

Stationary isothermic surfaces of the heat flow

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS 3]. Let $u = u(x, t)$ be the unique solution of the following problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in} \quad \Omega \times (0, +\infty), \quad (1.1)$$

$$u = 1 \quad \text{on} \quad \partial\Omega \times (0, +\infty), \quad (1.2)$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}, \quad (1.3)$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$.

A conjecture, posed in [Kl] by M.S. Klamkin and referred to by L. Zalcman in [Z] as the *Matzoh Ball Soup*, was settled affirmatively by G. Alessandrini in [A 1]-[A 2]. In [A 2], under the assumption that every point of $\partial\Omega$ is regular with respect to the Laplacian, it was proved that if all the spatial isothermic surfaces of u are *invariant with time* then Ω must be a *ball*. (Of course, the values of u vary with time on its spatial isothermic surfaces.)

The case where the homogeneous initial data in (1.3) is replaced by a function in the space $L^2(\Omega)$ was also considered in [A 1]-[A 2] and, with the help of J. Serrin's celebrated symmetry theorem for elliptic equations [Ser], was settled in the following terms: if all the spatial isothermic surfaces of the solution u of the heat equation with homogeneous Dirichlet boundary condition and initial data $\varphi \in L^2(\Omega)$ are invariant with time, then either φ is an eigenfunction of the Laplacian or Ω is a ball. The analogous question where condition (1.2) is replaced by the homogeneous Neumann boundary condition was examined and answered positively (see [Sak], Theorem 1) with the aid of the classification theorem for *isoparametric hypersurfaces in Euclidean*

space due to T. Levi-Civita and B. Segre (see [LC], [Seg]). The method used in [Sak] can be applied to give an alternative proof of Alessandrini's results.

An important observation is that, in order to prove Klamkin's conjecture [Kl], both methods employed in [A 1]-[A 2] and [Sak] need to assume that *infinitely many* isothermic surfaces of u are invariant with time. As a natural consequence of this remark, one may wonder if the requirement that a finite number (possibly only one) of level surfaces of u are invariant with time implies that Ω is a ball.

Our main result in this direction is the following.

Theorem 1.1 *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is a domain, with boundary ∂D , satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$. Assume that the solution u of problem (1.1)-(1.3) satisfies the following condition:*

$$u(x, t) = a(t), \quad (x, t) \in \partial D \times (0, +\infty), \quad (1.4)$$

for some function $a : (0, +\infty) \rightarrow (0, +\infty)$. Then Ω must be a ball.

We recall that Ω satisfies the *exterior sphere condition* if for every $y \in \partial\Omega$ there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, where $B_r(z)$ denotes an open ball centered at $z \in \mathbb{R}^N$ and with radius $r > 0$. Also, D satisfies the *interior cone condition* if for every $x \in \partial D$ there exists a finite right spherical cone K_x with vertex x such that $K_x \subset \overline{D}$ and $\overline{K_x} \cap \partial D = \{x\}$.

The proof of Theorem 1.1 exploits arguments different from the ones used in [A 1]-[A 2] and [Sak]. Our technique is essentially based on the following three ingredients. One ingredient is a careful study of the asymptotic behavior of $u(x, t)$ as $t \rightarrow 0$ which is based on the results of S. R. S. Varadhan [V] (see also [EI]). The second one is A. D. Aleksandrov's uniqueness theorem [Alek]. A special case of this theorem is the well-known *Soap-Bubble Theorem*. The third one is the following balance law proved in [MS 1]-[MS 2] (see [MS 3] for a shorter proof):

Theorem 1.2 (balance law) *Let G be a domain in $\mathbb{R}^N$, $N \geq 2$, let x_0 be a point in G and set $d_* = \text{dist}(x_0, \partial G)$. Suppose that $v = v(x, t)$ is a solution of the heat equation in $G \times (0, +\infty)$. Then the following hold:*

(i) $v(x_0, t) = 0$ for every $t \in (0, +\infty)$ if and only if

$$\int_{\partial B_r(x_0)} v(x, t) dS_x = 0 \text{ for every } (r, t) \in (0, d_*) \times (0, +\infty);$$

(ii) $\nabla v(x_0, t) = 0$ for every $t \in (0, +\infty)$ if and only if

$$\int_{\partial B_r(x_0)} (x - x_0)v(x, t) dS_x = 0 \text{ for every } (r, t) \in (0, d_*) \times (0, +\infty).$$

Section 2 is devoted to an outline of the proof of Theorem 1.1. In Section 3, we consider the case where the domain Ω is unbounded.

