

On the solution of the Time Dependent Ginzburg-Landau equations

BY

TAKAHIRO AKIYAMA[†], HIRONORI KASAI[†] AND MASAYOSHI TSUTSUMI[†]

([†]Department of Applied Physics, Waseda University,
[‡]Department of Mathematical Science, Waseda University.)

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with smooth boundary $\partial\Omega$. We consider the Time Dependent Ginzburg-Landau equations:

$$\psi_t = \mathbf{D}_A^2 \psi + \kappa(1 - |\psi|^2)\psi + i\Phi\psi, \quad \text{in } (0, \infty) \times \Omega, \quad (1)$$

$$\mathbf{A}_t = -\text{rot}^2 \mathbf{A} - \frac{i}{2} \{ \overline{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} + \nabla \Phi, \quad \text{in } (0, \infty) \times \Omega, \quad (2)$$

where $\mathbf{D}_A \psi = (\nabla \psi - i\mathbf{A}\psi)$, ψ is the complex order parameter, $\mathbf{A}$ is the magnetic vector potential, Φ is the scalar electric potential and κ is a real positive constant called the Ginzburg-Landau parameter of the substance. The system (1)-(2), which we call the TDGL equations below, was proposed by A.Schmid [8] or L.P.Gor'kov and G.M. Eliashberg [6]. The TDGL equations have an important property, namely, that of gauge invariance. Throughout this paper we consider the problem in the Coulomb gauge, namely,

$$\text{div} \mathbf{A} = 0, \quad \text{in } \Omega. \quad (3)$$

Then, the boundary conditions are prescribed as

$$\frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega, \quad (4)$$

$$\mathbf{A} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \quad (5)$$

$$\text{rot} \mathbf{A} \times \mathbf{n} = 0, \quad \text{on } \partial\Omega, \quad (6)$$

$$\frac{\partial \Phi}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega, \quad (7)$$

where $\mathbf{n}$ denotes the unit outer normal vector to the boundary $\partial\Omega$. The initial conditions are

$$\psi(0, x) = \psi_0, \quad \text{in } \Omega, \quad (8)$$

$$\mathbf{A}(0, x) = \mathbf{A}_0, \quad \text{in } \Omega. \quad (9)$$

In the following, in order to get the uniqueness of solutions, we assume that

$$\int_{\Omega} \Phi(t, x) dx = 0, \quad \forall t \geq 0. \quad (10)$$

Let $W^{m,p}(\Omega)$ be the standard Sobolev space of complex-valued functions, and as usual, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$. $\mathbf{W}^{m,p}(\Omega)$ and $\mathbf{H}^m(\Omega)$ denote the Sobolev spaces of real vector-valued functions. As usual, $L^2(\Omega) = H^0(\Omega)$ and $\mathbf{L}^2(\Omega) = \mathbf{H}^0(\Omega)$. We introduce the following function spaces :

$$\begin{aligned} C_n^\infty(\Omega) &= \{\psi \in C^\infty(\Omega); \frac{\partial\psi}{\partial\mathbf{n}} = 0 \text{ on } \partial\Omega\}, \\ \mathbf{H}_\sigma(\Omega) &= \{\mathbf{A} \in \mathbf{L}^2(\Omega); \operatorname{div}\mathbf{A} = 0 \text{ in } \Omega, \mathbf{A} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbf{H}_\sigma^1(\Omega) &= \{\mathbf{A} \in \mathbf{H}^1(\Omega); \operatorname{div}\mathbf{A} = 0 \text{ in } \Omega, \\ &\quad \operatorname{rot}\mathbf{A} \times \mathbf{n} = 0, \mathbf{A} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ C_\sigma^\infty(\Omega) &= \{\mathbf{A} \in \{C_0^\infty(\Omega)\}^d; \operatorname{div}\mathbf{A} = 0 \text{ in } \Omega, \}. \end{aligned}$$

Clearly, $C_n^\infty(\Omega)$ is dense in $L^2(\Omega)$ and $C_\sigma^\infty(\Omega)$ is dense in $\mathbf{H}_\sigma(\Omega)$ (See [11]).

The scalar products and the norms of $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ are denoted by

$$\begin{aligned} (\psi, \varphi)_{L^2} &= \int_{\Omega} \psi(x) \overline{\varphi(x)} dx, & \|\psi\|_{L^2} &= (\psi, \psi)_{L^2}^{1/2}, \\ (\mathbf{A}, \mathbf{B})_{\mathbf{L}^2} &= \int_{\Omega} \mathbf{A}(x) \cdot \mathbf{B}(x) dx, & \|\mathbf{A}\|_{\mathbf{L}^2} &= (\mathbf{A}, \mathbf{A})_{\mathbf{L}^2}^{1/2}. \end{aligned}$$

Also, the norms of $L^\infty(\Omega)$ and $\mathbf{L}^\infty(\Omega)$ are denoted by

$$\|\psi\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \Omega} |\psi(x)|, \quad \|\mathbf{A}\|_{\mathbf{L}^\infty} = \operatorname{ess\,sup}_{x \in \Omega} |\mathbf{A}(x)|.$$

$\mathcal{L}^2(\Omega)$ is defined by $L^2(\Omega) \oplus \mathbf{H}_\sigma(\Omega)$ where $\oplus$ implies direct product. Clearly $\mathcal{L}^2(\Omega)$ becomes a Hilbert space. The scalar product and the norm of $\mathcal{L}^2(\Omega)$ are defined by

$$\begin{aligned} (U, V)_{\mathcal{L}^2} &= \int_{\Omega} \psi(x) \overline{\varphi(x)} dx + \int_{\Omega} \mathbf{A}(x) \cdot \mathbf{B}(x) dx, \\ \|U\|_{\mathcal{L}^2}^2 &= \|\psi\|_{L^2}^2 + \|\mathbf{A}\|_{\mathbf{L}^2}^2 = \|\psi\|_{L^2}^2 + \|\mathbf{A}\|_{\mathbf{H}_\sigma}^2, \end{aligned}$$

where $U = (\psi, \mathbf{A})^t$ and $V = (\varphi, \mathbf{B})^t$. For all $1 \leq p < \infty$, $\mathcal{L}^p(\Omega)$ is defined by $L^p(\Omega) \oplus \mathbf{L}^p(\Omega)$. We denote the norm in $\mathcal{L}^p(\Omega)$ by

$$\|U\|_{\mathcal{L}^p}^p = \|\psi\|_{L^p}^p + \|\mathbf{A}\|_{\mathbf{L}^p}^p. \quad (11)$$

Analogously, $\mathcal{H}^m(\Omega)$, $\mathcal{W}^{m,p}(\Omega)$ and $\mathcal{L}^\infty(\Omega)$ are defined by $H^m(\Omega) \oplus \mathbf{H}^m(\Omega)$, $W^{m,p}(\Omega) \oplus \mathbf{W}^{m,p}(\Omega)$ and $L^\infty(\Omega) \oplus \mathbf{L}^\infty(\Omega)$, respectively. We introduce

$$\mathcal{H}_\sigma^\alpha(\Omega) = H^\alpha(\Omega) \oplus (\mathbf{H}^\alpha(\Omega) \cap \mathbf{H}_\sigma^1(\Omega)),$$

where $\alpha \geq 1$.

