

A generalization of the Liouville theorem to polyharmonic functions

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1 Introduction

Let $\mathbf{R}^n$ be the n -dimensional Euclidean space with a point $x = (x_1, x_2, \dots, x_n)$. For a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n,$$

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x_1} \right)^{\lambda_1} \left(\frac{\partial}{\partial x_2} \right)^{\lambda_2} \dots \left(\frac{\partial}{\partial x_n} \right)^{\lambda_n}.$$

We denote by rB^n the open ball centered at the origin with radius $r > 0$, whose boundary is denoted by rS^{n-1} .

A real valued function u is called polyharmonic of order m on $\mathbf{R}^n$ if $u \in C^{2m}$ and $\Delta^m u = 0$, where m is a positive integer, Δ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$. We denote by $H^m(\mathbf{R}^n)$ the space of polyharmonic functions of order m on $\mathbf{R}^n$. In particular, u is harmonic on $\mathbf{R}^n$ if $u \in H^1(\mathbf{R}^n)$.

The Liouville theorem for polyharmonic functions is known in several forms (cf. [1, 3, 4]).

THEOREM A. *Let $u \in H^m(\mathbf{R}^n)$ and $s > 2(m-1)$. Then u is a polynomial of degree less than s if one of the following conditions holds:*

- (i) $\lim_{r \rightarrow \infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ dS = 0$ (see [1]);
- (ii) $\lim_{r \rightarrow \infty} \frac{1}{r^{s+n}} \int_{rB^n} u^+ dx = 0$ (see [3]);
- (iii) $\limsup_{r \rightarrow \infty} \left(\max_{x \in rS^{n-1}} \frac{u(x)}{|x|^s} \right) \leq 0$ (see [4]).

Now we propose the following theorem.

THEOREM. Let $u \in H^m(\mathbf{R}^n)$ and $s > 2(m-1)$. Then u is a polynomial of degree at most s if and only if

$$\liminf_{r \rightarrow \infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ dS < \infty. \quad (1)$$

We here note that each condition of Theorem A implies (1), so that our theorem gives an improvement of Theorem A.

2 The main lemmas

Let us begin with the following lemma, which expands polyharmonic functions to harmonic functions (cf. [2], [5]).

LEMMA 1 (THE FINITE ALMANSI EXPANSION). A real valued function u on $\mathbf{R}^n$ belongs to $H^m(\mathbf{R}^n)$, then there exists a unique family $\{h_i\}_{i=1}^m \subset H^1(\mathbf{R}^n)$ such that

$$u(x) = \sum_{i=1}^m |x|^{2(i-1)} h_i(x) \quad (2)$$

for every $x \in \mathbf{R}^n$.

PROOF. We prove this lemma by induction on m . For $m = 1$ the conclusion is trivial. Suppose the conclusion is true for $m = k$, and let $u \in H^{k+1}(\mathbf{R}^n)$. Then there exists a family $\{g_i\}_{i=1}^k \subset H^1(\mathbf{R}^n)$ such that

$$\Delta u = \sum_{i=1}^k |x|^{2(i-1)} g_i(x). \quad (3)$$

If a family $\{h_i\}_{i=1}^{k+1} \subset H^1(\mathbf{R}^n)$ satisfies (2), then we should have

$$\begin{aligned} \Delta u &= \sum_{i=1}^{k+1} \Delta \left(|x|^{2(i-1)} h_i(x) \right) \\ &= \sum_{i=1}^k \Delta \left(|x|^{2i} h_{i+1}(x) \right). \end{aligned}$$

If we write $r = |x|$, then

$$\begin{aligned} \Delta \left(|x|^{2i} h_{i+1}(x) \right) &= \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} \right)^2 \left(r^{2i} h_{i+1}(x) \right) \\ &= \sum_{j=1}^n \left\{ \frac{\partial^2 r^{2i}}{\partial x_j^2} h_{i+1}(x) + 2 \frac{\partial r^{2i}}{\partial x_j} \frac{\partial h_{i+1}(x)}{\partial x_j} + r^{2i} \frac{\partial^2 h_{i+1}(x)}{\partial x_j^2} \right\} \end{aligned}$$

$$\begin{aligned}
&= 2ir^{2(i-1)} \left\{ (2i - 2 + n)h_{i+1}(x) + 2r \frac{\partial h_{i+1}}{\partial r}(x) \right\} \\
&= |x|^{2(i-1)} \left\{ 2i(2i - 2 + n)h_{i+1}(x) + 4ir \frac{\partial h_{i+1}}{\partial r}(x) \right\}.
\end{aligned}$$