2 Outline of the proof of Theorem 1.1

Define the function $W = W(x, s)$ by

$$W(x, s) = s \int_0^{+\infty} u(x, t) e^{-s t} dt, \quad s > 0. \quad (2.1)$$

Notice that W is the solution of the following elliptic boundary value problem:

$$\Delta W - s W = 0 \quad \text{in } \Omega, \quad (2.2)$$

$$W = 1 \quad \text{on } \partial\Omega. \quad (2.3)$$

A result in [V] (see also [EI]) shows that, as $s \rightarrow +\infty$, the function $-\frac{1}{\sqrt{s}} \log W(x, s)$ converges uniformly on $\bar{\Omega}$ to the function $d = d(x)$ defined by

$$d(x) = \text{dist}(x, \partial\Omega), \quad x \in \Omega. \quad (2.4)$$

(Since Ω enjoys the exterior sphere condition, we can apply the result in [V].) Moreover, if u satisfies (1.4), then for any fixed $s > 0$, W is constant on ∂D . Indeed,

$$W(x, s) = s \int_0^{+\infty} a(t) e^{-s t} dt := A(s), \quad x \in \partial D. \quad (2.5)$$

Thus, in view of the result in [V], we can define the positive number $R > 0$ by

$$R = \lim_{s \rightarrow +\infty} \left\{ -\frac{1}{\sqrt{s}} \log A(s) \right\}. \quad (2.6)$$

In Lemma 2.1 below, we prove analyticity of ∂D and $\partial\Omega$ by using our balance law.

Lemma 2.1 *The following assertions hold:*

- (i) *for every $x \in \partial D$, $d(x) = R$, where d is defined by (2.4);*
- (ii) *∂D is analytic;*
- (iii) *$\partial\Omega$ is analytic and $\partial\Omega = \{x \in \mathbb{R}^N : \text{dist}(x, D) = R\}$;*
- (iv) *the mapping: $\partial D \ni x \mapsto y(x) \equiv x - R\nu^*(x) \in \partial\Omega$ is a diffeomorphism, where $\nu^*(x)$ denotes the interior unit normal vector to ∂D at $x \in \partial D$;*
- (v) *for every $x \in \partial D$, $\nabla d(y(x)) = \nu^*(x)$ and $\overline{B_R(x)} \cap \partial\Omega = \{y(x)\}$;*
- (vi) *let $\kappa_j(y)$, $j = 1, \dots, N-1$ denote the j -th principal curvature at $y \in \partial\Omega$ of the analytic surface $\partial\Omega$ with respect to the interior normal direction to $\partial\Omega$. Then $\kappa_j(y) < \frac{1}{R}$, $j = 1, \dots, N-1$, for every $y \in \partial\Omega$.*

Proof. (i) The result in [V] and the definition (2.6) of R yield this assertion.

(ii) It suffices to show that, for every point $x \in \partial D$, there exists a time $t^* > 0$ such that $\nabla u(x, t^*) \neq 0$, since u is analytic with respect to the space variable.

Assume by contradiction that there exists a point $x_0 \in \partial D$ such that $\nabla u(x_0, t) = 0$ for every $t > 0$. Since u is continuous up to $\partial\Omega \times (0, +\infty)$, by Theorem 1.2 (ii), we can infer that

$$\int_{\partial B_R(x_0)} (x - x_0) \cdot u(x, t) dS_x = 0 \quad \text{for every } t > 0,$$

and hence

$$\int_{\partial B_R(x_0)} (x - x_0) \cdot W(x, s) dS_x = 0 \quad \text{for every } s > 0, \quad (2.7)$$

in view of (2.1).

On the other hand, since D satisfies the interior cone condition, there exists a finite right spherical cone K with vertex at x_0 such that $K \subset \overline{D}$ and $\overline{K} \cap \partial D = \{x_0\}$. By translating and rotating if needed, we can suppose that $x_0 = 0$ and that K is the set $\{x \in B_\rho(0) : x_N < -|x| \cos \theta\}$, where $\rho \in (0, R)$ and $\theta \in (0, \frac{\pi}{2})$.