Since $\operatorname{div} \mathbf{A} = 0$, the TDGL equations (1)-(2) are rewritten as

$$\psi_t = \Delta \psi - 2i(\mathbf{A} \cdot \nabla \psi) - |\mathbf{A}|^2 \psi + \kappa(1 - |\psi|^2)\psi + i\Phi \psi, \quad (12)$$

$$\mathbf{A}_t = \Delta \mathbf{A} - P\left[\frac{i}{2}\{\bar{\psi}(\nabla \psi) - \psi(\overline{\nabla \psi})\} + |\psi|^2 \mathbf{A}\right], \quad (13)$$

$$\Delta \Phi = \operatorname{div} \left\{ \frac{i}{2} \{ \bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} \right\}, \quad (14)$$

where P is the orthogonal projection $P : \mathbf{L}^2(\Omega) \rightarrow \mathbf{H}_\sigma(\Omega)$.

First, we discuss (14). We recall the Gagliardo-Nirenberg inequality

$$\|\partial^j f\|_{L^p} \leq C \|\partial^k f\|_{L^q}^s \|f\|_{L^r}^{1-s}, \quad (15)$$

where $\frac{1}{p} - \frac{j}{d} = s(\frac{1}{q} - \frac{k}{d}) + (1-s)\frac{1}{r}$, $j/k \leq s \leq 1$ (if $k - j - \frac{d}{q}$ is a non-negative integer, only $s < 1$ is allowed.) where d is a number of dimension. Hereafter, C denotes various positive constants which may change from line to line. Calculating the right-hand side of (14), we have

$$\operatorname{div} \left\{ \frac{i}{2} \{ \bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} \right\} = \frac{i}{2} \{ \bar{\psi} \Delta \psi - \psi \overline{\Delta \psi} - 2i(\nabla |\psi|^2) \mathbf{A} \}. \quad (16)$$

Using the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} & \left\| \operatorname{div} \left\{ \frac{i}{2} \{ \bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} \right\} \right\|_{L^2} \\ & \leq C \{ \|\psi\|_{L^\infty} \|\Delta \psi\|_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^4} \|\psi\|_{L^\infty} \|\nabla \psi\|_{L^4} \} \\ & \leq C \left\{ \|\psi\|_{H^2}^{7/4} \|\psi\|_{L^2}^{1/4} + \|\mathbf{A}\|_{\mathbf{L}^4} \|\psi\|_{H^2}^{13/8} \|\psi\|_{L^2}^{3/8} \right\}, \quad \text{for } d = 3, \end{aligned} \quad (17)$$

$$\begin{aligned} & \left\| \operatorname{div} \left\{ \frac{i}{2} \{ \bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} \right\} \right\|_{L^2} \\ & \leq C \left\{ \|\psi\|_{H^2}^{3/2} \|\psi\|_{L^2}^{1/2} + \|\mathbf{A}\|_{\mathbf{L}^3} \|\psi\|_{H^2}^{4/3} \|\psi\|_{L^2}^{2/3} \right\}, \quad \text{for } d = 2, \end{aligned} \quad (18)$$

where C is a constant depending on Ω . By the standard elliptic theorems, we have

Lemma 1 *We suppose that $(\psi, \mathbf{A}) \in \mathcal{H}_\sigma^2(\Omega)$ is given. Then, the boundary value problem*

$$\Delta \Phi = \operatorname{div} \left\{ \frac{i}{2} \{ \bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} \right\}, \quad \text{in } \Omega \quad (19)$$

$$\frac{\partial \Phi}{\partial n} = 0, \quad \text{on } \partial \Omega \quad (20)$$

$$\int_\Omega \Phi(x) dx = 0, \quad (21)$$

has a unique solution $\Phi(\psi, \mathbf{A}) \in H^2(\Omega)$.

We employ the semi-group theory. (12) and (13) with the initial conditions (8), (9) and the boundary conditions (4)-(7) are expressed as an abstract evolution equation of the form

$$\frac{d}{dt}U + \mathcal{A}U = \mathcal{B}(U), \quad (22)$$

$$U(0) = U_0 = (\psi_0, \mathbf{A}_0)^t, \quad (23)$$

where $U = (\psi, \mathbf{A})^t$ is a pair of functions, $\mathcal{A} = \begin{pmatrix} -\Delta + \beta I & 0 \\ 0 & -\Delta + \beta I \end{pmatrix}$, $D(\mathcal{A}) \equiv \mathcal{H}_\sigma^2(\Omega)$ and

$$\mathcal{B}(U) = \begin{pmatrix} (\beta + \kappa)\psi - 2i(\mathbf{A} \cdot \nabla \psi) - |\mathbf{A}|^2\psi - \kappa|\psi|^2\psi + i\Phi(\psi, \mathbf{A})\psi \\ \beta\mathbf{A} - P \left[\frac{i}{2} \left\{ \overline{\psi}(\nabla \psi) - \psi(\overline{\nabla \psi}) \right\} + |\psi|^2\mathbf{A} \right] \end{pmatrix},$$

where $D(\mathcal{B}) \supset D(\mathcal{A}^\alpha)$ ($\alpha \geq \frac{5}{8}$ ($d = 3$), $\alpha \geq \frac{1}{2}$ ($d = 2$)) (See below proposition 2, proposition 3).

If $-\mathcal{A}$ is invertible and the infinitesimal generator of an analytic semigroup, $\mathcal{A}^\alpha$ can be defined for $0 \leq \alpha \leq 1$ so that $\mathcal{A}^\alpha$ is a closed linear invertible operator with domain $D(\mathcal{A}^\alpha)$ dense in X . The closedness of $\mathcal{A}^\alpha$ implies that $D(\mathcal{A}^\alpha)$ endowed with the graph norm of $\mathcal{A}^\alpha$, (i.e. the norm $\|U\| = \|U\|_X + \|\mathcal{A}^\alpha U\|_X$), is a Banach space. Since $\mathcal{A}^\alpha$ is invertible, its graph norm $\|\cdot\|$ is equivalent to the norm $\|U\|_\alpha = \|\mathcal{A}^\alpha U\|_X$. Thus, $D(\mathcal{A}^\alpha)$ equipped with the norm $\|\cdot\|_\alpha$ is a Banach space, which we denote by X_α . From this definition it is clear that $0 < \alpha < \beta$ implies $X_\alpha \supset X_\beta$ and that the imbedding of X_β into X_α is continuous. The following theorem is well-known.

