

A MATHEMATICAL APPROACH TO INTERMITTENCY (1),(2)

Michiko Yuri

Department of Business Administration, Sapporo University

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1 Piecewise C^0 -invertible Systems

Let $(T, X, Q = \{X_i\}_{i \in I})$ be a piecewise C^0 -invertible system i.e., X is a compact metric space with metric d , $T : X \rightarrow X$ is a noninvertible map which is not necessarily continuous, and $Q = \{X_i\}_{i \in I}$ is a countable disjoint partition $Q = \{X_i\}_{i \in I}$ of X such that $\bigcup_{i \in I} \text{int} X_i$ is dense in X and satisfy the following properties.

- (01) For each $i \in I$ with $\text{int} X_i \neq \emptyset$, $T|_{\text{int} X_i} : \text{int} X_i \rightarrow T(\text{int} X_i)$ is a homeomorphism and $(T|_{\text{int} X_i})^{-1}$ extends to a homeomorphism v_i on $cl(T(\text{int} X_i))$.
- (02) $T(\bigcup_{\text{int} X_i = \emptyset} X_i) \subset \bigcup_{\text{int} X_i = \emptyset} X_i$.
- (03) $\{X_i\}_{i \in I}$ generates $\mathcal{F}$, the sigma algebra of Borel subsets of X .

Let $\underline{i} = (i_1 \dots i_n) \in I^n$ satisfy $\text{int}(X_{i_1} \cap T^{-1}X_{i_2} \cap \dots \cap T^{-(n-1)}X_{i_n}) \neq \emptyset$. Then we define $X_{\underline{i}} := X_{i_1} \cap T^{-1}X_{i_2} \cap \dots \cap T^{-(n-1)}X_{i_n}$ which is called a cylinder of rank n and write $|\underline{i}| = n$. By (01), $T^n|_{\text{int} X_{i_1 \dots i_n}} : \text{int} X_{i_1 \dots i_n} \rightarrow T^n(\text{int}(X_{i_1 \dots i_n}))$ is a homeomorphism and $(T^n|_{\text{int} X_{i_1 \dots i_n}})^{-1}$ extends to a homeomorphism $v_{i_1} \circ v_{i_2} \circ \dots \circ v_{i_n} = v_{i_1 \dots i_n} : cl(T^n(\text{int} X_{\underline{i}})) \rightarrow cl(\text{int} X_{\underline{i}})$.

We impose on (T, X, Q) the next condition which gives a nice countable states symbolic dynamics similar to sofic shifts (cf. [11]):

(Finite Range Structure) $\mathcal{U} = \{\text{int}(T^n X_{i_1 \dots i_n}) : \forall X_{i_1 \dots i_n}, \forall n > 0\}$ consists of finitely many open subsets $U_1 \dots U_N$ of X .

In particular, we say that (T, X, Q) satisfies Bernoulli property if $cl(T(\text{int} X_i)) = X (\forall i \in I)$ so that $\mathcal{U} = \{\text{int} X\}$ and that (T, X, Q) satisfies Markov property if $\text{int}(cl(\text{int} X_i) \cap cl(\text{int} T X_j)) \neq \emptyset$ implies $cl(\text{int} T X_j) \supset cl(\text{int} X_i)$. (T, X, Q) satisfying Bernoulli (Markov) property is called a piecewise C^0 -invertible Bernoulli (Markov) system respectively. We say that $X_i \in Q$ is a *full cylinder* if $cl(T(\text{int} X_i)) = X$. We assume further the next condition :

(**Transitivity**) $\text{int}X = \cup_{k=1}^N U_k$ and $\forall l \in \{1, 2, \dots, N\}, \exists 0 < s_l < \infty$ such that for each $k \in \{1, 2, \dots, N\}$, U_k contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank s_l such that $T^{s_l}(\text{int}X^{(k,l)}(s_l)) = U_l$.