From the uniqueness of Almansi expansion for Δu , it is necessary and sufficient to find a unique solution h_{i+1} for the equation

$$g_i(x) = 2i(2i - 2 + n)h_{i+1}(x) + 4ir \frac{\partial h_{i+1}}{\partial r}(x) \quad (4)$$

for each $i = 1, \dots, k$. We see that the unique solution for (4) is given by

$$h_{i+1}(x) = \frac{1}{4ir^{i-1+n/2}} \int_0^r t^{i-2+n/2} g_i(tx/r) dt.$$

Here we have only to show that h_{i+1} is harmonic on $\mathbf{R}^n$. Actually, putting $x = r\zeta$, where $r = |x|$ and $\zeta = x/|x| = x/r$, we have

$$\begin{aligned}
h_{i+1}(x) &= h_{i+1}(r\zeta) \\
&= \frac{r^{1-i-n/2}}{4i} \int_0^r t^{i-2+n/2} g_i(t\zeta) dt \\
&= \frac{r^{1-i-n/2}}{4i} \int_0^1 (rs)^{i-2+n/2} g_i(rs\zeta) r ds \\
&= \frac{1}{4i} \int_0^1 s^{i-2+n/2} g_i(xs) ds.
\end{aligned}$$

Since g_i is harmonic, we see that $\Delta h_{i+1}(x) = 0$ for $i = 1, \dots, k$. Now put

$$h_1(x) = u(x) - \sum_{i=2}^{k+1} |x|^{2(i-1)} h_i(x).$$

Then $\Delta h_1(x) = 0$ by (3), and the induction is completed.

Next we prepare the following lemma, which gives a relation between spherical means and derivatives for harmonic functions.

LEMMA 2. Suppose $u \in H^1(\mathbf{R}^n)$. For each multi-index λ , there exists a positive constant $C = C(\lambda)$ such that

$$\int_{rS^{n-1}} u x^\lambda dS = Cr^{2|\lambda|+n-1} D^\lambda u(0) + P_{2|\lambda|+n-3}(r) \quad (5)$$

for every $r > 0$, where $P_k(r)$ is a polynomial of degree at most k depends on u .

PROOF. We prove this lemma by induction on the length of λ . Assume first that $\lambda_n = 1$ and $\lambda_i = 0$ ($i = 1, \dots, n-1$). Using Green's formula and the mean-value property for harmonic functions, we have

$$\begin{aligned}
\int_{rS^{n-1}} ux^\lambda dS &= \int_{rS^{n-1}} ux_n dS \\
&= r \int_{rS^{n-1}} u \frac{x_n}{r} dS \\
&= r \int_{rB^n} \frac{\partial u}{\partial x_n} dx \\
&= \sigma_n r^{n+1} \frac{\partial u}{\partial x_n}(0),
\end{aligned}$$

where σ_n is the n -dimensional volume of the unit ball. Hence (5) holds for $|\lambda| = 1$.

Next suppose that (5) holds for $|\lambda| \leq k$, where k is a positive integer. Let $\mu = (\mu_1, \dots, \mu_n)$ such that $|\mu| = k + 1$. We may assume without loss of generality that $\mu_n \geq 2$, and set $\mu' = (\mu_1, \dots, \mu_{n-1}, \mu_n - 1)$. Then we write

$$\int_{rS^{n-1}} ux^\mu dS = r \int_{rS^{n-1}} ux^{\mu'} \frac{x_n}{r} dS.$$

From Green's formula we obtain

$$\begin{aligned}
\int_{rS^{n-1}} ux^\mu dS &= r \int_{rB^n} \frac{\partial(ux^{\mu'})}{\partial x_n} dx \\
&= r \int_{rB^n} \left(x^{\mu'} \frac{\partial u}{\partial x_n} + (\mu_n - 1)ux_1^{\mu_1} \cdots x_n^{\mu_n-2} \right) dx = (*).
\end{aligned}$$

Set $\mu'' = (\mu_1, \dots, \mu_{n-1}, \mu_n - 2)$. Since $|\mu'| = k$ and $|\mu''| = k - 1$, we find

$$\begin{aligned}
(*) &= r \int_0^r \left(\int_{tS^{n-1}} \left(x^{\mu'} \frac{\partial u}{\partial x_n} + (\mu_n - 1)ux^{\mu''} \right) dS \right) dt \\
&= r \int_0^r \left(\int_{tS^{n-1}} \frac{\partial u}{\partial x_n} x^{\mu'} dS \right) dt + (\mu_n - 1)r \int_0^r \left(\int_{tS^{n-1}} ux^{\mu''} dS \right) dt \\
&= r \int_0^r \left(C(\mu')t^{2|\mu'|+n-1} D^{\mu'} \left(\frac{\partial u}{\partial x_n} \right) (0) + P_{2|\mu'|+n-3}(t) \right) dt \\
&\quad + (\mu_n - 1)r \int_0^r \left(C(\mu'')t^{2|\mu''|+n-1} D^{\mu''} u(0) + P_{2|\mu''|+n-3}(t) \right) dt \\
&= C(\mu)r^{2k+n+1} D^\mu u(0) + P_{2k+n-1}(r),
\end{aligned}$$

where $C(\mu) = \frac{C(\mu')}{2k+n} > 0$ and P_ℓ denotes various polynomials of degree at most ℓ which may change from one occurrence to the next; throughout this note, we use this convention. Hence (5) also holds for $|\mu| = k + 1$. The induction is completed.

