

On q -analogues of multiple sine functions

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Basic back-ground problem: To calculate the sine function

$S_A(x) = \prod_{a \in A} (a-x)$ of a ring (or integral domain) A .

$\left(\prod_{a \in A} (a-x) \stackrel{''}{=} \exp \left(-\frac{\partial}{\partial s} \sum_{a \in A} (a-x)^{-s} \right) \Big|_{s=0} \right)$: regularized product

Example 1. $S_{\mathbb{Z}}(x) = \prod_{m=-\infty}^{\infty} (m-x) \quad (\text{Im } x > 0)$

$$\stackrel{\text{def.}}{=} \exp \left(-\frac{\partial}{\partial s} \sum_{m=-\infty}^{\infty} (m-x)^{-s} \Big|_{s=0} \right)$$

$$= 1 - e^{2\pi i x}$$

$$\sim 2s \sin(\pi x).$$

Example 2. Let τ be an imaginary quadratic integer,

$\text{Im } \tau > 0$, then

$$S_{\mathbb{Z}[\tau]}(x) = \prod_{m,n=-\infty}^{\infty} (m+n\tau+x)$$

$$= \exp \left(-\frac{\partial}{\partial s} \sum_{m,n} (m+n\tau+x)^{-s} \Big|_{s=0} \right)$$

$$= (1-q_x) \prod_{n=1}^{\infty} (1-q_{\tau}^n q_x) (1-q_{\tau}^n q_x^{-1})$$

for $0 < \text{Im } x < \text{Im } \tau$, where $q_x = e^{2\pi i x}$ and $q_{\tau} = e^{2\pi i \tau}$.

There are two proofs

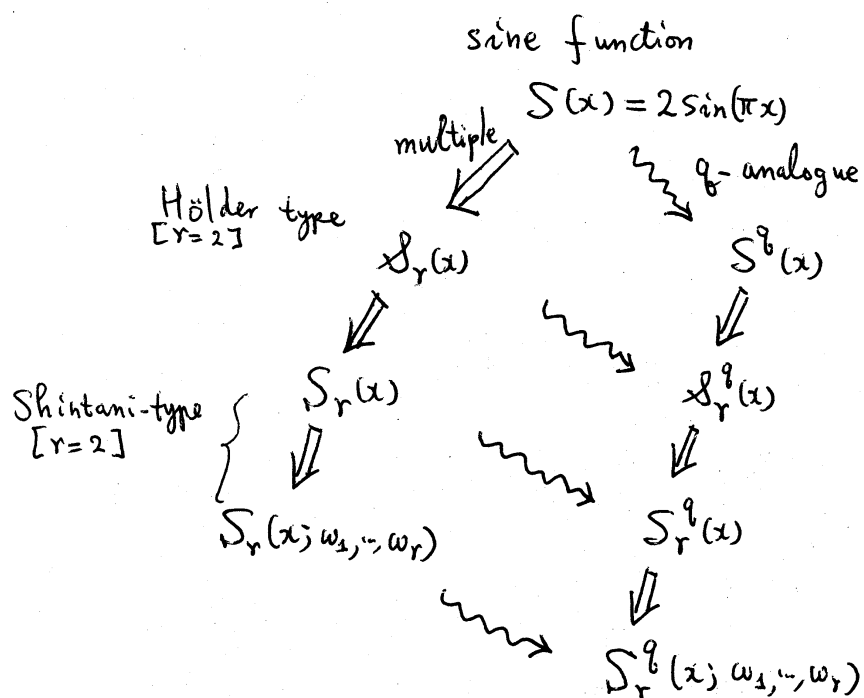
- 1) Fourier expansion of "non-absolute Eisenstein series"
- 2) double gamma function.

It turns out that $S_{\mathbb{Z}[\tau]}(x) = \sin_{q_2}(\pi_{q_2} x)$.

Problem $A = \mathcal{O}_K$ integerring of a number field K . Then, what is $S_A(x)$? Moreover, $K^{ab} = K(S_A(K))$?

- 1° K : totally real $\Rightarrow$ Shintani's approximation to $S_A(x)$ via multiple sine functions.
- 2° K : not totally real $\Rightarrow$ q -analogues of multiple sine functions.

generalizations



basic properties

- ⑥ periodicity
- ⑦ distribution property (multiplication formula)
- ⑧ relation to special values of zeta and L-functions
(Dirichlet's "class number formula", ...)
- ⑨ relation to gamma factors of $\left\{ \begin{array}{l} \text{Selberg} \\ \text{arithmetic} \end{array} \right\}$ zeta functions
- [④ addition formula
⑤ algebraicity of special values] ... "difficult" in general

§1. Survey of the sine function.