Theorem 1 (H. Fujita and T. Kato [5]) *We consider that*

$$\frac{d}{dt}u(t) + \mathcal{A}u(t) = B(u) \quad (t > 0), \quad (24)$$

$$u(0) = u_0. \quad (25)$$

Let $-\mathcal{A}$ be the infinitesimal generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ and assume further that $0 \in \rho(-\mathcal{A})$. Let $\mathcal{U}$ be an open subset of X_α . Let the function $B : \mathcal{U} \rightarrow X$ satisfy that for every $u \in \mathcal{U}$, there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ and constant $L > 0$, such that

$$\|B(u_1) - B(u_2)\|_X \leq L\|u_1 - u_2\|_\alpha, \quad \text{for all } u_i \in \mathcal{V} \ (i = 1, 2).$$

Then, for every initial data $u_0 \in \mathcal{U}$, the initial value problem (24), (25) has a unique local solution $u \in C([0, T_0) : X) \cap C((0, T_0) : D(\mathcal{A})) \cap C^1((0, T_0) : X)$ where $T_0 = T_0(\|u_0\|_\alpha) > 0$.

Applying Theorem 1, we have our main results.

Theorem 2 (Existence of local strong solutions.) *Let $U_0 \in D(\mathcal{A}^\alpha)$ with $\alpha \geq \frac{5}{8}$ ($d = 3$), $\frac{1}{2}$ ($d = 2$). The initial value problem of the TDGL equations (12)-(14) with the conditions (3)-(10), has a unique local strong solution $U \in C([0, T_0) : \mathcal{L}^2(\Omega)) \cap C((0, T_0) : D(\mathcal{A})) \cap C^1((0, T_0) : \mathcal{L}^2(\Omega))$ where $T_0 = T_0(\|U_0\|_\alpha) > 0$.*

For $d = 2$, we have next theorem.

Theorem 3 (Global strong solution) *If $d = 2$, then the solution in Theorem 2 is global. i.e. we may take $T_0 = \infty$.*

2 Proof of Theorem 2.

In order to prove Theorem 2, we prepare three propositions.

Proposition 1 *$-\mathcal{A}$ is the infinitesimal generator of C_0 semigroup of contraction on $\mathcal{L}^2(\Omega)$.*

Proof of Proposition 1. Since $\operatorname{div} \mathbf{A} = 0$, we have

$$|\nabla \mathbf{A}|^2 - |\operatorname{rot} \mathbf{A}|^2 = \operatorname{div} \{(\mathbf{A} \cdot \nabla) \mathbf{A}\}. \quad (26)$$

Therefore,

$$\int_{\Omega} |\nabla \mathbf{A}|^2 dx = \int_{\Omega} |\operatorname{rot} \mathbf{A}|^2 dx + \int_{\partial\Omega} \{(\mathbf{A} \cdot \nabla) \mathbf{A}\} \cdot \mathbf{n} dS. \quad (27)$$

We estimate the second term on the right-hand side of (27) as

$$\begin{aligned} \left| \sum_{j,k=1}^d \int_{\partial\Omega} A_j \frac{\partial A_k}{\partial x_j} \cdot \mathbf{n}_k dS \right| &= \left| \sum_{j,k=1}^d \int_{\partial\Omega} A_j A_k \frac{\partial \mathbf{n}_k}{\partial x_j} dS \right| \leq C \int_{\partial\Omega} |\mathbf{A}|^2 dS \\ &\leq \epsilon \|\nabla \mathbf{A}\|_{\mathbf{L}^2}^2 + C_\epsilon \|\mathbf{A}\|_{\mathbf{L}^2}^2, \end{aligned} \quad (28)$$

for any $\epsilon (<< 1)$. Here $\mathbf{A} = (A_1, A_2, A_3)$ ($d = 3$), $\mathbf{A} = (A_1, A_2)$ ($d = 2$) and $C_\epsilon (\geq \epsilon)$ is a positive constant depending on ϵ . Since (27) and (28), we have

$$(1 - \epsilon) \|\mathbf{A}\|_{\mathbf{H}^1}^2 \leq \int_{\Omega} |\operatorname{rot} \mathbf{A}|^2 dx + \beta \|\mathbf{A}\|_{\mathbf{L}^2}^2, \quad (29)$$

where $\beta = 1 - \epsilon + C_\epsilon (\geq 1)$.

We next consider the operator $\mathcal{A}$. Clearly $\mathcal{A}$ is strongly elliptic. Since $\Delta =$

$-\text{rot}^2 + \text{grad div}$, using (4)-(6) and (29), we have for $U(= (\psi, \mathbf{A})^t) \in C_n^\infty(\Omega) \oplus C_\sigma^\infty(\Omega)$,

$$\begin{aligned}
& (-\mathcal{A}U, U)_{\mathcal{L}^2} \\
&= \int_{\Omega} \Delta \psi(x) \overline{\psi(x)} dx + \int_{\Omega} \Delta \mathbf{A}(x) \cdot \mathbf{A}(x) dx - \beta(\psi, \psi)_{L^2} - \beta(\mathbf{A}, \mathbf{A})_{\mathbf{L}^2} \\
&= - \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi(x)} dx - \int_{\Omega} \text{rot} \mathbf{A}(x) \cdot \text{rot} \mathbf{A}(x) dx - \beta(\psi, \psi)_{L^2} - \beta(\mathbf{A}, \mathbf{A})_{\mathbf{L}^2} \\
&\leq -\|\psi\|_{H^1}^2 + \|\psi\|_{L^2}^2 - (1 - \epsilon)\|\mathbf{A}\|_{\mathbf{H}^1}^2 - \beta\|\psi\|_{L^2}^2 \\
&\leq -(1 - \epsilon)\|U\|_{\mathcal{H}^1}^2 \\
&\leq 0.
\end{aligned} \tag{30}$$

which holds valid for every $U \in \mathcal{H}_\sigma^2(\Omega)$. Therefore, $-\mathcal{A}$ is dissipative.