Since $K \subset \overline{D}$ and $\overline{K} \cap \partial D = \{0\}$, proposition (i) implies that

$$d(x) > R \quad \text{for every } x \in K. \quad (2.8)$$

The set defined by

$$V = \{x \in \partial B_R(0) : x_N \geq R \sin \theta\}, \quad (2.9)$$

is such that

$$\partial\Omega \cap \partial B_R(0) \subset V, \quad (2.10)$$

because, otherwise, there would be a point in K contradicting (2.8).

Thus, from (2.10) it follows that we can choose a number $\delta > 0$ such that

$$d(x) \geq 5\delta \text{ for every } x \in \partial B_R(0) \cap \{x_N \leq 0\}. \quad (2.11)$$

Since we know that $-\frac{1}{\sqrt{s}} \log W(x, s)$ converges uniformly on $\bar{\Omega}$ to $d(x)$ as $s \rightarrow +\infty$, we can choose $s^* > 0$ such that

$$\left| -\frac{1}{\sqrt{s}} \log W(x, s) - d(x) \right| < \delta,$$

for every $x \in \bar{\Omega}$ and every $s \geq s^*$. This latter inequality, together with (2.9), (2.10), and (2.11), gives, for every $s \geq s^*$, the following two estimates:

$$\begin{aligned} \int_{\partial B_R(0) \cap \{x_N \leq 0\}} x_N W(x, s) dS_x &\geq -\frac{1}{2} R e^{-4\delta\sqrt{s}} \mathcal{H}^{N-1}(\partial B_R(0)), \\ \int_{V \cap \bar{\Omega}_{2\delta}} x_N W(x, s) dS_x &\geq R \sin \theta e^{-3\delta\sqrt{s}} \mathcal{H}^{N-1}(V \cap \bar{\Omega}_{2\delta}). \end{aligned} \quad (2.12)$$

Here $\mathcal{H}^{N-1}(\cdot)$ denotes the $(N-1)$ -dimensional Hausdorff measure and $\Omega_{2\delta}$ is defined by

$$\Omega_{2\delta} = \{x \in \Omega : d(x) < 2\delta\}. \quad (2.13)$$

A consequence of (2.12) is that, for every $s \geq s^*$,

$$\begin{aligned} \int_{\partial B_R(0)} x_N W(x, s) dS_x &\geq \\ \int_{V \cap \bar{\Omega}_{2\delta}} x_N W(x, s) dS_x + \int_{\partial B_R(0) \cap \{x_N \leq 0\}} x_N W(x, s) dS_x &\geq \\ R e^{-3\delta\sqrt{s}} \left[\sin \theta \mathcal{H}^{N-1}(V \cap \bar{\Omega}_{2\delta}) - \frac{1}{2} e^{-\delta\sqrt{s}} \mathcal{H}^{N-1}(\partial B_R(0)) \right]. \end{aligned}$$

Therefore, we obtain a contradiction by observing that the first term of this chain of inequalities equals zero, by (2.7), while the last term can be made positive by choosing $s > 0$ sufficiently large.

(iii), (iv), and (v) Let

$$\Gamma = \{x \in \mathbb{R}^N : \text{dist}(x, D) = R\}.$$

It is clear that $\Gamma \subset \partial\Omega$. Take any point $x \in \partial D$. Then, there exists a unique point $y \in \partial\Omega$ such that $\overline{B_R(x)} \cap \partial\Omega = \{y\}$. Indeed, since ∂D is analytic by (ii), if $\tilde{y} \in \overline{B_R(x)} \cap \partial\Omega$ and $\tilde{y} \neq y$, then

$$\frac{y-x}{|y-x|} = -\nu^*(x) = \frac{\tilde{y}-x}{|\tilde{y}-x|},$$

where $\nu^*(x)$ is the interior unit normal vector to ∂D at x — a contradiction. Since Ω enjoys the exterior sphere property, there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, and hence $\overline{B_r(z)} \cap \overline{B_R(x)} = \{y\}$. Therefore,

$$\text{dist}(z, D) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \overline{D} = \{x\}. \quad (2.14)$$

Let κ_j^* , $j = 1, \dots, N-1$, denote the principal curvatures of the surface ∂D with respect to the interior normal direction to ∂D . Then (2.14) implies that

$$\kappa_j^*(x) \geq -\frac{1}{r+R}, \quad j = 1, \dots, N-1.$$

Since $\kappa_j^* > -\frac{1}{R}$ on ∂D , for every $j = 1, \dots, N-1$, Γ is an analytic hypersurface diffeomorphic to ∂D (see [GT], Lemma 14.16), and hence Γ equals $\partial\Omega$. Assertions (iii), (iv), and (v) then follow at once.