Let $S(x) = 2 \sin(\pi x)$. Then :

① $S'(x+1) = S'(x) \cdot (-1)$ with $(-1) = S_0(x)^{-1}$.

② $S'(Nx) = \prod_{k=0}^{N-1} S'(x + \frac{k}{N})$ for integers $N \geq 2$.

In particular $\prod_{k=1}^{N-1} S'(\frac{k}{N}) = N$.

③ Let χ be a primitive even Dirichlet character modulo $N \geq 2$, and $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ the Dirichlet L-function. Then $L(0, \chi) = 0$ and

$$L'(0, \chi) = -\frac{1}{2} \sum_{k=1}^{N-1} \chi(k) \log S'(\frac{k}{N}).$$

(Dirichlet's "class number formula": usually written for $L(1, \chi)$ via functional equation.)

④ Euler (discovery) : $\zeta(1-s) = \zeta(s) (2\pi)^s \Gamma(s) S'(\frac{s+1}{2})$

$\Downarrow$ symmetrize

§2. Survey of multiple sine functions (Kurokawa 1990)

Let $S_r(x; \underline{\omega}) = \Gamma_r(x, \underline{\omega})^{-1} \Gamma_r(|\underline{\omega}| - x, \underline{\omega})^{(-1)^r}$,

where $\underline{\omega} = (\omega_1, \dots, \omega_r)$, $|\underline{\omega}| = \omega_1 + \dots + \omega_r$

and

$$\Gamma_r(x, \underline{\omega})^{-1} = \prod_{\underline{n} \geq \underline{0}} (\underline{n} \cdot \underline{\omega} + x) \stackrel{\text{def.}}{=} \exp\left(-\frac{\partial}{\partial s} \zeta_r(0, x, \underline{\omega})\right)$$

with the multiple Hurwitz zeta function

$$\zeta_r(s, x, \underline{\omega}) = \sum_{\underline{n} \geq \underline{0}} (\underline{n} \cdot \underline{\omega} + x)^{-s}.$$

We put

$$S_r(x) = S_r(x; (1, \dots, 1))$$

and

$$\begin{aligned} S_r(x) &= e^{\frac{x^{r-1}}{r-1}} \prod_{n=-\infty}^{\infty} P_r\left(\frac{x}{n}\right) n^{r-1} \\ &= e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty} \left(P_r\left(\frac{x}{n}\right) P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}} \right) n^{r-1} \end{aligned}$$

for $S_1(x) = 2 \sin(\pi x)$; these are meromorphic of order r ,
 $r \geq 2$ and

Theorem $S_r(x) = C_r \prod_{k=1}^r S_k(x)^{c(r,k)}$
 with $C_r = \begin{cases} 1 & \dots r : \text{even} \\ e^{2\pi i(1-r)} & \dots r : \text{odd} \end{cases}$
 and $c(r, k) = \frac{1}{k} \sum_{l=1}^k (-1)^{l-1} \binom{k}{l} l^r$,
 $(c(r, 1) = 1, \dots, c(r, r) = (-1)^{r-1} (r-1)!)$

$$\textcircled{6} \quad S_x(x+\omega_i, \underline{\omega}) = S_r(x, \underline{\omega}) S_{r-1}(x, \underline{\omega}(i))^{-1}$$

where $\underline{\omega}(i) = (\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_r)$.

(similar for $S_r(x), \mathcal{S}_r(x)$ also)

$$\textcircled{1} \quad \begin{cases} S_r(Nx, \underline{\omega}) = \prod_{\substack{0 \leq k_i \leq N-1 \\ i=1, \dots, r}} S_x\left(x + \frac{k \cdot \underline{\omega}}{N}, \underline{\omega}\right), \\ \prod'_{\substack{0 \leq k_i \leq N-1 \\ i=1, \dots, r}} S_x\left(\frac{k \cdot \underline{\omega}}{N}, \underline{\omega}\right) = N. \end{cases}$$

$\textcircled{2}$ [and homogeneity: $S_x(cx, c\underline{\omega}) = S_x(x, \underline{\omega})$ for $c > 0$]

a) Let χ be a primitive character modulo $N \geq 2$

satisfying $\chi(-1) = (-1)^{r-1}$ for $r \geq 1$. Then

$$L'(1-r, \chi) = \sum_{\substack{1 \leq j \leq r \\ 1 \leq k \leq N-1}} c_{k,j}^N \chi(k) \log S_j\left(\frac{k}{N}\right).$$

(similar for $\mathcal{S}_j(\frac{k}{N})$ instead of $S_j(\frac{k}{N})$.)