We define a continuous sesquilinear form on $\mathcal{H}_\sigma^1(\Omega) \times \mathcal{H}_\sigma^1(\Omega)$ by

$$a(U, V) = \int_{\Omega} \nabla \psi \cdot \overline{\nabla \varphi} dx + \int_{\Omega} \text{rot} \mathbf{A} \cdot \text{rot} \mathbf{B} dx + \beta(\psi, \varphi)_{L^2} + \beta(\mathbf{A}, \mathbf{B})_{\mathbf{L}^2} + \lambda(U, V)_{\mathcal{L}^2} \tag{31}$$

where $U = (\psi, \mathbf{A})^t$, $V = (\varphi, \mathbf{B})^t$ and λ is a complex number. We have

$$\begin{aligned}
& |a(U, V)| \\
&\leq \left\{ \int_{\Omega} |\nabla \psi|^2 dx \right\}^{1/2} \left\{ \int_{\Omega} |\nabla \varphi|^2 dx \right\}^{1/2} + \left\{ \int_{\Omega} |\text{rot} \mathbf{A}|^2 dx \right\}^{1/2} \left\{ \int_{\Omega} |\text{rot} \mathbf{B}|^2 dx \right\}^{1/2} \\
&\quad + \beta\|\psi\|_{L^2} \|\varphi\|_{L^2} + \beta\|\mathbf{A}\|_{\mathbf{L}^2} \|\mathbf{B}\|_{\mathbf{L}^2} + |\lambda| \|U\|_{\mathcal{L}^2} \|V\|_{\mathcal{L}^2} \\
&\leq (C_\beta + |\lambda|) \|U\|_{\mathcal{H}^1} \|V\|_{\mathcal{H}^1}
\end{aligned} \tag{32}$$

where $C_\beta = \max\{2, \beta\}$. For $0 \leq |\lambda| < 1 - \epsilon$, we obtain

$$\begin{aligned}
|a(U, U)| &\geq \left| \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi(x)} dx + \int_{\Omega} \text{rot} \mathbf{A}(x) \cdot \text{rot} \mathbf{A}(x) dx \right. \\
&\quad \left. + \beta(\psi, \psi)_{L^2} + \beta(\mathbf{A}, \mathbf{A})_{\mathbf{L}^2} \right| - |\lambda| \|U\|_{\mathcal{L}^2}^2 \\
&\geq (1 - \epsilon - |\lambda|) \|U\|_{\mathcal{H}^1}^2.
\end{aligned} \tag{33}$$

If $0 \leq |\lambda| < 1 - \epsilon$, then $a(U, V)$ is coercive. We may apply the classical Lax-Milgram theorem to obtain a unique weak solution $U \in \mathcal{H}_\sigma^1(\Omega)$ to the boundary value problem

$$\mathcal{A}U + \lambda U = f \tag{34}$$

for all $f \in \mathcal{L}^2(\Omega)$ and $0 \leq |\lambda| < 1 - \epsilon$. Especially, this solution belongs to $H^2(\Omega) \oplus \mathbf{H}^2(\Omega)$. This implies that the range of $\lambda I + \mathcal{A}$ is $\mathcal{L}^2(\Omega)$ for all $0 < |\lambda| < 1 - \epsilon$.

Clearly, $D(-\mathcal{A}) \supset C_n^\infty(\Omega) \oplus C_\sigma^\infty(\Omega)$. Since $C_n^\infty(\Omega) \oplus C_\sigma^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$, it follows that $D(-\mathcal{A})$ is dense in $\mathcal{L}^2(\Omega)$.

From the Lumer-Phillips theorem, it follows that $-\mathcal{A}$ is the infinitesimal generator of C_0 semigroup of contraction on $\mathcal{L}^2(\Omega)$. Q.E.D.

Since $\mathcal{A}$ is self-adjoint and non-negative, operator $-\mathcal{A}$ is infinitesimal of an analytic semigroup on $\mathcal{L}^2(\Omega)$. Therefore we may define fractional powers of $\mathcal{A}$. For $0 < \alpha < 1$, we define the path of integration $\mathcal{C}$ into the upper and lower sides of negative real axis and obtain

$$\mathcal{A}^{-\alpha}U \equiv \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{-\alpha} (tI + \mathcal{A})^{-1} U \, dt, \quad 0 < \alpha < 1. \quad (35)$$

If $U \in D(\mathcal{A})$, then we define

$$\mathcal{A}^\alpha U \equiv \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{\alpha-1} \mathcal{A}(tI + \mathcal{A})^{-1} U \, dt, \quad 0 < \alpha < 1. \quad (36)$$

Using (30) and Schwarz's inequality, we have

$$\begin{aligned} \|\mathcal{A}U\|_{\mathcal{L}^2} \|U\|_{\mathcal{L}^2} &\geq (\mathcal{A}U, U)_{\mathcal{L}^2} \\ &\geq (1 - \epsilon) \|U\|_{\mathcal{H}^1}^2 \geq (1 - \epsilon) \|U\|_{\mathcal{L}^2}^2 \\ &\geq \delta \|U\|_{\mathcal{L}^2}^2, \end{aligned} \quad (37)$$

for some $0 < \delta \leq 1 - \epsilon$. Therefore,

$$\|\mathcal{A}^{-1}U\|_{\mathcal{L}^2} \leq \delta^{-1} \|U\|_{\mathcal{L}^2}. \quad (38)$$

$D(\mathcal{A}^\alpha)$ endowed with the graph norm of $\mathcal{A}^\alpha$, is a Banach space. Since $\mathcal{A}^\alpha$ is invertible, its graph norm $\|\cdot\|$ is equivalent to the norm $\|U\|_\alpha = \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}$. Thus, $D(\mathcal{A}^\alpha)$ equipped with the norm $\|\cdot\|_\alpha$ is a Banach space, which we also denote by $D(\mathcal{A}^\alpha)$. It is clearly that $D(\mathcal{A}^\alpha) \simeq \mathcal{H}_\sigma^{2\alpha}(\Omega)$ and

$$\|U\|_{\mathcal{H}_\sigma^{2\alpha}} \leq C \|U\|_\alpha = C \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad (39)$$

(See [12]). When $d = 3$, using Sobolev's inequality and Hölder's inequality, we have

$$\|U\|_{\mathcal{W}^{1,12/5}} \leq C \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha \geq \frac{5}{8}, \quad (40)$$

$$\|U\|_{\mathcal{L}^p} \leq C \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha \geq \frac{5}{8}, \quad (41)$$

for any $1 \leq p \leq 12$. When $d = 2$, we have

$$\|U\|_{\mathcal{L}^p} \leq C \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha \geq \frac{1}{2}, \quad (42)$$

$$\|U\|_{\mathcal{H}^1} \leq C \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha \geq \frac{1}{2} \quad (43)$$

for any $1 \leq p \leq +\infty$.