2 Topological pressure for potentials of weak bounded variation

Definition We say that ϕ is a potential of *weak bounded variation*(WBV) if there exists a sequence of positive numbers $\{C_n\}$ satisfying $\lim_{n \rightarrow \infty} (1/n) \log C_n = 0$ and $\forall n \geq 1, \forall X_{i_1 \dots i_n} \in V_{j=0}^{n-1} T^{-j}Q$,

$$\frac{\sup_{x \in X_{i_1 \dots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x))}{\inf_{x \in X_{i_1 \dots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x))} \leq C_n.$$

(C.f.[11,13,15-19])

We define a partition function for each $n > 0$ and for each $U_k \in \mathcal{U}$ as follows :

$$Z_n(U_k, \phi) := \sum_{\substack{\mathbf{i}: |\mathbf{i}|=n, \text{int}(TX_{i_n})=U_k \supset \text{int}X_{i_1}}} \sum_{\substack{\mathbf{v}: x=x \in \text{cl}(\text{int}X_{\mathbf{i}})}} \exp[\sum_{h=0}^{n-1} \phi T^h(x)].$$

We further define :

$$\bar{Z}_n(U_k, \phi) = \sum_{\substack{\mathbf{i}: |\mathbf{i}|=n, \text{int}(TX_{i_n})=U_k \supset \text{int}X_{i_1}}} \sup_{x \in X_{\mathbf{i}}} \exp[\sum_{h=0}^{n-1} \phi T^h(x)]$$

and

$$\underline{Z}_n(U_k, \phi) = \sum_{\substack{\mathbf{i}: |\mathbf{i}|=n, \text{int}(TX_{i_n})=U_k \supset \text{int}X_{i_1}}} \inf_{x \in X_{\mathbf{i}}} \exp[\sum_{h=0}^{n-1} \phi T^h(x)].$$

Lemma 2.1 ([17]) *Let (T, X, Q) be a piecewise C^0 -invertible Markov system with finite range structure satisfying the transitivity. Let ϕ be a potential of WBV. For each $U_k \in \mathcal{U}$, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \bar{Z}_n(U_k, \phi)$, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \underline{Z}_n(U_k, \phi)$ exist and the limits does not depend on k . Furthermore,*

$$\begin{aligned} P_{\text{top}}(T, \phi) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(X, \phi) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(U_k, \phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \bar{Z}_n(U_k, \phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \underline{Z}_n(U_k, \phi), \end{aligned}$$

where

$$\log Z_n(X, \phi) := \sum_{\substack{\mathbf{i}: |\mathbf{i}|=n, \text{int}(TX_{i_n}) \supset \text{int}X_{i_1}}} \sum_{\substack{\mathbf{v}: x=x \in \text{cl}(\text{int}X_{\mathbf{i}})}} \exp[\sum_{h=0}^{n-1} \phi T^h(x)].$$

We define

$$\mathcal{W}_0(T) := \{\phi : X \rightarrow \mathbf{R} | \phi \text{ satisfies WBV and } P_{\text{top}}(T, \phi) < \infty\}.$$

Then we can easily see that the pressure function $P_{\text{top}}(T, \cdot) : \mathcal{W}_0(T) \rightarrow \mathbf{R}$ satisfies continuity for bounded functions and convexity.

3 Weak Gibbs measures associated to potentials of WBV

Definition ([7],[11],[13],[15-19]) A Borel probability measure ν is called a *weak Gibbs measure* for a function ϕ with a constant P if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n \rightarrow \infty} (1/n) \log K_n = 0$ such that ν -a.e. x ,

$$K_n^{-1} \leq \frac{\nu(X_{i_1 \dots i_n}(x))}{\exp(\sum_{i=0}^{n-1} \phi T^i(x) + nP)} \leq K_n,$$

where $X_{i_1 \dots i_n}(x)$ denotes the cylinder containing x .

Definition A Borel probability measure ν on X is called a *f-conformal measure* if $\frac{d(\nu T)|_{X_i}}{d\nu|_{X_i}} = f|_{X_i} (\forall i \in I)$.

Lemma 3.1 ([17]) Let (T, X, Q) be a piecewise C^0 -invertible Markov system with FRS satisfying the transitivity and $\text{int}X \in \mathcal{U}$. Let $\phi \in \mathcal{W}_0(T)$ and ν be an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure. Then ν is a weak Gibbs measure for ϕ with $-P_{\text{top}}(T, \phi)$.

For $\phi : X \rightarrow \mathbf{R}$ we define the Ruelle-Perron-Frobenius operator $\mathcal{L}_\phi$ by

$$\mathcal{L}_\phi g(x) = \sum_{i \in I} \exp[\phi(v_i(x))] g(v_i(x)) \quad (\forall g \in C(X), \forall x \in X).$$

Lemma 3.2 ([11],[13]) If there exist $p > 0$ and a Borel probability measure ν on X satisfying $\mathcal{L}_\phi^* \nu = p\nu$, then ν is an $\exp[\log p - \phi]$ -conformal measure and $p = \exp[P_{\text{top}}(T, \phi)]$.

