

Discrete final-offer arbitration model

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Abstract

A bargaining problem with two players Labor (player L) and Management (player M) is considered. The players must decide the monthly wage payed to L by M . At the begining players L and M submit their offers s_1 and s_2 . If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator is called in and he chooses the offer which is nearest for his solution α . We suppose that a solution α is concentrated in two points $a, 1 - a$ at the interval $[0, 1]$ with probabilities $p, q = 1 - p$. The equilibrium in the arbitration game among pure and mixed strategies is derived.

Key words: bargaining problem, arbitration, equilibrium strategy.

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1 Introduction

We consider a zero-sum game related with a model of the labor-management negotiations using an arbitration procedure. Imagine that two players: Labor (player L) and Management (player M) bargain on a wage bill which has to be in the range $[0, 1]$ where the current wage bill is normalised at zero, and the known maximum management ability to pay is at 1. Player L is interested to maximize a wage bill as much as possible and the player M has the opposite goal.

At the begining the players L and M submit their offers s_1 and s_2 respectively, $s_1, s_2 \in [0, 1]$. If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator A is called in and he has to choose one of the decisions.

There are different approaches in analyzing the arbitration models [1-6]. We consider here the final-offer arbitration procedure [3] which allows the arbitrator only to choose one of the two final offers made by the players. We suppose here that the arbitrator imposes a solution α which is random variable being concentrated in two points a and $b = 1 - a$ with different probabilities p and $q = 1 - p$, $0 \leq a, p \leq 1$. The arbitrator chooses the offer which is nearest for his solution α . The solution of this game with equal $p = q = 1/2$ was obtained in [6]. In this paper we obtain the solution of this game where p and q can be non-equal.

So, we have a zero-sum game determined in the unit square where the strategies of players L and M are the real numbers $s_1, s_2 \in [0, 1]$ and payoff function in this game has form $H(s_1, s_2) = EH_\alpha(s_1, s_2)$, where

$$H_\alpha(s_1, s_2) = \begin{cases} (s_1 + s_2)/2, & \text{if } s_1 \leq s_2 \\ s_1, & \text{if } s_1 > s_2, |s_1 - \alpha| < |s_2 - \alpha| \\ s_2, & \text{if } s_1 > s_2, |s_1 - \alpha| > |s_2 - \alpha| \\ \alpha, & \text{if } s_1 > s_2, |s_1 - \alpha| = |s_2 - \alpha| \end{cases} \quad (1)$$

Below we show that the equilibrium in this game in dependence on value a can be among pure (section 2) and mixed (sections 3-4) strategies.

2 Solution of the game. Pure strategies

Theorem 1. Let $p \in (0, 0.5]$ and $a \in [0, p/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1, s_2^* = 0$. The value of the game $v = q$.

Proof. Let player II uses $s_2 = 0$. The payoff of player I is equal to:

for $s_1 \in [0, 2a)$ $H(s_1, 0) = ps_1 + qs_1 = s_1 < 2a \leq p \leq q$,

for $s_1 = 2a$ $H(2a, 0) = pa + (1 - p)2a = (2 - p)a < 2a \leq p \leq q$,

for $s_1 \in (2a, 1]$ $H(s_1, 0) = p0 + qs_1 = qs_1$.

The maximum of the function is reached for $s_1 = 1$ and equals to q . Now, suppose that player I uses $s_1 = 1$. For $s_2 \in [0, 1 - 2a)$ $H(1, s_2) = ps_2 + q$. Minimum of this function lies in $s_2 = 0$ and equal to q . For $s_2 = 1 - 2a$ $H(1, 1 - 2a) = p(1 - 2a) + q(1 - a) = 1 - a - ap$. Because $p/(1 + p) > p/2 \geq a$, it follows $p > a + ap$ and $1 - a - ap > 1 - p = q$. For $s_2 \in (1 - 2a, 1]$ $H(s_1, s_2) = ps_2 + qs_2 = s_2$. According to condition $p \geq 2a$ we have $s_2 \geq 1 - 2a > 1 - p = q$. So, for all s_2 $H(1, s_2) \geq q$ and $H(s_1, 0) \leq q$ for all s_1 . Hence, $\{s_1 = 1, s_2 = 0\}$ is an equilibrium in the game and $v = q$.

