, 1, 1, 0]	[3, 5, 3, 2, 1, 1, 1]	[3, 5, 3, 1, 1, 0, 0]
0	[4, 6, 3, 1, 1, 0, 0]	[4, 6, 2, 1, 1, 0, 0] a_1	[4, 6, 3, 1, 1, 1, 0]	[4, 6, 3, 2, 1, 1, 1]	[4, 6, 2, 1, 1, 0, 0]
0	[4, 8, 2, 1, 1, 0, 0]	[4, 7, 3, 1, 1, 0, 0]	[4, 8, 2, 1, 1, 1, 0]	[4, 8, 2, 2, 1, 1, 1]	0
[4, 6, 1, 1, 1, 1, 0]	[4, 6, 1, 1, 1, 0, 0] c_4	0	[4, 6, 1, 1, 2, 1, 0]	[4, 6, 2, 2, 1, 1, 1]	0
0	[5, 8, 2, 1, 1, 0, 0]	0	[5, 8, 2, 1, 1, 1, 0]	[5, 8, 3, 2, 1, 1, 1]	0
[3, 11, 0, 1, 0, 1, 0]	[3, 11, 1, 1, 1, 0, 0]	2[3, 9, 1, 1, 1, 0, 0]	[3, 11, 0, 1, 0, 1, 0] b_5	[3, 11, 1, 2, 0, 1, 1]	0
0	[4, 11, 1, 1, 1, 0, 0]	2[4, 9, 2, 1, 1, 0, 0]	2[4, 11, 1, 1, 1, 1, 0]	[4, 11, 2, 2, 1, 1, 1]	0
0	0	[4, 11, 1, 1, 1, 0, 0]	$\frac{1}{2}$ [4, 13, 0, 1, 1, 1, 0]	$\frac{1}{6}$ [4, 13, 1, 2, 1, 1, 1]	0
0	0	0	$\frac{1}{2}$ [5, 13, 1, 1, 0, 1, 0]	$\frac{1}{2}$ [5, 13, 2, 2, 0, 1, 1]	0
0	0	0	0	$\frac{1}{3}$ [6, 13, 3, 1, 1, 1, 0]	0

Here, let $[a, b, c, d, e, f, g] = (q-1)^a q^b (q+1)^c (q^2-q+1)^d (q^2+1)^e (q^2+q+1)^f (q^2-q+1)^g$ and

$$\begin{array}{lll}
 a_1 = q+2, & c_1 = 2q^3 + 2q + 1, & d_1 = 2q^4 + q^3 + 2q^2 + q + 1, \\
 a_2 = 2q+1, & c_2 = q^3 + q^2 + 1, & d_2 = q^4 + q^3 + 2q^2 + 2q + 1, \\
 b_1 = 2q^2 + 2q + 1, & c_3 = 2q^3 + 5q^2 + 3q + 1, & d_3 = q^4 + 2q^3 + 3q^2 + q + 1, \\
 b_2 = 2q^2 + 4q + 1, & c_4 = q^3 + 2q^2 + q + 1, & d_4 = 3q^4 + 8q^3 + 10q^2 + 4q + 1, \\
 b_3 = 2q^2 + 3q + 2, & c_5 = 2q^3 + q^2 + q + 1, & d_5 = q^4 + q^2 + q + 1, \\
 b_4 = 2q^2 + q + 1, & c_6 = q^3 + q + 1, & e_1 = q^5 + 4q^4 + 4q^3 + 3q^2 + 2q + 1, \\
 b_5 = q^2 + 2, & & e_2 = 2q^5 + 4q^4 + 4q^3 + 5q^2 + 3q + 1, \\
 & & e_3 = q^5 + 5q^4 + 6q^3 + 6q^2 + 3q + 1, \\
 & & e_4 = q^5 + 2q^4 + 3q^3 + 4q^2 + 2q + 1
 \end{array}$$

By this result and Proposition 3, the representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, \dots, e_{21}$ can be calculated.

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Graduate School of Mathematics
 Kobe University
 1-1 Rokkodai-cho, Nada-ku, Kobe, 657-8501
 JAPAN
 E-mail address: k.ishimoto1024@gmail.com