We now turn to the nonlinear term of (22). We have

Proposition 2 *If $d = 3$, $\alpha \geq \frac{5}{8}$ and $U \in D(\mathcal{A})$, then $\mathcal{B}(U)$ is well-defined and*

$$\|\mathcal{B}(U)\|_{\mathcal{L}^2} \leq C \left\{ \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^2 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^3 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^4 \right\}, \quad (44)$$

where C is a constant depending on β , κ , and Ω . Moreover, if $U, V \in D(\mathcal{A})$, then

$$\|\mathcal{B}(U) - \mathcal{B}(V)\|_{\mathcal{L}^2} \leq C \|\mathcal{A}^\alpha U - \mathcal{A}^\alpha V\|_{\mathcal{L}^2}, \quad (45)$$

where C is a constant depending on β , κ , $\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}$, $\|\mathcal{A}^\alpha V\|_{\mathcal{L}^2}$ and Ω .

Proof of Proposition 2. We note that $D(\mathcal{A}) \subset \mathcal{H}^2(\Omega)$. From Sobolev's theorem, it follows that $U \in \mathcal{L}^\infty(\Omega)$. Therefore, $\mathcal{B}(U) \in \mathcal{L}^2(\Omega)$ and is well-defined. Let $U = (\psi, \mathbf{A})^t$ and $V = (\varphi, \mathbf{B})^t$. Note that, from Lemma 1, Φ is uniquely defined by $(\psi, \mathbf{A})$. Let

$$\Phi_1 = \Phi(\psi, \mathbf{A}), \quad \Phi_2 = \Phi(\varphi, \mathbf{B}).$$

From (14), we have

$$\begin{aligned} \|\nabla \Phi_1\|_{\mathbf{L}^2}^2 &= - \int_{\Omega} \Phi_1 \operatorname{div} \left\{ \frac{i}{2} \{ \bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} \right\} dx \\ &= \int_{\Omega} \nabla \Phi_1 \cdot \left\{ \frac{i}{2} \{ \bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \} \right\} dx \\ &\leq \|\nabla \Phi_1\|_{\mathbf{L}^2} \|\bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi})\|_{\mathbf{L}^2}. \end{aligned}$$

Therefore,

$$\|\nabla \Phi_1\|_{\mathbf{L}^2} \leq \|\bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi})\|_{\mathbf{L}^2}, \quad (46)$$

$$\|\Phi_1\|_{L^2} \leq C \|\bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi})\|_{\mathbf{L}^2}. \quad (47)$$

From Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned} \|\Phi_1\|_{L^3} &\leq \|\nabla \Phi_1\|_{\mathbf{L}^2}^{1/2} \|\Phi_1\|_{L^2}^{1/2} \leq C \|\bar{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi})\|_{\mathbf{L}^2} \\ &\leq C \left\{ 2\|\psi\|_{L^{12}} \|\psi\|_{W^{1,12/5}} + \|\psi\|_{L^6}^2 \|\mathbf{A}\|_{\mathbf{L}^6} \right\}. \end{aligned} \quad (48)$$

We now estimate

$$\begin{aligned} \|\mathcal{B}(U)\|_{\mathcal{L}^2} &\leq C \left\{ \|\psi\|_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^2} + \|\mathbf{A} \cdot \nabla \psi\|_{L^2} + \| |\mathbf{A}|^2 \psi \|_{L^2} + \| |\psi|^2 \psi \|_{L^2} \right. \\ &\quad \left. + \|\Phi_1 \psi\|_{L^2} + 2\|\psi \nabla \bar{\psi}\|_{\mathbf{L}^2} + \| |\psi|^2 \mathbf{A} \|_{\mathbf{L}^2} \right\}. \end{aligned} \quad (49)$$

Using (40) and (41), we have

$$\begin{aligned}
\|\mathcal{B}(U)\|_{\mathcal{L}^2} &\leq C \left\{ \|\psi\|_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^2} + \|\mathbf{A}\|_{\mathbf{L}^{12}} \|\psi\|_{W^{1,12/5}} + \|\mathbf{A}\|_{\mathbf{L}^6}^2 \|\psi\|_{L^6} \right. \\
&\quad + \|\psi\|_{L^6}^3 + (2\|\psi\|_{L^{12}} \|\psi\|_{W^{1,12/5}} + \|\psi\|_{L^6}^2 \|\mathbf{A}\|_{\mathbf{L}^6}) \|\psi\|_{L^6} \\
&\quad \left. + 2\|\psi\|_{L^{12}} \|\psi\|_{W^{1,12/5}} + \|\psi\|_{L^6}^2 \|\mathbf{A}\|_{\mathbf{L}^6} \right\} \\
&\leq C \left\{ \|U\|_{\mathcal{L}^2} + \|U\|_{\mathcal{L}^2} + \|U\|_{\mathcal{L}^{12}} \|U\|_{W^{1,12/5}} + \|U\|_{\mathcal{L}^6}^2 \|U\|_{\mathcal{L}^6} \right. \\
&\quad + \|U\|_{\mathcal{L}^6}^3 + (2\|U\|_{\mathcal{L}^{12}} \|U\|_{W^{1,12/5}} + \|U\|_{\mathcal{L}^6}^3) \|U\|_{\mathcal{L}^6} \\
&\quad \left. + 2\|U\|_{\mathcal{L}^{12}} \|U\|_{W^{1,12/5}} + \|U\|_{\mathcal{L}^6}^3 \right\} \\
&\leq C' \left\{ \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^2 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^3 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^4 \right\}, \tag{50}
\end{aligned}$$

for all $\alpha \geq \frac{5}{8}$. Thus we have the first assertion.