Analogous arguments leads to

Theorem 2. Let $p \in (0.5, 1)$ and $a \in [0, q/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1, s_2^* = 0$, and value of the game $v = q$.

3 Method for obtaining the equilibrium among mixed strategies

In case $a > \min\{p/2, q/2\}$ equilibrium consists of mixed strategies, i.e. randomised strategies of players L and M . Denote $F_1(s_1)$ and $F_2(s_2)$ distribution functions of the strategies for L and M , respectively. Suppose, that $F_1(s_1) \left[F_2(s_2) \right]$ is continuous and its support consists of two intervals $(\alpha_1; \alpha_2]$ and $(\alpha_3; \alpha_4] \left[(\beta_1; \beta_2], (\beta_3; \beta_4] \right]$ at the $[0; 1]$ with $\alpha_2 \leq \alpha_3 \left[\beta_2 \leq \beta_3 \right]$.

In extreme points of the interval $[0; 1]$ functions $F_1(s_1)$ and $F_2(s_2)$ can have a gap. Let also $\beta_4 \leq \alpha_1$, $F_1(\alpha_1) = 0$ and $F_2(\beta_4) = 1$.

Let $F_{1,12}(s_1)$ and $F_{1,34}(s_1)$ denote the form of $F_1(s_1)$ at the intervals $(\alpha_1; \alpha_2]$ and $(\alpha_3; \alpha_4]$; and, respectively, $F_{2,12}(s_2)$ and $F_{2,34}(s_2)$ – for the function $F_2(s_2)$ at $(\beta_1; \beta_2]$ and $(\beta_3; \beta_4]$.

Firstly, consider the case $p \leq 0.5$. Admit, that the intervals $(\alpha_1; \alpha_2]$ and $(\beta_1; \beta_2]$ are symmetric in respect on the point a and the intervals $(\alpha_3; \alpha_4]$ and $(\beta_3; \beta_4]$ are symmetric in respect on b . Otherwords,

$$\alpha_1 = 2a - \beta_2, \quad \beta_1 = 2a - \alpha_2, \quad \alpha_4 = 2b - \beta_3, \quad \beta_4 = 2b - \alpha_3. \quad (2)$$

Suppose, that player L (M) uses a mixed strategy $F_1(s_1)$ ($F_2(s_2)$) and consider the payoffs of the players.

For $s_1 \in (\alpha_1; \alpha_2]$,

$$H(s_1, F_2(s_2)) = p \left\{ s_1 F_{2,12}(2a - s_1) + \int_{2a-s_1}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\} + qs_1. \quad (3)$$

For $s_1 \in (\alpha_3; \alpha_4]$,

$$\begin{aligned} H(s_1, F_2(s_2)) = p \left\{ 0 \cdot F_2(0) + \int_{2a-\alpha_2}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\} \\ + q \left\{ s_1 F_{2,34}(2b - s_1) + \int_{2b-s_1}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\}. \end{aligned} \quad (4)$$

For $s_2 \in (\beta_1; \beta_2]$,

$$\begin{aligned} H(F_1(s_1), s_2) = p \left\{ \int_{2a-\beta_2}^{2a-s_2} s_1 dF_{1,12}(s_1) + s_2(1 - F_{1,12}(2a - s_2)) \right\} \\ + q \left\{ \int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \int_{\alpha_3}^{2b-\beta_3} s_1 dF_{1,34}(s_1) + 1 \cdot (1 - F_1(1)) \right\}. \end{aligned} \quad (5)$$

