

Schubert Eisenstein Series for $GL(3)$

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1 Schubert Eisenstein series

We try to explain the concept of Schubert Eisenstein series and possible arithmetic implication. This is joint work with D. Bump and a survey talk at RIMS. The original work can be found in [1]

1.1 Schubert Eisenstein series

Let us start with a general set up.

Let G be a split reductive algebraic group over a global field F . Let $\hat{T}$ be the maximal torus of the group $\hat{G}$ with opposite root data, so that $\hat{G}(\mathbb{C})$ is the connected Langlands L-group. Let $\nu \in \hat{T}(\mathbb{C})$. Then ν parametrizes a character χ_ν of $T(\mathbb{A})/T(F)$, where $\mathbb{A}$ is the adele ring of F . Extending χ_ν to the Borel subgroup $B(\mathbb{A})$, let f_ν be an element of the corresponding induced representation, so that

$$f_\nu(bg) = (\delta^{1/2}\chi_\nu)(b)f(g), \quad b \in B(\mathbb{A}). \quad (1)$$

The flag variety $X = B \backslash G$ is a projective variety. We recall its decomposition into Schubert cells. We have the Bruhat decomposition $G = \bigcup BwB$, a disjoint union over $w \in W$, and let Y_w be the image of BwB in X . The Schubert cell X_w is the Zariski closure of Y_w . It equals

$$\bigcup_{\substack{u \in W \\ u \leq w}} Y_u,$$

where $u \leq w$ is the Bruhat order. Let G_w be the subset of G that is the union of BuB for $u \leq w$. It is not a subgroup. Let $X_w(F)$ be the set of $\gamma \in B_F \backslash G_F$ belonging to X_w . Thus $X_w(F) = B_F \backslash G_w(F)$. We may now define the *Schubert Eisenstein series*

$$E_w(g, f, \chi) = \sum_{\gamma \in X_w(F)} f(\gamma g).$$

1.2 Bott-Samelson varieties

Let us recall the Bott-Samelson varieties and their relationship with Schubert varieties. We will denote by α_i and s_i the simple roots and corresponding simple reflections. Let $w \in W$ and let $\mathfrak{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$ be a reduced decomposition of w into a product of simple reflections: $w = s_{i_1} \cdots s_{i_k}$. Let P_j be the minimal parabolic subgroup generated by B and s_j . We define a left action of B^k on $P_{i_1} \times \cdots \times P_{i_k}$ by

$$(b_1, \dots, b_k) \cdot (p_{i_1}, \dots, p_{i_k}) = (b_1 p_{i_1} b_2^{-1}, b_2 p_{i_2} b_3^{-1}, \dots, b_k p_{i_k}). \quad (2)$$

The quotient $B^k \backslash (P_{i_1} \times \cdots \times P_{i_k})$ is the *Bott-Samelson variety* $Z_{\mathfrak{w}}$. There is a morphism $\text{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \rightarrow X_w$ induced by the multiplication map that sends

$$(p_{i_1}, \dots, p_{i_k}) \mapsto p_{i_1} \cdots p_{i_k}.$$

This map is a surjective birational morphism.

It is known that Bott-Samelson varieties are always nonsingular, so this gives a resolution of the singularities of the Schubert variety X_w . The map $\text{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \rightarrow X_w$ may not be an isomorphism. In special cases where it is an isomorphism, every element of X_w has a unique representation as a product $i_{\alpha_1}(\gamma_1) \cdots i_{\alpha_k}(\gamma_k)$, where if α is a root (in this case a simple root) i_{α} is the Chevalley embedding of $\text{SL}(2)$ into G corresponding to α , so the image of i_{α_i} lies in the Levi subgroup of P_{i_i} . Beyond these special cases where $\text{BS}_{\mathfrak{w}}$ is an isomorphism, in every case each element of X_w has such a factorization, and if the element is in general position, it is unique, since $\text{BS}_{\mathfrak{w}}$ is birational. Let us call this a *Bott-Samelson factorization*. This means that we may write

$$E_{s_1 \cdots s_k}(g, \nu) = \sum_{\gamma_k \in B_{\text{SL}_2}(F) \backslash \text{SL}_2(F)} E_{s_1 \cdots s_{k-1}}(i_{\alpha_k}(\gamma_k)g, \nu), \quad (3)$$

building up the Schubert Eisenstein series by repeated SL_2 summations. If $\text{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \rightarrow X_w$ is not an isomorphism, a modification of this method should be applicable.

