

On the fundamental solution of partial differential
operators of Schrodinger's type

Daisuke FUJIWARA

Dept. Math. Univ. of Tokyo

§ 1 Preliminaries

We shall construct the fundamental solution of partial differential operators of Schrodinger's type;

$$L = (\hbar/i) \partial / \partial t + \frac{1}{2} \sum 1/\sqrt{g(x)} (\hbar/i) \partial / \partial x_j (\sqrt{g(x)} g^{jk} (\hbar/i) \partial / \partial x^k + V(x),$$

where $\hbar$ is a positive constant and $(g^{jk}(x))_{jk}$ is a positive definite matrix-valued function of class $C^\infty(\mathbb{R}^n)$. The notion of Feynman

integral has been explained mathematically by several authors (for example [1] and [5][7] and their references.) as a limit of analytic continuation of Wiener integrals. Direct treatment of it was proposed by Ito [6] in introducing an "ideal uniform measure on $R^{[0,t]}$ ".

In this note we prove that the Riemannian sum approximation of Feynman's path integral that Feynman himself defined in [2] actually converges in the operator norm to fundamental solution of the operator L if the function $\exp(i/\hbar)S$, S being the classical action, oscillates rapidly. Note that our method enables one to treat the case that $g^{jk}(x)$ are not constant.

§ 2 assumptions

The Lagrangean function is of the form

$$L(q, \dot{q}) = \frac{1}{2} \sum_{jk} g_{jk}(q) \dot{q}_j \dot{q}_k - V(q),$$

where (g_{jk}) is a positive definite matrix valued function, i.e.,

$ds^2 = \sum_{jk} g_{jk}(x) dx^j dx^k$ is a Riemannian metric in R^n . The Hamiltonian

function is $H(p, q) = \dot{q} \cdot p - L$, where $\dot{q} \cdot p = \sum_{j=1}^n \dot{q}_j p_j$. We denote

by $q(t, y, \xi)$ and $p(t, y, \xi)$ the solution of Hamilton equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = - \frac{\partial H}{\partial q}$$

satisfying initial conditions $q = y, p = \xi$ at $t = 0$.

Our first assumption is

(A-I) There exists a constant $\delta > 0$ such that the canonical transformation

$$\chi_t; (y, \xi) \rightarrow (x, \eta) = (q(t, y, \xi), p(t, y, \xi))$$

induces a diffeomorphism of the configuration space for any $t \in [0, \delta]$.

Let $x^0 = x^0(t, x, \xi)$ be the unique solution of $q(t, x^0, \xi) = x$.

Then a generating function of χ_t is given by

$$S_0(t, x, \xi) = \int_0^t L(\dot{q}, q) ds + x^0 \cdot \xi,$$

where the integral should be made along the classical orbit from x^0 to x .

Denoting the Euclidean length of a vector x in R^n by $|x|$, we shall further make the following assumptions;

$$(A-II) \quad \Phi = \left| \text{grad}_{\xi} (S(x, \xi) - S(z, \xi)) \right| \geq \Phi_1(x, z, \xi) \theta(|x-z|),$$

$$\Psi = \left| \text{grad}_x (S(x, \xi) - S(x, \eta)) \right| \geq \Phi_2(x, \xi, \eta) \theta(|\xi - \eta|),$$

where $\Phi_1(x, z, \xi)$ and $\Phi_2(x, \xi, \eta)$ are smooth functions with a positive

lower bound and $\theta(t)$ is a function such that $\theta(t) = 0(t)$ near $t=0$

and $\theta(t) = t^\sigma$ for $t > 1$, with some $\sigma > 0$.

(A-III) For any multi-index α , there exists a constant $C > 0$

such that we have

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha (S(x, \xi) - S(z, \xi)) \right| \leq C \bar{\varphi},$$

$$\left| \left(\frac{\partial}{\partial \eta} \right)^\alpha (S(x, \xi) - S(x, \eta)) \right| \leq C \bar{\psi}.$$

Let $Y(t, x, \xi) = \det \frac{\partial q(t, y, \xi)}{\partial y}$. Then $Y \neq 0$ for $0 \leq t \leq \delta$ by (A-I). Our last assumption is

(A-IV) For any multi-index α there exists a constant $C > 0$ such that we have estimates;

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha (Y(t, x, \xi)^{-1} Y(t, x, \eta)^{-1}) \right| \leq C \bar{\varphi}_2(x, \xi, \eta)$$

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha (Y(t, x, \xi)^{-1} Y(t, y, \xi)^{-1}) \right| \leq C \bar{\varphi}_1(x, y, \xi).$$

§ 3 Results

We set $E_0(t, x, \xi, y) = \left(\frac{g(y)}{g(x)} \right)^{\frac{1}{4}} Y(t, x, \xi)^{-\frac{1}{2}} \exp i h^{-1} S(t, x, \xi, y)$, where $S(t, x, \xi, y) = S_0(t, x, \xi) - y, \xi$. Then E satisfies $(h/i) \partial / \partial t + H(h/i \partial / \partial x, x) E(t, x, \xi, y) = h^2 F(t, x, \xi, y)$, where $F(t, x, \xi) = \frac{1}{2} \Delta Y(t, x, \xi)^{-\frac{1}{2}} \exp i h^{-1} S(tx, \xi)$.