For $s_2 \in (\beta_3; \beta_4]$,

$$\begin{aligned} H(F_1(s_1), s_2) = ps_2 + q \left\{ \int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \right. \\ \left. + \int_{\alpha_3}^{2b-s_2} s_1 dF_{1,34}(s_1) + s_2(1 - F_{1,34}(2b - s_2)) \right\}. \end{aligned} \quad (6)$$

If $F_1^*(s_1)$, $F_2^*(s_2)$ are optimal then the equations $H(s_1, F_2^*(s_2)) = v$ and $H(F_1^*(s_1), s_2) = v$, must be satisfied in the support-intervals where v -value of the game. Hence,

$$H(s_1, F_2^*(s_2)) = v, \quad s_1 \in (\alpha_1; \alpha_2] \cup (\alpha_3; \alpha_4],$$

$$H(F_1^*(s_1), s_2) = v, \quad s_2 \in (\beta_1; \beta_2] \cup (\beta_3; \beta_4].$$

From here,

$$\frac{\partial H(s_1, F_2^*(s_2))}{\partial s_1} = 0, \quad s_1 \in (\alpha_1; \alpha_2] \cup (\alpha_3; \alpha_4],$$

$$\frac{\partial H(F_1^*(s_1), s_2)}{\partial s_2} = 0, \quad s_2 \in (\beta_1; \beta_2] \cup (\beta_3; \beta_4].$$

Finding the derivative of (3-4) in s_1 and putting it equal to 0, and using the admission that $F_2^*(\beta_4) = 1$ and $F_2^*(s_2)$ is continuous at $[\beta_2; \beta_3]$, consequently, $F_2^*(\beta_2) = F_2^*(\beta_3)$, we obtain the system of differential equations with boundary conditions:

$$\begin{aligned} p \{2(s_1 - a)F_{2,12}'(2a - s_1) - F_{2,12}^*(2a - s_1)\} - q &= 0, \quad s_1 \in (\alpha_1; \alpha_2], \\ q \{2(b - s_1)F_{2,34}'(2b - s_1) + F_{2,34}^*(2b - s_1)\} &= 0, \quad s_1 \in (\alpha_3; \alpha_4], \\ F_{2,34}^*(\beta_4) &= 1, \quad F_{2,12}^*(\beta_2) = F_{2,34}^*(\beta_3). \end{aligned}$$

Changing the arguments $t_1 = 2a - s_1$, $t_1 \in (\beta_1; \beta_2]$ in the first equation and $t_2 = 2b - s_1$, $t_2 \in (\beta_3; \beta_4]$ in the second one we obtain the system:

$$\frac{dt_1}{2(a - t_1)} = \frac{dF_{2,12}^*}{F_{2,12}^* + p/q}, \quad \frac{dt_2}{2(b - t_2)} = \frac{dF_{2,34}^*}{F_{2,34}^*}.$$

The solution which satisfies the boundary conditions has the following form

$$F_2^*(s_2) = \begin{cases} 0, & \text{if } s_2 \leq 2a - \alpha_2, \\ \left(\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + \frac{q}{p}\right) \frac{\sqrt{a - \beta_2}}{\sqrt{a - s_2}} - \frac{q}{p}, & \text{if } 2a - \alpha_2 < s_2 \leq \beta_2, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}, & \text{if } \beta_2 < s_2 \leq \beta_3, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - s_2}}, & \text{if } \beta_3 < s_2 \leq 2b - \alpha_3, \\ 1, & \text{if } 2b - \alpha_3 < s_2. \end{cases} \quad (7)$$