(vi) Take any point $y \in \partial\Omega$. Propositions (iii) and (iv) imply that there exists a unique $x \in \partial D$ such that $\overline{B_R(y)} \cap \overline{D} = \{x\}$. Since ∂D is analytic, D satisfies the interior sphere condition, that is there exists a ball $B_r(z) \subset D$ such that $\overline{B_r(z)} \cap \partial D = \{x\}$. Therefore,

$$d(z) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \partial\Omega = \{y\}, \quad (2.15)$$

and consequently

$$\kappa_j(y) \leq \frac{1}{r+R}, \quad j = 1, \dots, N-1.$$

Assertion (vi) is proved. $\square$

Let us show that the two functions

$$W_\varepsilon^\pm(x, s) = \exp\{-\sqrt{s(1 \mp \varepsilon)} d(x)\}, \quad (2.16)$$

where $d(x)$ is defined by (2.4), provide respectively an upper and a lower barrier for W in Ω for large values of s .

Lemma 2.2 *For every $\varepsilon > 0$, there exists a positive number s_ε such that*

$$W_\varepsilon^-(x, s) \leq W(x, s) \leq W_\varepsilon^+(x, s) \quad (2.17)$$

for every $x \in \overline{\Omega}$ and every $s \geq s_\varepsilon$.

Proof. Choose a number $\delta > 0$ such that the function $d = d(x)$ defined in (2.4) is of class C^2 in the set $\overline{\Omega}_\delta$ where

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}. \quad (2.18)$$

Let $W_\epsilon^\pm(x, s)$ be given by (2.16). A straightforward computation gives

$$\Delta W_\epsilon^\pm - s W_\epsilon^\pm = \mp \epsilon \sqrt{s} \left\{ \sqrt{s} \pm \frac{\sqrt{(1 \mp \epsilon)}}{\epsilon} \Delta d \right\} W_\epsilon^\pm \quad \text{in } \Omega_\delta.$$

Set $M_\delta = \max_{\overline{\Omega}_\delta} |\Delta d|$. If $s \geq \frac{1+\epsilon}{\epsilon^2} M_\delta^2$, then

$$\begin{aligned} \Delta W_\epsilon^+ - s W_\epsilon^+ &\leq 0 \\ \Delta W_\epsilon^- - s W_\epsilon^- &\geq 0 \end{aligned} \quad \text{in } \Omega_\delta. \quad (2.19)$$

Since the function $-\frac{1}{\sqrt{s}} \log W(x, s)$ converges uniformly on $\overline{\Omega}$ to $d(x)$ as $s \rightarrow +\infty$, there exists a number $s^* > 0$ such that

$$-\delta(1 - \sqrt{1 - \epsilon}) \leq -\frac{1}{\sqrt{s}} \log W(x, s) - d(x) \leq \delta(\sqrt{1 + \epsilon} - 1), \quad x \in \overline{\Omega},$$

for every $s \geq s^*$. Hence, since $d(x) \geq \delta$ for every $x \in \Omega \setminus \Omega_\delta$, we obtain

$$W_\epsilon^-(x, s) \leq W(x, s) \leq W_\epsilon^+(x, s), \quad x \in \Omega \setminus \Omega_\delta, \quad (2.20)$$

for every $s \geq s^*$. Moreover,

$$W_\epsilon^-(x, s) = W(x, s) = W_\epsilon^+(x, s) = 1, \quad x \in \partial\Omega, \quad (2.21)$$

for every $s > 0$, clearly.