Next, we show (45). Suppose that

$$\begin{aligned}
\|\mathcal{B}(U) - \mathcal{B}(V)\|_{\mathcal{L}^2} &\leq C \left\{ \|\psi - \varphi\|_{L^2} + \|\mathbf{A} - \mathbf{B}\|_{\mathbf{L}^2} \right. \\
&\quad + \|\mathbf{A} \cdot \nabla \psi - \mathbf{B} \cdot \nabla \varphi\|_{L^2} + \| |\mathbf{A}|^2 \psi - |\mathbf{B}|^2 \varphi \|_{L^2} \\
&\quad + \| |\varphi|^2 \varphi - |\psi|^2 \psi \|_{L^2} + \|\Phi_1 \psi - \Phi_2 \varphi\|_{L^2} \\
&\quad \left. + 2\|\psi(\overline{\nabla \psi}) - \varphi(\overline{\nabla \varphi})\|_{\mathbf{L}^2} + \| |\psi|^2 \mathbf{A} - |\varphi|^2 \mathbf{B} \|_{\mathbf{L}^2} \right\} \\
&= C \left\{ I_0 + I_1 + I_2 + I_3 + I_4 + 2I_5 + I_6 \right\}, \tag{51}
\end{aligned}$$

where C is a constant depending on β and λ ,

$$\begin{aligned}
I_0 &= \|\psi - \varphi\|_{L^2} + \|\mathbf{A} - \mathbf{B}\|_{\mathbf{L}^2}, \\
I_1 &= \|\mathbf{A} \cdot \nabla \psi - \mathbf{B} \cdot \nabla \varphi\|_{L^2}, \\
I_2 &= \| |\mathbf{A}|^2 \psi - |\mathbf{B}|^2 \varphi \|_{L^2}, \\
I_3 &= \| |\varphi|^2 \varphi - |\psi|^2 \psi \|_{L^2}, \\
I_4 &= \|\Phi_1 \psi - \Phi_2 \varphi\|_{L^2}, \\
I_5 &= \|\psi(\overline{\nabla \psi}) - \varphi(\overline{\nabla \varphi})\|_{\mathbf{L}^2}, \\
I_6 &= \| |\psi|^2 \mathbf{A} - |\varphi|^2 \mathbf{B} \|_{\mathbf{L}^2}.
\end{aligned}$$

I_0 is estimated as follows:

$$I_0 = \|U - V\|_{\mathcal{L}^2} \leq \|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \quad \text{for all } \gamma \geq \frac{5}{8}. \tag{52}$$

I_1 and I_5 are estimated as follows:

$$\begin{aligned}
I_1 &\leq \| \mathbf{A} \cdot \nabla \psi - \mathbf{A} \cdot \nabla \varphi \|_{L^2} + \| \mathbf{A} \cdot \nabla \varphi - \mathbf{B} \cdot \nabla \varphi \|_{L^2} \\
&= \| \mathbf{A} \cdot (\nabla \psi - \nabla \varphi) \|_{L^2} + \| (\mathbf{A} - \mathbf{B}) \cdot \nabla \varphi \|_{L^2} \\
&\leq \| \mathbf{A} \|_{\mathbf{L}^{12}} \| \nabla(\psi - \varphi) \|_{\mathbf{L}^{12/5}} + \| \mathbf{A} - \mathbf{B} \|_{\mathbf{L}^{12}} \| \nabla \varphi \|_{\mathbf{L}^{12/5}} \\
&\leq \| U \|_{\mathcal{L}^{12}} \| \nabla(U - V) \|_{\mathcal{L}^{12/5}} + \| U - V \|_{\mathcal{L}^{12}} \| \nabla V \|_{\mathcal{L}^{12/5}}, \\
&\leq \| U \|_{\mathcal{L}^{12}} \| U - V \|_{\mathcal{W}^{1,12/5}} + \| U - V \|_{\mathcal{L}^{12}} \| V \|_{\mathcal{W}^{1,12/5}}.
\end{aligned} \tag{53}$$

Using (40) and (41), we have

$$\begin{aligned}
I_1 &\leq C \left\{ \| \mathcal{A}^\gamma U \|_{\mathcal{L}^2} \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2} + \| \mathcal{A}^\gamma V \|_{\mathcal{L}^2} \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2} \right\} \\
&\leq C_1 \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}, \quad \text{for all } \gamma \geq \frac{5}{8}.
\end{aligned} \tag{54}$$

$$\begin{aligned}
I_5 &\leq \| \psi(\overline{\nabla \psi}) - \psi(\overline{\nabla \varphi}) \|_{\mathbf{L}^2} + \| \psi(\overline{\nabla \varphi}) - \varphi(\overline{\nabla \varphi}) \|_{\mathbf{L}^2} \\
&\leq \| \psi \|_{L^{12}} \| \overline{\nabla(\psi - \varphi)} \|_{\mathbf{L}^{12/5}} + \| \overline{\nabla \varphi} \|_{\mathbf{L}^{12/5}} \| \psi - \varphi \|_{L^{12}} \\
&\leq \| U \|_{\mathcal{L}^{12}} \| \overline{\nabla(U - V)} \|_{\mathcal{L}^{12/5}} + \| \overline{\nabla U} \|_{\mathcal{L}^{12/5}} \| U - V \|_{\mathcal{L}^{12}} \\
&\leq \| U \|_{\mathcal{L}^{12}} \| \overline{U - V} \|_{\mathcal{W}^{1,12/5}} + \| \overline{U} \|_{\mathcal{W}^{1,12/5}} \| U - V \|_{\mathcal{L}^{12}} \\
&\leq C \left\{ \| \mathcal{A}^\gamma U \|_{\mathcal{L}^2} \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2} + \| \mathcal{A}^\gamma V \|_{\mathcal{L}^2} \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2} \right\} \\
&\leq C_5 \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}, \quad \text{for all } \gamma \geq \frac{5}{8}.
\end{aligned} \tag{55}$$

Using (41), we obtain the following estimates of I_2 and I_6 :

$$\begin{aligned}
I_2 &\leq \| |\mathbf{A}|^2 \psi - |\mathbf{B}|^2 \psi \|_{L^2} + \| |\mathbf{B}|^2 \psi - |\mathbf{B}|^2 \varphi \|_{L^2} \\
&\leq \| |\mathbf{A}|^2 - |\mathbf{B}|^2 \|_{\mathbf{L}^3} \| \psi \|_{L^6} + \| |\mathbf{B}|^2 \|_{\mathbf{L}^3} \| \psi - \varphi \|_{L^6} \\
&\leq (\| \mathbf{A} \|_{\mathbf{L}^6} + \| \mathbf{B} \|_{\mathbf{L}^6}) \| \mathbf{A} - \mathbf{B} \|_{\mathbf{L}^6} \| \psi \|_{L^6} + \| \mathbf{B} \|_{\mathbf{L}^6}^2 \| \psi - \varphi \|_{L^6} \\
&\leq C \| U - V \|_{\mathcal{L}^6} \leq C_2 \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}, \quad \text{for all } \gamma \geq \frac{5}{8},
\end{aligned} \tag{56}$$