Finding the derivative of (5-6) in s_2 and putting it equal to 0, and using the admission $F_1^*(\alpha_1) = 0$ and $F_1^*(\alpha_2) = F_1^*(\alpha_3)$, we obtain the system:

$$\begin{aligned} p \{1 - F_{1,12}^*(2a - s_2) - 2(a - s_2)F_{1,12}'(2a - s_2)\} &= 0, \quad s_2 \in (\beta_1; \beta_2], \\ p + q \{1 - F_{1,34}^*(2b - s_2) - 2(b - s_2)F_{1,34}'(2b - s_2)\} &= 0, \quad s_2 \in (\beta_3; \beta_4], \\ F_{1,12}^*(\alpha_1) &= 0, \quad F_{1,12}^*(\alpha_2) = F_{1,34}^*(\alpha_3). \end{aligned}$$

Let change the arguments $t_1 = 2a - s_2$, $t_1 \in (\alpha_1; \alpha_2]$ in the first equation, and $t_2 = 2b - s_2$, $t_2 \in (\alpha_3; \alpha_4]$ in the second equation:

$$\frac{dt_1}{2(t_1 - a)} = \frac{dF_{1,12}^*}{1 - F_{1,12}^*}, \quad \frac{dt_2}{2(t_2 - b)} = \frac{dF_{1,34}^*}{1 + p/q + F_{1,34}^*}.$$

The solution of the system:

$$F_1^*(s_1) = \begin{cases} 0, & \text{if } s_1 \leq 2a - \beta_2, \\ 1 - \frac{\sqrt{a - \beta_2}}{\sqrt{s_1 - a}}, & \text{if } 2a - \beta_2 < s_1 \leq \alpha_2, \\ 1 - \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}, & \text{if } \alpha_2 < s_1 \leq \alpha_3, \\ 1 + \frac{p}{q} - \left(\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + \frac{p}{q}\right) \frac{\sqrt{\alpha_3 - b}}{\sqrt{s_1 - b}}, & \text{if } \alpha_3 < s_1 \leq 2b - \beta_3, \\ 1, & \text{if } 2b - \beta_3 < s_1. \end{cases} \quad (8)$$

Now let us substitute the functions (7)–(8) to (3) – (6). For $s_1 \in (\alpha_1; \alpha_2]$,

$$H_1 = H(s_1, F_2^*(s_2)) = p \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} ((2a - \beta_2) - (2b - \beta_3)) + p\alpha_3 + q(2a - \beta_2).$$

For $s_1 \in (\alpha_3; \alpha_4]$,

$$H_2 = H(s_1, F_2^*(s_2)) = p \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} ((2a - \beta_2) - (2b - \beta_3)) + p\alpha_3 + q(2a - \beta_2) - \\ - p\alpha_2 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - q\alpha_2 \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3.$$

For $s_2 \in (\beta_1; \beta_2]$,

$$H_3 = H(F_1^*(s_1), s_2) = q \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} ((\alpha_2 - 2a) - (\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b) - \\ - q\beta_3 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - p\beta_3 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + p\beta_2 + q\theta$$

For $s_2 \in (\beta_3; \beta_4]$,

$$H_4 = H(F_1^*(s_1), s_2) = q \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} ((\alpha_2 - 2a) - (\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b),$$

where

$$\theta = \begin{cases} 0, & \text{if } F_1^*(1) = 1, \\ -\frac{p}{q} + \left(\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + \frac{p}{q} \right) \frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}, & \text{if } F_1^*(1) < 1. \end{cases}$$

So, take place

$$H_2 = H_1 + \chi_1,$$

$$H_3 = H_4 + \chi_2,$$

where

$$\chi_1 = -p\alpha_2 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - q\alpha_2 \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3,$$

$$\chi_2 = -q\beta_3 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - p\beta_3 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + p\beta_2 + q\theta.$$

But must be $H_1 = H_2 = v$ and $H_3 = H_4 = v$, hence

$$\chi_1 = 0,$$

$$\chi_2 = 0,$$

$$H_1 = H_4.$$

Below we will find a solution of the system (9) in different cases. The value of the equal

$$v = p \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \left((2a - \beta_2) - (2b - \beta_3) \right) + p\alpha_3 + q(2a - \beta_2).$$