We define two operators $E(t)$ and $F(t)$ by

$$(1) \quad E(t) f(x) = (2\pi h)^{-n} \iint E(t, x, \xi, y) f(y) dy d\xi,$$

$$F(t) f(x) = (2\pi h)^{-n} h^2 \iint F(t, x, \xi, y) f(y) dy d\xi.$$

These are defined at least for any f in $C_0^\infty(\mathbb{R}^n)$. It is easy to see

$E(0) f(x) = f(x)$. Moreover we have

Lemma We have the following estimate

$$\|E(t)\| \leq C \quad \text{and} \quad \|F(t)\| \leq C h^2, \quad \text{for any } t \text{ in } [0, \delta].$$

~~This~~ This is an immediate consequence of our previous work [4].

$$\text{We set } E(t) = E_0(t) + (i/h) \int_0^t E_0(t-s) F(s) ds.$$

Let T_t be the one parameter unitary group generated by $H(-ih \partial/\partial x, x)$.
Our main result is

THEOREM (Feynman [2])

$$\lim_{k \rightarrow \infty} \| E(t/k) E(t/k) \dots E(t/k) - T_t \| = 0.$$

$k \rightarrow \infty$

Proof

~~First, note that~~ Note that $-ih \partial/\partial x + H(-ih \partial/\partial x, x)E(t)$
 $= -ih^{-1} \int_0^t F(t-s) F(s) ds$ and that

$$h^{-1} \left\| \int_0^t F(t-s) F(s) ds \right\| \leq C |t| h^3 \text{ for } t \in [0, \delta].$$

Hence the difference $R(t) = T_t - E(t)$ is estimated as $\|R(t)\| \leq C |t|^2 h^4$.

Now let t be any positive number. Take k so large as t/k belongs to

$[0, \delta]$. Then $\|R(t/k)\| \leq C h^4 (t/k)^2$. This implies that
 $\| E(t/k) E(t/k) \dots E(t/k) - T_t \| \leq (1 + \|R(t/k)\|)^k - 1,$

and this converges to 0.

§ 4 Space time Approach

If we integrate first by ξ in (1) and use stationary phase method, we can prove that

$$E(t) f(x) = \int a(t, x, y) \exp i h^{-1} \mathcal{P}(x, y) f(y) dy,$$

where $a(t, x, y) = (hi/2\pi)^{-\frac{1}{2}n} (\det \text{Hess}_{\xi} S)^{-\frac{1}{2}} \left(\frac{g(y)}{g(x)} \right)^{\frac{1}{4}} Y(tx \xi(x, y, t))^{\frac{1}{2}},$

where $\mathcal{P}(x, y)$ is the classical action $\int_x^y L ds$ along classical path

from y to x . $q(t, x, \xi)$ is the solution of $q(t, x, \xi) = y$ and
 $\text{Hess}_{\xi} S$ = the Hessian matrix of $S_0(t, x, \xi)$ with respect to ξ variables at
 $\xi = \xi(x, y, t)$. Starting with this expression of $E(t)$, we can
 discuss everything and prove our result in the configuration space and
 time.

References

- [1] Babbitt, D.G., Asymptotic summation procedure for certain Feynman integrals. J. Math. Physics, vol. 4, pp 36-41 (1963).
- [2] Feynman, R., Space time approach to non-relativistic quantum mechanics. Reviews of Modern Physics. vol. 20, pp317-384, (1948).
- [3] Fujiwara, D., Fundamental solution of partial differential operators of Schrodinger's type I and II. Proc. Japan Acad. 1974.
- [4] -----, On the boundedness of integral transformations with highly oscillatory kernels. ibid.
- [5] Gelfand, I.M. and Yaglom, A.M., Integration in functional spaces and its application in quantum physics. English translation. J. Math. Physics, vol 1, pp 48-69 (1960).
- [6] Ito, K., Wiener integral and Feynman integral.

5

4th Berkeley symposium, pp227-238.

[7] Nelson, E., Feynman integrals and the Schrodinger equations.

J. Math. Physics, vol. 5, pp332-343. (1964).

[8] C. Morette, *On the definition and
Approximation of Feynman's path integrals.*