$$\begin{aligned}
I_6 &\leq \| (|\psi|^2 - |\varphi|^2) \mathbf{A} \|_{\mathbf{L}^2} + \| |\varphi|^2 (\mathbf{A} - \mathbf{B}) \|_{\mathbf{L}^2} \\
&\leq \| |\psi|^2 - |\varphi|^2 \|_{L^3} \| \mathbf{A} \|_{\mathbf{L}^6} + \| |\varphi|^2 \|_{L^3} \| \mathbf{A} - \mathbf{B} \|_{\mathbf{L}^6} \\
&\leq \| |\psi| - |\varphi| \|_{L^6} \| |\psi| + |\varphi| \|_{L^6} \| \mathbf{A} \|_{\mathbf{L}^6} + \| \varphi \|_{L^6}^2 \| \mathbf{A} - \mathbf{B} \|_{\mathbf{L}^6} \\
&\leq (\| \psi \|_{L^6} + \| \varphi \|_{L^6}) \| \psi - \varphi \|_{L^6} \| \mathbf{A} \|_{\mathbf{L}^6} + \| \varphi \|_{L^6}^2 \| \mathbf{A} - \mathbf{B} \|_{\mathbf{L}^6} \\
&\leq C \| U - V \|_{\mathcal{L}^6} \leq C_6 \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}, \quad \text{for all } \gamma \geq \frac{5}{8}.
\end{aligned} \tag{57}$$

Next, we estimate I_3 as follows:

$$\begin{aligned}
I_3 &\leq \| |\varphi|^2 \varphi - |\varphi|^2 \psi \|_{L^2} + \| |\varphi|^2 \psi - |\psi|^2 \psi \|_{L^2} \\
&\leq \| \varphi \|_{L^6}^2 \| \varphi - \psi \|_{L^6} + \| \psi \|_{L^6} \| |\varphi| - |\psi| \|_{L^6} \| |\varphi| + |\psi| \|_{L^6} \\
&\leq \| \varphi \|_{L^6}^2 \| \varphi - \psi \|_{L^6} + \| \psi \|_{L^6} \| \varphi - \psi \|_{L^6} (\| \varphi \|_{L^6} + \| \psi \|_{L^6}) \\
&\leq C \| U - V \|_{\mathcal{L}^6} \leq C_6 \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}, \quad \text{for all } \gamma \geq \frac{5}{8}.
\end{aligned} \tag{58}$$

Finally, we estimate I_4 . From Poincaré's inequality, we have

$$\| \Phi_1 - \Phi_2 \|_{L^2} \leq C' \| \nabla(\Phi_1 - \Phi_2) \|_{\mathbf{L}^2}, \tag{59}$$

where C' is a constant depending on Ω . Suppose that

$$f = \frac{i}{2} \left\{ \overline{\psi}(\mathbf{D}_A \psi) - \psi(\overline{\mathbf{D}_A \psi}) \right\} \tag{60}$$

$$g = \frac{i}{2} \left\{ \overline{\varphi}(\mathbf{D}_B \varphi) - \varphi(\overline{\mathbf{D}_B \varphi}) \right\} \tag{61}$$

From (14), we have

$$\Delta(\Phi_1 - \Phi_2) = \operatorname{div}(f - g). \tag{62}$$

Now,

$$\begin{aligned}
\| \nabla(\Phi_1 - \Phi_2) \|_{\mathbf{L}^2}^2 &= - \int_{\Omega} (\Phi_1 - \Phi_2) \operatorname{div}(f - g) \, dx \\
&= \int_{\Omega} (\nabla(\Phi_1 - \Phi_2)) (f - g) \, dx \\
&\leq \| \nabla(\Phi_1 - \Phi_2) \|_{\mathbf{L}^2} \| f - g \|_{\mathbf{L}^2}
\end{aligned} \tag{63}$$

From (59), (63) and estimates of I_5 and I_6 , we have

$$\| \nabla(\Phi_1 - \Phi_2) \|_{\mathbf{L}^2} \leq \| f - g \|_{\mathbf{L}^2} \leq C_5 I_5 + C_6 I_6 \leq K \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}, \tag{64}$$

$$\| \Phi_1 - \Phi_2 \|_{\mathbf{L}^2} \leq C' \| f - g \|_{\mathbf{L}^2} \leq C' C_5 I_5 + C' C_6 I_6 \leq K' \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}. \tag{65}$$

for all $\gamma \geq 5/8$, where K and K' are constants. Therefore, from (46), (47), (64), (65) and Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned}
I_4 &\leq \| \Phi_1 \psi - \Phi_2 \psi \|_{L^2} + \| \Phi_2 \psi - \Phi_2 \varphi \|_{L^2} \\
&\leq \| \Phi_1 - \Phi_2 \|_{L^{12/5}} \| \psi \|_{L^{12}} + \| \Phi_2 \|_{L^{12/5}} \| \psi - \varphi \|_{L^{12}} \\
&\leq \| \nabla(\Phi_1 - \Phi_2) \|_{\mathbf{L}^2}^{1/4} \| \Phi_1 - \Phi_2 \|_{\mathbf{L}^2}^{3/4} \| \psi \|_{L^{12}} + \| \nabla \Phi_2 \|_{\mathbf{L}^2}^{1/4} \| \Phi_2 \|_{\mathbf{L}^2}^{3/4} \| \psi - \varphi \|_{L^{12}} \\
&\leq C \| \mathcal{A}^\gamma U - \mathcal{A}^\gamma V \|_{\mathcal{L}^2}, \quad \text{for all } \gamma \geq \frac{5}{8}.
\end{aligned} \tag{66}$$

Combining (52), (54)-(58) and (66), the proof of Proposition 3 is completed. Q.E.D.

Using (42) and (43), we have

Proposition 3 *If $d = 2$, $\alpha \geq 1/2$ and $U \in D(\mathcal{A})$, then $\mathcal{B}(U)$ is well-defined and*

$$\|\mathcal{B}(U)\|_{\mathcal{L}^2} \leq C \left\{ \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^2 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^3 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^4 \right\}, \quad (67)$$

where C is a constant depending on β , κ , and Ω . Moreover, if $U, V \in D(\mathcal{A})$ then

$$\|\mathcal{B}(U) - \mathcal{B}(V)\|_{\mathcal{L}^2} \leq C \|\mathcal{A}^\alpha U - \mathcal{A}^\alpha V\|_{\mathcal{L}^2}, \quad (68)$$

where C is a constant depending on β , κ , $\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}$, $\|\mathcal{A}^\alpha V\|_{\mathcal{L}^2}$ and Ω .

The proof is similar to those of the case of $d = 3$ and omit it.