Denote $\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} = \frac{1}{x}$, $\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} = y$. After symplifications (9) can be rewritten:

$$\begin{aligned} -\frac{\alpha_2}{x}(py + q) + q\alpha_3 &= 0, \\ -\beta_3y(q/x + p) + p\beta_2 + q\theta &= 0, \\ p(y(2a - 2b - \beta_2 + \beta_3) + 2\alpha_3 - 2b) &= q \left(\frac{1}{x}(\alpha_2 - \alpha_3 - 2a + 2b) + 2\beta_2 - 2a \right). \end{aligned} \quad (11)$$

If $F_1^*(1) = 1$ (or, $F_{1,34}^*(2b - \beta_3) = 1$, or $\theta = 0$), then $y(q/x + p) = p$. Substituting it to (11) we receive $\beta_2 = \beta_3$. If $F_1^*(1) < 1$ ($2b - \beta_3 = 1$), then $\beta_3 = 2b - 1$, $y = \frac{\sqrt{\alpha_3-b}}{\sqrt{a}}$ and $q\theta = -p + (q/x + p)y$.

Analogously, if $F_2^*(0+) = 0$ ($F_{2,12}^*(2a - \alpha_2) = 0$), then $1/x(py + q) = q$. Substituting to (11), we receive $\alpha_2 = \alpha_3$. If $F_2^*(0+) > 0$ ($2a - \alpha_2 = 0$), then $\alpha_2 = 2a$ and $1/x = \frac{\sqrt{a-\beta_2}}{\sqrt{a}}$. Thus, take place $F_1^*(1) = 1 \implies \beta_2 = \beta_3$ and $F_2^*(0+) = 0 \implies \alpha_2 = \alpha_3$.

Varying different collections of the values $F_1^*(1)$ and $F_2^*(0+)$ and demanding that the support of optimal strategies belongs to $[0; 1]$, we will obtain the form of optimal strategies depending on values of a and p (see Fig. 1).

4 Solution of the game. Mixed Strategies

4.1 Equilibrium for $(p, a) \in D_1$

Suppose that $F_1^*(1) = 1$ and $F_2^*(0+) = 0$ (i.e. $\alpha_2 = \alpha_3 = A$, $\beta_2 = \beta_3 = B$). From the equations $\frac{1}{x} = \frac{\sqrt{a-B}}{\sqrt{A-a}}$, $y = \frac{\sqrt{A-b}}{\sqrt{b-B}}$ it follows

$$\alpha_2 = \alpha_3 = A = \frac{bx^2(1+y^2) - ay^2(1+x^2)}{x^2 - y^2}, \quad \beta_2 = \beta_3 = B = \frac{a(1+x^2) - b(1+y^2)}{x^2 - y^2}. \quad (12)$$

The first two equations in (11) give

$$\begin{cases} qx = py + q, \\ y \left(\frac{q}{x} + p \right) = p, \end{cases}$$

which positive solution is

$$x = \frac{p^2 + pq - q^2 + \sqrt{p^4 + 2p^3q - p^2q^2 + 2pq^3 + q^4}}{2pq}, \quad (13)$$

$$y = \frac{p^2 - pq - q^2 + \sqrt{p^4 + 2p^3q - p^2q^2 + 2pq^3 + q^4}}{2p^2}. \quad (14)$$

It is not difficult to check that it satisfies to the third equation in (11).

The values x, y and (12) give the solution of the game iff the following system of inequalities be satisfied

$$\beta_1 \geq 0, \quad \alpha_4 \leq 1,$$

or

$$a \geq \frac{1+y^2}{3+2y^2-x^2y^2}, \quad a \geq \frac{1+x^2}{3+2x^2-x^2y^2}.$$

The solution of this system is the inequality $a \geq \frac{1+x^2}{3+2x^2-x^2y^2}$ ($\alpha_4 \leq 1$). It determines some region on the plane (p, a) , denote it D_1 (see. Fig.1) with the lower border $a_1(p) = \frac{1+x^2}{3+2x^2-x^2y^2}$.