3 Proof of the theorem

First we show that our theorem is valid under the two sided condition on spherical means for polyharmonic functions.

LEMMA 3. *Let $u \in H^m(\mathbf{R}^n)$ and $s > 2(m-1)$. Then u is a polynomial of degree at most s if*

$$\liminf_{r \rightarrow \infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} |u| dS < \infty. \quad (6)$$

PROOF. By (6) we can find a sequence $\{r_j\}_{j=1}^{\infty}$ such that $r_j \rightarrow \infty$ and

$$\sup_j \left(r_j^{-s-n+1} \int_{r_j S^{n-1}} |u| dS \right) < \infty. \quad (7)$$

Using (2) and Lemma 2, we have

$$\begin{aligned} \int_{rS^{n-1}} u x^\lambda dS &= \int_{rS^{n-1}} \left(\sum_{i=1}^m |x|^{2(i-1)} h_i(x) \right) x^\lambda dS \\ &= \sum_{i=1}^m r^{2(i-1)} \int_{rS^{n-1}} h_i(x) x^\lambda dS \\ &= \sum_{i=1}^m r^{2(i-1)} \left(C_i r^{2|\lambda|+n-1} D^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right), \end{aligned}$$

where $C_i = C_i(\lambda)$ is a positive constant and $P_{i,k}$ denotes various polynomials of degree at most k depends on h_i . Hence it follows that

$$r^{|\lambda|} \int_{rS^{n-1}} |u| dS \geq \left| \sum_{i=1}^m r^{2(i-1)} \left(C_i r^{2|\lambda|+n-1} D^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right) \right|,$$

so that we obtain

$$r_j^{-s-n+1} \int_{r_j S^{n-1}} |u| dS \geq r_j^{|\lambda|-s+2(m-1)} \left| C_m D^\lambda h_m(0) + O(r_j^{-2}) \right|$$

as $r_j \rightarrow \infty$. By (7), we find

$$D^\lambda h_m(0) = 0$$

for all $|\lambda| > s - 2(m-1)$. By analyticity of harmonic functions, we see that h_m is a polynomial of degree at most $s - 2(m-1)$. Hence we note that

$$r^{2(m-1)} \int_{rS^{n-1}} h_m(x) x^\lambda dS = O(r^{s+|\lambda|+n-1}) \quad \text{as } r \rightarrow \infty.$$

Consequently,

$$r_j^{-s-n+1} \int_{r_j S^{n-1}} |u| dS \geq r_j^{|\lambda|-s+2(m-2)} \left| C_{m-1} D^\lambda h_{m-1}(0) + O(r_j^{-2}) \right| + O(1)$$

as $r_j \rightarrow \infty$. This implies that $D^\lambda h_{m-1}(0) = 0$ for $|\lambda| > s - 2(m - 2)$, so that h_{m-1} is a polynomial of degree at most $s - 2(m - 2)$. By repeating this arguments, we see that each h_i is a polynomial of degree at most $s - 2(i - 1)$ ($i = 1, \dots, m$). Thus it follows that u is a polynomial. In view of (2), the degree of u is at most $2(i - 1) + s - 2(i - 1) = s$.

PROOF OF THE THEOREM. If $u \in H^m(\mathbf{R}^n)$, then we see from (2) that

$$\frac{1}{\omega_n r^{n-1}} \int_{rS^{n-1}} u \, dS = \sum_{i=1}^m r^{2(i-1)} h_i(0),$$

where ω_n denotes the surface measure of S^{n-1} .

Since $|u| = 2u^+ - u$, we have

$$\begin{aligned} & \liminf_{r \rightarrow \infty} r^{-s-n+1} \int_{rS^{n-1}} |u| \, dS \\ &= \liminf_{r \rightarrow \infty} \left(2r^{-s-n+1} \int_{rS^{n-1}} u^+ \, dS - r^{-s-n+1} \int_{rS^{n-1}} u \, dS \right) \\ &= \liminf_{r \rightarrow \infty} \left(2r^{-s-n+1} \int_{rS^{n-1}} u^+ \, dS - r^{-s} P_{2(m-1)}(r) \right). \end{aligned}$$

Hence (1) implies (6) since $s > 2(m - 1)$, so that the present theorem follows from Lemma 3.

References

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