

INTEGRAL MEANS OF THE FRACTIONAL DERIVATIVE OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. By using the definition of fractional derivative (cf., [2]), we investigate the sharp integral means inequalities of the fractional derivatives of univalent functions with negative coefficients and extend the sharp results of H. Silverman [5, Theorem 2.2].

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of $f(z)$ normalized by

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Then a function $f(z)$ belonging to the class $\mathcal{S}$ is said to be in the class $\mathcal{K}$ if and only if

$$(1.2) \quad \operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > 0 \quad (z \in \mathcal{U}).$$

We denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ whose functions may be represented by

$$(1.3) \quad f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0).$$

Silverman [4] showed that f of the form (1.3) is in $\mathcal{T}$ if and only if $\sum_{k=2}^{\infty} k a_k \leq 1$, and that the extreme points of $\mathcal{T}$ are

$$(1.4) \quad f_1(z) = z \quad \text{and} \quad f_m(z) = z - z^m/m, \quad m = 2, 3, \dots$$

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Further a function f of the form (1.3) is in $\mathcal{C} = \mathcal{T} \cap \mathcal{K}$ if and only if $\sum_{k=2}^{\infty} k^2 a_k \leq 1$, and that the extreme points of $\mathcal{C}$ are $g_1(z) = z$ and $g_2(z) = z - z^m/m^2$ ($m = 2, 3, \dots$).

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ so that $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathcal{U}$) and $g(z) = h(w(z))$, we denote this subordination by $g(z) \prec h(z)$.

Many essentially equivalent definition of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (*cf.*, *e.g.*, [3], [6, p 45] and [7]). We find it to be convenient to recall here the following definition which were used recently by Owa [2] (and by Srivastava and Owa [7]).

Definition 1. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$(1.5) \quad \mathcal{D}_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),$$

where $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring for $\log(z-\zeta)$ to be real for $z > \zeta$.

Definition 2. Under the hypotheses of Definition 1, the fractional derivative of order $n + \lambda$ is defined by

$$(1.6) \quad \mathcal{D}_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} \mathcal{D}_z^\lambda f(z) \quad (0 \leq \lambda < 1; n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}).$$

In [5] it is proven that

$$(1.7) \quad \int_0^{2\pi} |f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\beta d\theta$$

for all $f \in \mathcal{T}$, $\beta > 0$ and $0 < r < 1$. In this paper, by using the fractional derivative, we prove that

$$(1.8) \quad \int_0^{2\pi} |\mathcal{D}_z^\lambda f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^\lambda f_2(re^{i\theta})|^\beta d\theta$$

for all $f \in \mathcal{T}$, $\beta > 0$, $0 < r < 1$ and $0 \leq \lambda < 1$. We also obtain the integral means inequality for $\mathcal{D}_z^{n+\lambda} f(z)$ ($n = 1, 2$) if $f \in \mathcal{C}$ (or $\mathcal{T}$).

2. Main Results

The following result will be required in our investigation.

Lemma. (Littlewood [1]) *If f and g are analytic in $\mathcal{U}$ with $g \prec f$, then, for $\beta > 0$ and $0 < r < 1$,*

$$(2.1) \quad \int_0^{2\pi} |g(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\beta d\theta.$$

Applying the above lemma, we prove

Theorem 1. *Let $\beta > 0$ and $f_2(z)$ is defined by (1.4). If $f \in \mathcal{T}$, then for $z = re^{i\theta}$ and $0 < r < 1$,*

$$(i) \quad \int_0^{2\pi} |\mathcal{D}_z^\lambda f(z)|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^\lambda f_2(z)|^\beta d\theta \quad (0 \leq \lambda < 1)$$

$$(ii) \quad \int_0^{2\pi} |\mathcal{D}_z^{2+\lambda} f(z)|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^{2+\lambda} f_2(z)|^\beta d\theta \quad (0 < \lambda < 1).$$

Proof. We prove (i). The proof of (ii) is similar and will be omitted. If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($a_k \geq 0$), then

$$\mathcal{D}_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \sum_{k=2}^{\infty} \Phi(k) k a_k z^{k-1} \right),$$

where

$$(2.2) \quad \Phi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geq 2).$$

Note that $\Phi(k)$ is a non-increasing function of k ,

$$(2.3) \quad 0 < \Phi(k) \leq \Phi(2) = \frac{1}{2-\lambda}.$$

Since

$$\mathcal{D}_z^\lambda f_2(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{1}{2-\lambda} z \right),$$

we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} \Phi(k) k a_k z^{k-1} \right|^\beta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1}{2-\lambda} z \right|^\beta d\theta.$$

By Lemma, it suffices to prove that

$$1 - \sum_{k=2}^{\infty} \Phi(k) k a_k z^{k-1} \prec 1 - \frac{1}{2-\lambda} z.$$

Setting

$$(2.4) \quad 1 - \sum_{k=2}^{\infty} \Phi(k) k a_k z^{k-1} = 1 - \frac{w(z)}{2 - \lambda}.$$

From (2.3) and (2.4), we obtain

$$|w(z)| \leq \left| \sum_{k=2}^{\infty} (2 - \lambda) \Phi(2) k a_k z^{k-1} \right| \leq |z| \sum_{k=2}^{\infty} k a_k \leq |z|.$$

This completes the proof of (i).