examples

$$\begin{aligned} L'(-1, \chi) &= \frac{1}{2} \log \prod_{k=1}^{N-1} \left(\frac{\mathcal{S}_2(\frac{k}{N})^N}{\mathcal{S}_1(\frac{k}{N})^k} \right)^{\chi(k)} \\ &= -\frac{1}{2} \log \prod_{k=1}^{N-1} \left(S_2(\frac{k}{N})^N S_1(\frac{k}{N})^k \right)^{\chi(k)}, \end{aligned}$$

$$\begin{aligned} L'(-2, \chi) &= \frac{1}{2} \log \prod_{k=1}^{N-1} \left(\frac{\mathcal{S}_3(\frac{k}{N})^{N^2} \mathcal{S}_1(\frac{k}{N})^{k^2}}{\mathcal{S}_2(\frac{k}{N})^{2Nk}} \right)^{\chi(k)} \\ &= -\frac{1}{2} \log \prod_{k=1}^{N-1} \left(S_3(\frac{k}{N})^{2N^2} S_2(\frac{k}{N})^{2Nk-3N^2} S_1(\frac{k}{N})^{k^2} \right)^{\chi(k)}. \end{aligned}$$

two proofs $\left\{ \begin{array}{l} 1) \text{poly-logarithm} \xleftrightarrow[\text{relation}]{(*)} \mathcal{S}_r(x) \Rightarrow L(r, \chi) \Rightarrow \text{f.t. eq.} \\ 2) \zeta(s-r+1, x) \xleftrightarrow[\text{relation}]{} \zeta_r(s, x) \Rightarrow L'(1-x, \chi). \end{array} \right.$

$$(*) \quad \mathcal{S}_r(x) = \exp \left(- \frac{(r-1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(2\pi i)^k}{k!} x^k \operatorname{Li}_{r-k}(e^{-2\pi i x}) \right. \\ \left. + \frac{\pi i}{r} x^r + \frac{(r-1)!}{(2\pi i)^{r-1}} \zeta(x) \right)$$

for $\operatorname{Im} x < 0$ (or $0 < x < 1$); "Debye polylogarithm" appears.

b) $K/\mathbb{Q}$ totally real, χ certain type Dirichlet character,

$$L'_K(0, \chi) \doteq \sum_{\substack{k \leq [K:\mathbb{Q}] \\ j: \text{finite index}}} c_{k,j} \log S_k(\alpha_j, \omega_j)$$

for some $\alpha_j \in K$; originally due to Shimura via $\Gamma_k(\alpha_j, \omega_j)$.

③ gamma factors of Selberg zeta functions.

Let $M = \Gamma \backslash G/K$ be a compact locally symmetric space of rank one. Assume that $\dim M$ is even ($\Leftrightarrow$ the gamma factor is non-trivial $\Leftrightarrow G \not\subset SO(1, 2n-1)$).

Then, the Selberg zeta function $Z_M(s)$ has a meromorphic continuation to $s \in \mathbb{C}$ of order $\dim M$ with the functional equation:

$$Z_M(2\rho_M - s) = Z_M(s) \exp\left(\text{vol}(M) \int_0^{s-\rho_M} \mu_M(it) dt\right)$$

where $\mu_M(t)$ is the Plancherel measure. This is due to Selberg (1956 for $G = \text{SL}_2(\mathbb{R}) \sim \text{SO}(1, 2)$) and Gangoli (1977, general G). The problem is to calculate $\exp(\dots)$ explicitly and obtain the gamma factor

$$\Gamma_M(s) \text{ satisfying } \exp(\dots) = \frac{\Gamma_M(s)}{\Gamma_M(2\rho_M - s)}.$$

Theorem. Let

$$S_M(s) \stackrel{\text{def}}{=} \left(\frac{\det(\sqrt{\Delta_{M'}} + \rho_M^2 + (s - \rho_M))}{\det(\sqrt{\Delta_{M'}} + \rho_M^2 - (s - \rho_M))} \right)^{\text{vol}(M) (-1)^{\dim M/2}},$$

where $M' = G'/K$ is the compact dual symmetric space and $\Delta_{M'}$ is the Laplacian. Then

$$\begin{aligned} S_M(s) &= \exp\left(\text{vol}(M) \int_0^{s-\rho_M} \mu_M(it) dt\right) \\ &= \begin{cases} (S_{2n}(s) S_{2n}(s+1))^{\text{vol}(M) (-1)^n} & \dots G = \text{SO}(1, 2n), \\ \left(\prod_{k=0}^n S_{2n}(s+k) \binom{n}{k}^2 \right)^{\text{vol}(M) (-1)^n} & \dots G = \text{SU}(1, n), \\ \left(\prod_{k=0}^{2n-1} S_{4n}(s+k) \frac{1}{2n} \binom{2n}{k} \binom{2n}{k+1} \right)^{\text{vol}(M)} & \dots G = \text{Sp}(1, n), \\ (S_{16}(s) S_{16}(s+1)^{10} S_{16}(s+2)^{28} S_{16}(s+3)^{28} S_{16}(s+4)^{10} S_{16}(s+5))^{\text{vol}(M)} & \dots G = F_4. \end{cases} \end{aligned}$$