From (44) and (67), it follows that the domain of $\mathcal{B}$ can be extended by continuity to $D(\mathcal{A}^\alpha)$ ($\alpha \geq \frac{5}{8}$ ($d = 3$), $\alpha \geq 1/2$ ($d = 2$)). Therefore, (44), (45), (67) and (68) hold for every $U, V \in D(\mathcal{A}^\alpha)$. Therefore, the condition of Theorem 1 is satisfied. We complete the proof of Theorem 2. Q.E.D.

3 Proof of Theorem 3.

For $d = 2$ and $U_0 \in D(\mathcal{A}^{1/2})$, from Theorem 2, we have a local strong solution $U \in C([0, T_0) : \mathcal{L}^2(\Omega)) \cap C((0, T_0) : D(\mathcal{A})) \cap C^1((0, T_0) : \mathcal{L}^2(\Omega))$ $T_0 = T_0(\|U_0\|_{1/2})$. In order to establish global existence of solution, it is sufficient to obtain a propri estimate

$$\sup_{0 \leq t < T} \|\mathcal{A}^{1/2} U\|_{\mathcal{L}^2} \leq C, \quad (69)$$

for any $T > 0$.

From (1)-(2) with (3)-(10), we have

$$\begin{aligned} & \|\mathbf{D}_A \psi(t)\|_{\mathbf{L}^2}^2 + \|\text{rot} \mathbf{A}(t)\|_{\mathbf{L}^2}^2 + \frac{\kappa}{2} \|1 - |\psi(t)|^2\|_{L^2}^2 \\ & + \int_0^t \|\psi_t - i\Phi\psi\|_{L^2}^2 dt + 2 \int_0^t \|\mathbf{A}_t - \nabla \Phi\|_{\mathbf{L}^2}^2 dt \\ & = \|\mathbf{D}_{A_0} \psi_0\|_{\mathbf{L}^2}^2 + \|\text{rot} \mathbf{A}\|_{\mathbf{L}^2}^2 + \frac{\kappa}{2} \|1 - |\psi_0|^2\|_{L^2}^2 \\ & \leq 2\|\nabla \psi_0\|_{\mathbf{L}^2}^2 + 2\|\mathbf{A}_0\|_{\mathbf{L}^4}^2 \|\psi_0\|_{L^4}^2 + 2\|\nabla \mathbf{A}_0\|_{\mathbf{L}^2}^2 + \kappa|\Omega| + \kappa\|\psi_0\|_{L^4}^4. \end{aligned} \quad (70)$$

Thus, we have

$$\sup_{0 \leq t < \infty} \|\mathbf{D}_A \psi\|_{\mathbf{L}^2} \leq C, \quad (71)$$

$$\sup_{0 \leq t < \infty} \|\text{rot} \mathbf{A}\|_{\mathbf{L}^2} \leq C, \quad (72)$$

$$\sup_{0 \leq t < \infty} \|1 - |\psi|^2\|_{L^2} \leq C, \quad (73)$$

$$\int_0^T \|\psi_t - i\Phi\psi\|_{L^2}^2 dt \leq C, \quad (74)$$

$$\int_0^T \|\mathbf{A}_t - \nabla\Phi\|_{\mathbf{L}^2}^2 dt \leq C, \quad (75)$$

for any $T > 0$, here C is a constant which depends only on $\|U_0\|_{\mathcal{H}^1}$ and Ω . Since $\|\nabla\mathbf{A}\|_{\mathbf{L}^2}$ and $\|\text{rot}\mathbf{A}\|_{\mathbf{L}^2}$ are equivalent norms in $\mathbf{H}^1(\Omega) \cap \mathbf{H}_\sigma(\Omega)$ and $\|\mathbf{A}\|_{\mathbf{L}^2} \leq C\|\text{rot}\mathbf{A}\|_{\mathbf{L}^2}$ for any $\mathbf{A} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_\sigma(\Omega)$, (72) yields that

$$\sup_{0 \leq t < \infty} \|\mathbf{A}(t)\|_{\mathbf{H}^1} \leq C. \quad (76)$$

Using Sobolev's inequality, we have

$$\sup_{0 \leq t < \infty} \|\mathbf{A}(t)\|_{\mathbf{L}^p} \leq C, \quad \text{for any } 1 \leq p \leq 6 \ (d=3), \quad (77)$$

$$\sup_{0 \leq t < \infty} \|\mathbf{A}(t)\|_{\mathbf{L}^p} \leq C, \quad \text{for any } 1 \leq p \leq \infty \ (d=2). \quad (78)$$

Hence, we have

$$\|\psi(t)\|_{L^4}^4 \leq 2\|1 - |\psi|^2\|_{L^2}^2 + 2|\Omega| \quad (79)$$

From (73), we get

$$\sup_{0 \leq t < \infty} \|\psi(t)\|_{L^4} \leq C. \quad (80)$$

Now,

$$\begin{aligned} \|\nabla\psi\|_{\mathbf{L}^2} &\leq \|\nabla\psi - i\mathbf{A}\psi\|_{\mathbf{L}^2} + \|\mathbf{A}\psi\|_{\mathbf{L}^2} \\ &\leq \|\mathbf{D}_A\psi\|_{\mathbf{L}^2} + 2\|\mathbf{A}\|_{\mathbf{L}^4}^2 + 2\|\psi\|_{L^4}^2 \end{aligned} \quad (81)$$

Therefore, from (71), (77) and (80), we have

$$\sup_{0 \leq t < \infty} \|\nabla\psi(t)\|_{\mathbf{L}^2} \leq C. \quad (82)$$

Since, $\|\psi\|_{L^2} \leq |\Omega|^{1/4}\|\psi\|_{L^4}$, we have

$$\sup_{0 \leq t < \infty} \|\psi(t)\|_{L^2} \leq C. \quad (83)$$

Therefore, from (82) and (83), it follows that

$$\sup_{0 \leq t < \infty} \|\psi(t)\|_{H^1} \leq C. \quad (84)$$

Hence, we obtain

$$\sup_{0 \leq t < \infty} \|U(t)\|_{\mathcal{H}^1}^2 \leq \sup_{0 \leq t < \infty} \|\psi(t)\|_{H^1}^2 + \sup_{0 \leq t < \infty} \|\mathbf{A}(t)\|_{\mathbf{H}^1}^2 \leq M_0, \quad (85)$$

where M_0 is a constant which depends only on $\|U_0\|_{\mathcal{H}^1}$ and Ω . but not on t . This implies

$$\sup_{0 \leq t < \infty} \|\mathcal{A}^{1/2} U\|_{\mathcal{L}^2} < C. \quad (86)$$

This completes the proof of Theorem 3.

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