4 Indifferent periodic points associated to potentials of WBV

Lemma 4.1 $P_{\text{top}}(T, \phi) \geq \frac{1}{q} \sum_{h=0}^{q-1} \phi T^h(x_0) (\forall x_0 \in X, T^q x_0 = x_0)$.

Definition x_0 is called an *indifferent periodic point* with period q with respect to ϕ if $P_{\text{top}}(T, \phi) = \frac{1}{q} \sum_{h=0}^{q-1} \phi T^h(x_0)$. If there exists an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure ν , then x_0 satisfies

$$\frac{d(\nu T^q)}{d\nu}|_{X_{i_1 \dots i_q}(x_0)}(x_0) = \exp[qP_{\text{top}}(T, \phi) - \sum_{h=0}^{q-1} \phi T^h(x_0)] = 1.$$

If x_0 is not indifferent, then we call x_0 a *repelling periodic point*.

Proposition 4.1 ([16-17]) Let x_0 be an indifferent periodic point with period q with respect to $\phi \in \mathcal{W}_0(T)$. Let ν be an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure. Then

(i) $\forall s \geq 1, P_{\text{top}}(T, s\phi) = sP_{\text{top}}(T, \phi)$ and $\forall s < 1, P_{\text{top}}(T, s\phi) \geq sP_{\text{top}}(T, \phi)$.

(ii) $\nu(X_{i_1 \dots i_n}(x_0))$ decays subexponentially fast.

5 Jump transformations

Let J be a subset of the index set I and let $B_1 = \bigcup_{i \in J} X_i$. Define $\mathcal{B}_1 := \{X_i \in Q : X_i \subset B_1\}$ and for each $n > 1$ $\mathcal{B}_n := \{X_{i_1 \dots i_n} \in \bigvee_{i=0}^{n-1} T^{-i} Q : X_{i_k} \subset B_1^c (k = 1, \dots, n-1), X_{i_n} \subset B_1\}$. Define a function $R : X \rightarrow \mathbf{N} \cup \{\infty\}$ by $R(x) = \inf\{n \geq 0 : T^n x \in B_1\} + 1$. Then we see that $B_n := \{x \in X | R(x) = n\} = \bigcup_{X_{i_1 \dots i_n} \in \mathcal{B}_n} X_{i_1 \dots i_n}$ and $D_n := \{x \in X | R(x) > n\} = \bigcap_{i=0}^n T^{-i} B_1^c$. Now we define the jump transformation $T^* : \bigcup_{n=1}^{\infty} B_n \rightarrow X$ by $T^* x = T^{R(x)} x$. We denote $X^* := X \setminus (\bigcup_{i=0}^{\infty} T^{*-i}(\bigcap_{n \geq 0} D_n))$ and $I^* := \bigcup_{n \geq 1} \{(i_1 \dots i_n) \in I^n : X_{i_1 \dots i_n} \subseteq B_n\}$. Then it is easy to see that $(T^*, X^*, Q^* = \{X_{\underline{i}}\}_{\underline{i} \in I^*})$ is a piecewise C^0 -invertible Markov system with FRS and the property (1) : $B_{n+1} = D_n \cap T^{-n} B_1$ is valid for $n \geq 1$. Let $\phi : X \rightarrow \mathbf{R}$ be a potential of WBV with $P_{\text{top}}(T, \phi) < \infty$. We assume further the next condition :

(Local Bounded Distortion) $\exists \theta > 0$ and $\forall X_{i_1 \dots i_n} \in \mathcal{B}_n, \exists 0 < L_\phi(i_1 \dots i_n) < \infty$ such that

$$|\phi v_{i_1 \dots i_n}(x) - \phi v_{i_1 \dots i_n}(y)| \leq L_\phi(i_1 \dots i_n) d(x, y)^\theta$$

and

$$\sup_{n \geq 1} \sup_{X_{i_1 \dots i_n} \in \mathcal{B}_n} \sum_{j=0}^{n-1} L_\phi(i_{j+1} \dots i_n) < \infty.$$

Define $\phi^* : \bigcup_{n=1}^{\infty} B_n \rightarrow \mathbf{R}$ by $\phi^*(x) = \sum_{i=0}^{R(x)-1} \phi T^i(x)$ and denote the local inverses to $T^*|_{X_{\underline{i}} (\underline{i} \in I^*)}$ by $v_{\underline{i}}$. Then $\{\phi^* v_{\underline{i}}\}$ is a family of equi-Hölder continuous functions and if T^* satisfies the next property then ϕ^* satisfies summability of variation.