Choose $s_\epsilon = \max(s^*, \frac{1+\epsilon}{\epsilon^2} M_\delta^2)$. Then by the comparison principle, from (2.19), (2.20) and (2.21), we have

$$W_\epsilon^-(x, s) \leq W(x, s) \leq W_\epsilon^+(x, s), \quad x \in \Omega_\delta, \quad (2.22)$$

for every $s \geq s_\epsilon$. Combining (2.22) with (2.20) yields (2.17). $\square$

With the help of Lemma 2.1, we obtain

Lemma 2.3 *Let $x_0 \in \partial D$ and put $y_0 = y(x_0) \in \partial\Omega$, where $y(x_0)$ is given in Lemma 2.1 (see (iv) and (v)). Then*

$$\lim_{s \rightarrow +\infty} s^{\frac{N-1}{4}} \int_{\partial B_R(x_0)} e^{-\sqrt{s(1 \pm \varepsilon)} d(x)} dS_x = \left(\frac{2\pi}{\sqrt{1 \pm \varepsilon}} \right)^{\frac{N-1}{2}} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y_0) \right] \right\}^{-\frac{1}{2}}, \quad (2.23)$$

where $\kappa_j(y)$, $j = 1, \dots, N-1$ denotes the j -th principal curvature at $y \in \partial\Omega$ of the analytic surface $\partial\Omega$ with respect to the interior normal direction to $\partial\Omega$.

Proof. In view of proposition (vi) of Lemma 2.1, in order to prove this lemma we can use Laplace's method (see [deB], p. 71 for example) or the stationary phase method (see [Ev], pp. 208 – 217 for example). See [MS 3] for details. $\square$

Combining Lemma 2.3 with Lemma 2.2 yields

Lemma 2.4 *Let $x_0 \in \partial D$ and put $y_0 = y(x_0) \in \partial\Omega$, where $y(x_0)$ is given in Lemma 2.1 (see (iv) and (v)). Then*

$$\lim_{s \rightarrow +\infty} s^{\frac{N-1}{4}} \int_{\partial B_R(x_0)} W(x, s) dS_x = (2\pi)^{\frac{N-1}{2}} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y_0) \right] \right\}^{-\frac{1}{2}}. \quad (2.24)$$

The last lemma is

Lemma 2.5 *We have*

$$\prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y) \right] = \text{a constant} > 0, \quad \text{for every } y \in \partial\Omega, \quad (2.25)$$

where $\kappa_j(y)$, $j = 1, \dots, N-1$ denotes the j -th principal curvature at $y \in \partial\Omega$ of the analytic surface $\partial\Omega$ with respect to the interior normal direction to $\partial\Omega$. In particular, if $N = 2$, Ω must be a ball.

Proof. Let p and q be two distinct points in $\partial\Omega$. Propositions (iv) and (v) from Lemma 2.1 guarantee that there exist two distinct points P, Q in ∂D such that $p = y(P)$ and $q = y(Q)$ in (iv).

For $x \in B_R(0)$, consider the function

$$v(x, t) = u(x + P, t) - u(x + Q, t). \quad (2.26)$$

Then $v = v(x, t)$ satisfies the heat equation in $B_R(0) \times (0, +\infty)$ and by (1.4)

$$v(0, t) = u(P, t) - u(Q, t) = 0,$$

for every $t > 0$. Since v is continuous up to $\partial B_R(0) \times (0, +\infty)$, by Theorem 1.2 (i) we obtain

$$\int_{\partial B_R(0)} v(x, t) dS_x = 0$$

for every $t > 0$, and hence

$$\int_{\partial B_R(P)} u(x, t) dS_x = \int_{\partial B_R(Q)} u(x, t) dS_x$$

for every $t > 0$. Therefore, in view of (2.1), we have

$$\int_{\partial B_R(P)} W(x, s) dS_x = \int_{\partial B_R(Q)} W(x, s) dS_x \quad (2.27)$$

for every $s > 0$. With the help of Lemma 2.4, by multiplying both sides of (2.27) by $s^{\frac{N-1}{4}}$, we can take the limits as $s \rightarrow +\infty$. Therefore, since $p = y(P)$ and $q = y(Q)$, after some manipulation, we obtain:

$$\prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(p) \right] = \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(q) \right],$$

that is, (2.25) holds. $\square$

We quote A.D. Aleksandrov's uniqueness theorem from [Alek], p. 412, adjusted to our notations. A special case of this theorem is the well-known *Soap-Bubble Theorem* (see also [R]).