Hence we obtain the gamma factor

$$\begin{aligned} \Gamma_M(s) &= \det(\sqrt{\Delta_M + \rho_M^2} + (s - \rho_M))^{vol(M)(-1)^{\dim M/2}} \\ &= \begin{cases} (\Gamma_{2n}(s) \Gamma_{2n}(s+1))^{vol(M)(-1)^{n-1}} & \dots G = SO(1, 2n) \\ \left(\prod_{k=0}^n \Gamma_{2n}(s+k) \binom{n}{k}^2 \right)^{vol(M)(-1)^{n-1}} & \dots G = SU(1, n) \\ \left(\prod_{k=0}^{2n-1} \Gamma_{4n}(s+k) \frac{1}{2n} \binom{2n}{k} \binom{2n}{k+1} \right)^{vol(M)} & \dots G = Sp(1, n) \\ (\Gamma_{16}(s) \Gamma_{16}(s+1)^{10} \Gamma_{16}(s+2)^{28} \Gamma_{16}(s+3)^{28} \Gamma_{16}(s+4)^{10} \Gamma_{16}(s+5))^{-vol(M)} & \dots G = F_4 \end{cases} \end{aligned}$$

§3. q -analogue of the sine function

We assume $q > 1$ for simplicity.

$$S^q(x) \stackrel{\text{def}}{=} \frac{\Gamma^q(\frac{1}{2})^2}{\Gamma^q(x) \Gamma^q(1-x)} = \bar{S}_q(x) q^{-(x-\frac{1}{2})^2} \prod_{n=1}^{\infty} (1 - q^{-(n+\frac{1}{2})})^{-2}$$

where $\Gamma^q(x)$ is the Jackson's q -gamma function and

$$\begin{aligned} \bar{S}_q(x) &= (1 - q^{-x}) \prod_{n=1}^{\infty} (1 - q^{-(n+x)}) (1 - q^{-(n-x)}) \\ &\sim \sigma(-\tau, -\tau x) \text{ if } q = e^{2\pi i \tau} \text{ with } \tau \in H_- (\text{Im } \tau < 0). \end{aligned}$$

$$\text{Let } L_q(s, \chi) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \chi(n) [n]^{-s}$$

where $[n] = [n]_q = \frac{q^n - 1}{q - 1}$ is the q -integer.

① periodicity, and ② distribution property are due to Jackson.

② Theorem Let χ be a primitive even Dirichlet character modulo $N \geq 2$. Then $L_q(s, \chi)$ is meromorphic on $\mathbb{C}$ and $L_q(0, \chi) = 0$. Moreover

$$L'_q(0, \chi) = -\frac{1}{2} \sum_{k=1}^{N-1} \chi(k) \log S_q^N\left(\frac{k}{N}\right).$$

§4. q -multiple sine functions. ($q > 1$)

We notice only the most basic case.

Let

$$\bar{S}_r^q(x, \underline{\omega}) \stackrel{\text{def}}{=} \prod_{n \geq 0} (1 - q^{-(n \cdot \underline{\omega} + x)}) (1 - q^{-(n \cdot \underline{\omega} + |\underline{\omega}| - x)})^{(-1)^{r-1}}$$

for $\underline{\omega} = (\omega_1, \dots, \omega_r)$. Then:

$$\textcircled{0} \quad \bar{S}_r^q(x + \omega_i, \underline{\omega}) = \bar{S}_r^q(x, \underline{\omega}) \bar{S}_{r-1}^q(x, \underline{\omega}(i))^{-1}.$$

$$\textcircled{1} \quad \bar{S}_r^q(Nx, \underline{\omega}) = \prod_{\substack{0 \leq k_i \leq N-1 \\ i=1, \dots, r}} \bar{S}_r^{q^N}\left(x + \frac{k_i \cdot \omega_i}{N}, \underline{\omega}\right)$$

and

$$\bar{S}_r^q(cx, c\underline{\omega}) = \bar{S}_r^{q^c}(x, \underline{\omega}) \quad \text{for } c > 0.$$

We have results similar to §2. The above $\bar{S}_r^q(x, \underline{\omega})$ can be considered as multi-parametric q -analogue of the sine function by setting $(q_1, \dots, q_r) = (q^{\omega_1}, \dots, q^{\omega_r})$.

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