(Exponential Instability) $\sigma^*(n) := \sup_{\underline{i} \in I^* : |\underline{i}|=n} \text{diam} X_{\underline{i}}$ decays exponentially fast as $n \rightarrow \infty$.

The summable variation allows one to show the existence of an unique equilibrium Gibbs state μ^* for ϕ^* under the existence of an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure ν on X with $\nu(\bigcap_{n \geq 0} D_n) = 0$ and $\mu^* \sim \nu|_{X^*}$. The following formula gives a T -invariant σ -finite measure $\mu \sim \nu$.

$$(2) : \mu(E) = \sum_{n=0}^{\infty} \mu^*(D_n \cap T^{-n} E).$$

If $\sum_{n=0}^{\infty} \nu(D_n) < \infty$, then μ is finite. In particular, $\mu(B_1) = \mu^*(X^*) > 0$, since $\nu(X^*) = 1$. If the reference measure ν is ergodic, then both μ, μ^* are ergodic, too.

Theorem 5.1 (A construction of conformal measures) ([17]) *Let (T, X, Q) be a piecewise C^0 -invertible Markov system with FRS satisfying transitivity. Let T^* be the jump transformation associated to a union of full cylinders of rank 1 which satisfies exponential instability. Let $\phi : X \rightarrow \mathbf{R}$ be a potential of WBV satisfying (LBD), $P_{\text{top}}(T, \phi) < \infty$ and*

$\|\mathcal{L}_\phi \cdot 1\| < \infty$. Suppose either $P_{top}(T^*, \phi^*) \geq 0$ or $\|\mathcal{L}_{(\phi - P_{top}(T^*, \phi^*)) \cdot 1}\| < \infty$. Then there exists a Borel probability measure ν on X supported on X^* satisfying

$$\frac{d\nu T}{d\nu}|_{X_i} = \exp[P_{top}(T, \phi) - \phi](\forall i \in I)$$

and $\nu(\bigcup_{i \in I} \partial X_i) = 0$.

We can associate the indifferent periodic points x_0 with respect to ϕ to the Marginal sets $\bigcap_{n \geq 0} D_n$.

Proposition 5.1 ([17])

(i) (Failure of bounded distortion)

$$C_{nq}(x_0) := \sup_{x, y \in X_{i_1 \dots i_{nq}(x_0)}} \frac{\exp[\sum_{i=0}^{nq-1} \phi T^i(x)]}{\exp[\sum_{i=0}^{nq-1} \phi T^i(y)]} \rightarrow \infty$$

monotonically as $n \rightarrow \infty$.

(ii) (Singularity of the invariant density) $x_0 \in \bigcap_{n \geq 0} D_n$ and $\frac{d\mu}{d\nu}(x_0) = \infty$.

For a T -invariant probability measure m on $(X, \mathcal{F})$, I_m denotes the conditional information of Q with respect to $T^{-1}\mathcal{F}$.

Theorem 5.2 (Variational principle) ([17]) Let ν be the $\exp[P_{top}(T, \phi) - \phi]$ -conformal measure obtained under assumptions in Theorem 5.1. We assume further that $\Gamma := \bigcap_{n \geq 0} D_n$ consists of periodic points. If $\int_X R d\nu < \infty$ and $H_\nu(Q^*) < \infty$, then there exists a T -invariant ergodic probability measure μ equivalent to ν which satisfies the following variational principle.

$$P_{top}(T, \phi) = h_\mu(T) + \int_X \phi d\mu \geq h_m(T) + \int_X \phi dm$$

for all T -invariant ergodic probability measure m on X with $I_m + \phi \in L^1(m)$ satisfying $h_m(T) < \infty$ or $\int_X \phi dm > -\infty$.

Corollary 5.1 (Phase transition) We assume all conditions in Theorem 5.2. If Γ consists of indifferent periodic points with respect to ϕ , then the set of equilibrium states for ϕ is the convex hull of μ and the set of invariant Borel probability measures supported on Γ .

6 Slow decay of correlations

We denote $v'_{i_1 \dots i_n}(x) = \frac{d(\mu v_{i_1 \dots i_n})}{d\mu}(x)$ and let $P_\mu : L^1(\mu) \rightarrow L^1(\mu)$ be the normalized transfer operator with respect to μ , i.e.,

$$P_\mu f(x) = \sum_{i \in I} v'_i(x) f(\psi_i(x)) 1_{TX_i}(x) (\forall f \in L^1(\mu)).$$

In this section, we shall establish bounds on the L^1 -convergence of iterated transfer operators $\{P_\mu^n\}_{n \geq 1}$ and bounds on the decay of correlations relative to bounded functions f satisfying a weak Lipschitz-type condition defined by :

(6-1) $\exists 0 < L_f < \infty$ such that

$$\sup_{X_{i(m)} \subset D_m^c} \sup_{x, y \in X_{i(m)}} |f(x) - f(y)| \leq L_f \sigma(m) \quad (\forall m > 0)$$

under the following conditions.