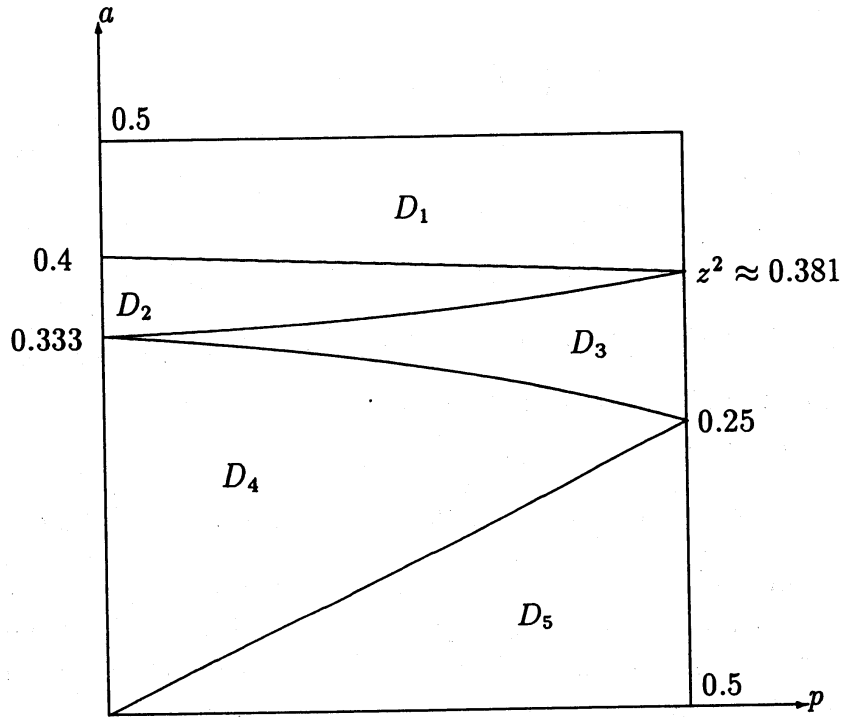


Fig. 1

Theorem 3. For $(p, a) \in D_1$ the equilibrium is (F_1^*, F_2^*) of the form (7-8) with parameters determined by (12-14). The value of the game : $v = q(2a - \beta_2) + p\alpha_3 - 2p(2b - 1) \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_2}}$.

Notice some properties of the solution:

$$\lim_{p \rightarrow 0} a_1(p) = 0.4, \quad \lim_{p \rightarrow 0.5} a_1(p) = z^2,$$

where z is the "golden section" of the interval $[0, 1]$. It follows from

$$\lim_{p \rightarrow 0+} x = 1, \quad \lim_{p \rightarrow 0+} y = 0, \quad \lim_{p \rightarrow 0.5-} x = \frac{\sqrt{5} + 1}{2}, \quad \lim_{p \rightarrow 0.5-} y = z = \frac{\sqrt{5} - 1}{2}.$$

Notice also, that for fixed p if a decreases then α_4 increases to 1 and reaches it for $a = a_1(p)$ (to obtain it we can substitute $a_1(p)$ instead of a to $\alpha_4 = 2 - 2a - \beta_3$). For values $a \leq a_1(p)$, the solution of the game is different.