1.3 GL_3 Schubert Eisenstein series, Explicit Computation

Let us be more precise : let $G = \text{GL}_3$ and let

$$\zeta^*(s) = |D_F|^{\frac{s}{2}} \prod_v \zeta_v(s), \quad \zeta_v(s) = \begin{cases} (1 - q_v^{-s})^{-1} & \text{if } v \text{ is nonarchimedean,} \\ \Gamma_v(s) & \text{if } v \text{ is archimedean} \end{cases}$$

where we recall that D_F is the discriminant of F . For simplicity we will assume that the character χ is unramified at every place. Find $\nu_1, \nu_2 \in \mathbb{C}$ such that

$$(\delta^{1/2}\chi) \begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} = |y_1|^{2\nu_1+\nu_2} |y_2|^{\nu_2-\nu_1} |y_3|^{-\nu_1-2\nu_2}.$$

We will denote this character χ_{ν_1, ν_2} . Also, take $f = f^\circ$ where

$$f^\circ(g) = f_{\nu_1, \nu_2}^\circ(g) = \prod_v f_v^\circ(g_v).$$

Thus if $k \in K$

$$f_{\nu_1, \nu_2}^\circ \left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k \right) = |y_1|^{2\nu_1+\nu_2} |y_2|^{\nu_2-\nu_1} |y_3|^{-\nu_1-2\nu_2}.$$

For each $w \in W$ normalize the Schubert Eisenstein series and denote

$$E_w^*(g; \nu_1, \nu_2) = \zeta^*(3\nu_1) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) E_w(g; \nu_1, \nu_2)$$

and

$$\hat{E}_{s_1 s_2}^*(\nu_1, \nu_2) = E_{s_1 s_2}^*(\nu_1, \nu_2) - E_{s_2}^*(\nu_1, \nu_2) - E_{s_2}^*(\nu_2, 1 - \nu_1 + \nu_2). \quad (4)$$

Similarly

$$\hat{E}_{s_2 s_1}^*(\nu_1, \nu_2) = E_{s_2 s_1}^*(\nu_1, \nu_2) - E_{s_1}^*(\nu_1, \nu_2) - E_{s_1}^*(1 - \nu_1 + \nu_2, \nu_1). \quad (5)$$

Theorem 1 [1] $E_{s_1 s_2}^*(g; \nu_1, \nu_2)$ has meromorphic continuation to all ν_1, ν_2 . It has a functional equation

$$E_{s_1 s_2}^*(g; \nu_1, \nu_2) = E_{s_1 s_2}^* \left(g; \frac{2}{3} - \nu_1, \nu_1 + \nu_2 - \frac{1}{3} \right).$$

Moreover $\hat{E}_{s_1 s_2}^{**}(g; \nu_1, \nu_2)$ is an entire function.

The similar result holds for $\hat{E}_{s_2 s_1}^*(\nu_1, \nu_2)$.

1.4 Kronecker Limit Formula

Bump and Goldfeld proved the following result. If $K/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class of K one may associate with $\mathfrak{a}$ a compact torus of GL_3 , and if $L_{\mathfrak{a}}$ is the period of $\kappa(g)$ over this torus, then the Taylor expansion of the L-function $L(s, \mathfrak{a})$ has the form $\rho s^{-1} + L_{\mathfrak{a}} + \dots$. Therefore if θ is a character of the ideal class group then $L(s, \theta) = \sum \theta(\mathfrak{a}) L_{\mathfrak{a}}$. The proof involves showing that the torus period of the Eisenstein series equals a Rankin-Selberg integral of a Hilbert modular Eisenstein series.

Considering Taylor expansions of E_w for various w at $\nu_1 = \nu_2 = 0$ we get

Theorem 2 [1] *We have*

$$\kappa(g) = \frac{\rho}{3} \zeta^*(2) \left[\hat{E}_{s_2 s_1}^{**}(g; 0, 0) + E_{s_1}^{**}(g; 1, 0) \right] + c_0.$$

Furthermore

$$\kappa(g) = \frac{\rho}{3} \zeta^*(2) \left[\hat{E}_{s_1 s_2}^{**}(g; 1, 0) + \phi_{s_2}(g) \right] + c'_0.$$

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References

- [1] Daniel Bump and YoungJu Choie, Schubert Eisenstein Series for $GL(3)$, Preprint (2011)