(6-2) $\Delta_1(k) := \sup_{n \geq 1} \sup_{i(n) \in \mathcal{A}_n} \sup_{X_{j(k)} \subset D_k^c} \sup_{x, y \in X_{j(k)}} |1 - \frac{\psi'_{i(n)}(x)}{\psi'_{i(n)}(y)}| \rightarrow 0$ as $k \rightarrow \infty$.

(6-3) $\Delta_2(k) := \sup_{X_{j(k)} \subset D_k^c} \sup_{x, y \in X_{j(k)}} |1 - \frac{(d\mu/d\nu)(x)}{(d\mu/d\nu)(y)}| \rightarrow 0$ as $k \rightarrow \infty$.

Here $\sigma(m) := \sup_{i(m) \in \mathcal{A}_m} \text{diam} X_{i(m)}$, $i(m)$ denotes a sequence $i_1 \dots i_m$ of length m and D_m^c denotes $X \setminus D_m$.

Remark (1) If $d\mu^*/d\nu$ is Hölder continuous (with exponent θ), $\Delta_2(m)$ can be bounded from above by $O(\Delta_1(m)) + O(\sigma(m)^\theta)$. For all examples, we can easily estimate both $\Delta_1(m)$ and $\sigma(m)$.

Remark (2) The condition (6-1) is milder than the usual Lipschitz condition. For example, for $S_\beta(x) = x + x^{1+\beta} \pmod{1}$ $f(x) = x^{-\delta}$ for any $0 < \delta < \beta$ is a non-Lipschitz unbounded function satisfying (6-1).

We denote $\Delta(k) := \max_{i=1,2} \Delta_i(k)$.

Theorem 6.1 (Polynomial bounds) *Let (T, X, Q) be a piecewise C^0 -invertible Bernoulli system and let ν and μ be the probability measures obtained in Theorems 5.1 and 5.2 respectively. Suppose that (6-2) and (6-3) are satisfied. Assume further that all $\mu(D_n)$, $\Delta(n)$ and $\sigma(n)$ decay polynomially fast. Then $\forall f \in L^\infty(\mu)$ satisfying (6-1) we have the following results.*

1. (Rates of L^1 -convergence of $\{P_\mu^n f\}_{n \geq 1}$) $\forall n \geq 1$ and $\forall 0 < \epsilon < 1$

$$\|P_\mu^n f - \int_X f d\mu\|_1 \leq \max\{O(\mu(D_{[n^\epsilon]})), O(\Delta([n^\epsilon])), O(\sigma(2[n^\epsilon]))\}.$$

2.(Decay of correlations) $\forall g \in L^\infty(\mu)$ and $\forall 0 < \epsilon < 1$

$$|\int_X f(gT^m)d\mu - \int_X f d\mu \int_X g d\mu| \leq \max\{O(\mu(D_{[n^\epsilon]})), O(\Delta([n^\epsilon])), O(\sigma(2[n^\epsilon]))\}.$$

The next result gives sufficient conditions for (6-2).

Lemma 6.1 Suppose that $\{\phi\psi_i\}_{i \in I}$, $\{\phi^*\psi_i^*\}_{i \in I^*}$ are equi-Hölder continuous with exponents θ_1, θ_2 respectively. Then $\forall X_{i_1 \dots i_m} \subset D_m^c$ and $\forall (j_1 \dots j_n) \in \mathcal{A}_n$ such that $X_{j_k} \subset B_1$ and $X_{j_{k+1} \dots j_n} \subset D_{n-k}$ and $\forall x, y \in X$ we have

$$\begin{aligned} & \left| 1 - \frac{\psi'_{j_1 \dots j_n}(\psi_{i_1 \dots i_m} x)}{\psi'_{j_1 \dots j_n}(\psi_{i_1 \dots i_m} y)} \right| \\ & \leq \max\{O(\sigma(m+n-k)^{\theta_2}), O(\sum_{i=[m/2]}^{\infty} \sup_{X_{i_1 \dots i_i} \subset B_i} \{diam X_{i_1 \dots i_i}\}^{\theta_1}), O(\sigma(\lfloor \frac{m}{2} \rfloor)^{\theta_2})\}. \end{aligned}$$

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