Remark. If $\lambda = 0$ in (i) of Theorem 1, then it would immediately yield the result of Silverman [5, Theorem 2.2].

For the fractional derivative of order $1 + \lambda$, we have

Theorem 2. If $f \in \mathcal{C}$ and $\beta > 0$, then for $z = re^{i\theta}$ and $0 < r < 1$,

$$(i) \quad \int_0^{2\pi} |\mathcal{D}_z^{1+\lambda} f(z)|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^{1+\lambda} f_2(z)|^\beta d\theta \quad (0 \leq \lambda < 1)$$

$$(ii) \quad \int_0^{2\pi} |\mathcal{D}_z^{1+\lambda} f(z)|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^{2+\lambda} g_2(z)|^\beta d\theta \quad (0 \leq \lambda \leq 2/3).$$

Proof. (i) From the definition (1.6), we have

$$(2.5) \quad \mathcal{D}_z^{1+\lambda} f(z) = \frac{z^{-\lambda}}{\Gamma(1-\lambda)} \left(1 - \sum_{k=2}^{\infty} \Psi(k) k(k-1) a_k z^{k-1} \right),$$

where

$$\Psi(k) = \frac{\Gamma(k-1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)} \quad (k \geq 2).$$

Note that $0 < \Psi(k) \leq \Psi(2) = 1/(1-\lambda)$.

Since

$$\mathcal{D}_z^{1+\lambda} f_2(z) = \frac{z^{-\lambda}}{\Gamma(1-\lambda)} \left(1 - \frac{1}{1-\lambda} z \right),$$

it suffices to show that

$$1 - \sum_{k=2}^{\infty} \Psi(k) k(k-1) a_k z^{k-1} \prec 1 - \frac{1}{1-\lambda} z.$$

Setting

$$1 - \sum_{k=2}^{\infty} \Psi(k)k(k-1)a_k z^{k-1} = 1 - \frac{w(z)}{1-\lambda},$$

$$|w(z)| \leq \left| \sum_{k=2}^{\infty} k(k-1)a_k z^{k-1} \right| \leq |z| \sum_{k=2}^{\infty} k^2 a_k \leq |z|.$$

By Lemma, the proof of (i) is completed.

(ii) Making use of (1.6) and (2.5), we obtain

$$\mathcal{D}_z^{1+\lambda} f(z) = \frac{z^{-\lambda}}{\Gamma(1-\lambda)} \left(1 - \sum_{k=2}^{\infty} \Theta(k)k^2 a_k z^{k-1} \right),$$

where

$$\Theta(k) = \frac{\Gamma(k)\Gamma(1-\lambda)}{k\Gamma(k-\lambda)} \quad (k \geq 2).$$

We note that $0 < \Theta(k) \leq \Theta(2) = 1/2(1-\lambda)$ for $0 \leq \lambda \leq 2/3$. Thus the proof of (ii) is much akin to that of (i), and we omit the details involved.

Denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, $0 \leq \alpha < 1$, the subclasses of $\mathcal{T}$ that are, respectively, starlike of order α and convex of order α . In [4], Silverman showed that $f \in \mathcal{T}^*(\alpha)$ if and only if $\sum_{k=2}^{\infty} ((k-\alpha)/(1-\alpha))a_k \leq 1$ and $f \in \mathcal{C}(\alpha)$ if and only if $\sum_{k=2}^{\infty} (k(k-\alpha)/(1-\alpha))a_k \leq 1$. In addition, the extreme points of $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are $h_m(z) = z - ((1-\alpha)/(m-\alpha))z^m$ and $k_m(z) = z - ((1-\alpha)/m(m-\alpha))z^m$ for $m \geq 2$.

For the cases of $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, the proof is much akin to that of Theorem 1 and Theorem 2, and we omit the details involved.

Theorem 3. (i) If $f \in \mathcal{T}^*(\alpha)$ and $\beta > 0$, then for $0 < r < 1$,

$$\int_0^{2\pi} |\mathcal{D}_z^\lambda f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^\lambda h_2(re^{i\theta})|^\beta d\theta \quad (0 \leq \lambda < 1)$$

and

$$\int_0^{2\pi} |\mathcal{D}_z^{2+\lambda} f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^{2+\lambda} h_2(re^{i\theta})|^\beta d\theta \quad (0 < \lambda < 1).$$

(ii) If $f \in \mathcal{C}(\alpha)$ and $\beta > 0$, then for $0 < r < 1$,

$$\int_0^{2\pi} |\mathcal{D}_z^\lambda f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^\lambda k_2(re^{i\theta})|^\beta d\theta \quad (0 \leq \lambda < 1),$$

$$\int_0^{2\pi} |\mathcal{D}_z^{1+\lambda} f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^{1+\lambda} h_2(re^{i\theta})|^\beta d\theta \quad (0 \leq \lambda < 1),$$

$$\int_0^{2\pi} |\mathcal{D}_z^{1+\lambda} f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^{1+\lambda} k_2(re^{i\theta})|^\beta d\theta \quad (0 \leq \lambda \leq 2/3)$$

and

$$\int_0^{2\pi} |\mathcal{D}_z^{2+\lambda} f(re^{i\theta})|^\beta d\theta \leq \int_0^{2\pi} |\mathcal{D}_z^{2+\lambda} k_2(re^{i\theta})|^\beta d\theta \quad (0 < \lambda < 1).$$

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