Theorem 2.6 (Aleksandrov) *Let $\Phi = \Phi(\kappa_1, \dots, \kappa_{N-1})$ be a continuously differentiable function, defined for $\kappa_1 \geq \dots \geq \kappa_{N-1}$, and subject to the condition $\frac{\partial \Phi}{\partial \kappa_i} > 0$ ($i = 1, \dots, N-1$).*

Suppose that in $\mathbb{R}^N$ we have a twice-differentiable closed surface S without self-intersections and with bounded principal curvatures.

If on the surface S the function Φ of its principal curvatures $\kappa_1, \dots, \kappa_{N-1}$ has at all points one and the same value, then S is a sphere.

Proof of Theorem 1.1. By Lemma 2.5, it suffices to consider the case where $N \geq 3$.

We set

$$\Phi = \Phi(\kappa_1, \dots, \kappa_{N-1}) = - \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j \right] \quad (2.28)$$

and observe that

$$\frac{\partial \Phi}{\partial \kappa_i} > 0 \quad (i = 1, \dots, N-1), \text{ if } \max_{1 \leq j \leq N-1} \kappa_j < \frac{1}{R}.$$

Since condition (2.25) holds by Lemma 2.5, we infer that the function Φ is constant on $\partial\Omega$. Therefore, by applying Theorem 2.6 to each connected component of $\partial\Omega$, we conclude that $\partial\Omega$ must be a sphere. $\square$

Remark. The method of proof of Theorem 2.6 is called *Aleksandrov's reflection principle* or *the method of moving planes*, which is based on the maximum principle for elliptic partial differential equations of second order. In fact, by using local coordinates, the condition $\Phi(\kappa_1, \dots, \kappa_{N-1}) = \text{constant}$ on the surface S can be converted into a second order partial differential equation which is of elliptic type, since $\frac{\partial \Phi}{\partial \kappa_i} > 0$ ($i = 1, \dots, N-1$). In the case the function Φ is given by (2.28), we obtain an equation of Monge-Ampère type.

3 Concluding remarks

By the same method as in the proof of Theorem 1.1, we see that the following theorem also holds.

Theorem 3.1 *Let Ω be an exterior domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is an exterior domain, with boundary ∂D , satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$.*

Assume that the solution u to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0, +\infty) \rightarrow (0, +\infty)$.

Then $\partial\Omega$ must be a sphere. That is, Ω must be the exterior of a ball.

Since both $\partial\Omega$ and ∂D are compact, it follows from the barrier arguments with the help of Varadhan's result that inequality (2.17) holds for x in an arbitrary bounded neighborhood of $\partial\Omega$ and for sufficiently large s . Therefore, we get the same relation

of the principal curvatures of $\partial\Omega$. Hence each connected component of $\partial\Omega$ is a sphere with the same radius. Moreover, by analyticity, $u(x, t)$ must be radially symmetric in x with respect to each center of each connected component of $\partial\Omega$. Thus $\partial\Omega$ must be a sphere.

Professor Messoud A. Efendiev gave us the following conjecture:

Consider domains Ω whose boundary $\partial\Omega$ is not compact. In particular, let Ω be a unbounded domain above a Lipschitz graph $x_N = \varphi(x_1, \dots, x_{N-1})$ over $\mathbb{R}^{N-1}$. Suppose that there exists an invariant isothermic surface. Then $\partial\Omega$ must be a hyperplane.

Our answer to this conjecture is the following theorem:

Theorem 3.2 *Let Ω be a unbounded domain above a locally Lipschitz graph $x_N = \varphi(x_1, \dots, x_{N-1})$ over $\mathbb{R}^{N-1}$ such that*

$$\nabla\varphi(x) = o(|x|^{\frac{1}{2}}) \text{ near infinity.} \quad (3.1)$$

Suppose that Ω satisfies the uniform exterior sphere condition, that is, there exists $r > 0$ such that for every $x \in \partial\Omega$ there exists a ball $B_r(z)$ with $\overline{B_r(z)} \cap \overline{\Omega} = \{x\}$. Assume that there exists a domain D with $\overline{D} \subset \Omega$ such that the solution u to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0, +\infty) \rightarrow (0, +\infty)$.

Then $\partial\Omega$ must be a hyperplane.

With the help of curvature estimates in a Bernstein's theorem due to L. Caffarelli, L. Nirenberg, and J. Spruck (see Theorem 2" and its proof in [CNS]), we can prove this theorem. The details will be given in a forthcoming paper.

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