4.2 Equilibrium for $(p, a) \in D_2$

If $F_1^*(1) < 1$ and $F_2^*(0+) = 0$ (or, equivalently, $\alpha_2 = \alpha_3 = A$, $\beta_2 = B$, $\beta_3 = 2b - 1$), then from the equations $\frac{1}{x} = \frac{\sqrt{a-B}}{\sqrt{A-a}}$ and $y = \frac{\sqrt{A-b}}{\sqrt{a}}$ we obtain

$$\alpha_2 = \alpha_3 = A = ay^2 + b, \quad \beta_2 = B = \frac{a(1+x^2) - (ay^2 + b)}{x^2}. \quad (15)$$

The first two equations of (11) take form

$$\begin{cases} qx = py + q, \\ 2ay \left(\frac{q}{x} + p \right) = p(1 - B). \end{cases} \quad (16)$$

From the first equation it follows $x = \frac{py+q}{q}$. Substituting it to the second equation we receive after symplification

$$\begin{aligned} & (2y^3ap^3 + (-p^3 + 4ap^2q + paq^2 + ap^3)y^2 + \\ & + (2aq^3 - 2p^2q + 2paq^2 + 2ap^2q)y + 3paq^2 - 2pq^2) / (py + q)^2 = 0. \end{aligned} \quad (17)$$

Substituting it to the third equation in (11) we obtain

$$\begin{aligned} & y(2y^3ap^3 + (-p^3 + 4ap^2q + paq^2 + ap^3)y^2 + \\ & + (2aq^3 - 2p^2q + 2paq^2 + 2ap^2q)y + 3paq^2 - 2pq^2) / (py + q)^2 = 0. \end{aligned}$$

It is suffucient to find only positive roots of (17).

Denoting $\lambda = p/q$ we have

$$2a\lambda^3y^3 + \lambda(a + 4a\lambda - \lambda^2 + a\lambda^2)y^2 + 2(a + a\lambda - \lambda^2 + a\lambda^2)y + \lambda(3a - 2) = 0. \quad (18)$$

Denote the cubic polynomial at the left side of (18) as $\nu(y)$, $\nu(0) = \lambda(3a - 2) < 0$, $a \in [0; 0.5)$. The coefficient in higher degree of y in (18) is positive, hence, at least one postive root exists. From here also follows that the maximum lies before minimum. The function $\nu = \nu(y)$ has two extreme points $y_1 = \frac{1}{3} \left(\frac{1}{a} - \frac{1+\lambda+\lambda^2}{\lambda^2} \right)$ and $y_2 = -\frac{1}{\lambda} < 0$. With $\nu(0) < 0$ it gives the uniqueness of the positive root of (18).

The solution takes place in case of $\beta_1 \geq 0$, or $a(3 - y^2) \geq 1$. It determines the lower border $a_2(p)$ of the region D_2 on the plane (p, a) .

Theorem 4. For $(p, a) \in D_2$ the equilibrium is (F_1^*, F_2^*) of the form (7-8) with parameters determined by (15-17). The value of the game: $v = q(2a - \beta_2) + p\alpha_3 - p(2b - 1 + \beta_2) \frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}$.

In case $a < a_2(p)$ the following solution will take place.

4.3 Equilibrium for $(p, a) \in D_3$

If $F_1^*(1) < 1$ and $F_2^*(0+) > 0$ (or, equivalently, $\alpha_2 = 2a$, $\beta_3 = 2b - 1$, $\alpha_3 = A$, $\beta_2 = B$), the first two equations in (11) with $1/x = \frac{\sqrt{a-B}}{\sqrt{a}}$ and $y = \frac{\sqrt{A-b}}{\sqrt{a}}$ (or, $\beta_2 = B = a - a/x^2$ and $\alpha_3 = A = ay^2 + b$) take the form

$$\begin{cases} 2a(py + q) = q(ay^2 + b)x, \\ 2ay \left(\frac{q}{x} + p \right) = p \left(b + \frac{a}{x^2} \right). \end{cases} \quad (19)$$

From the first equation in (19) it follows $x = \frac{2a(py+q)}{q(ay^2+b)}$. Substituting it to the second equation in (19) and the third equation in (11) we obtain

$$(3a^2y^4q^2p + (8a^2p^3 + 4a^2q^3)y^3 + (-2pa^2q^2 - 4p^3a + 16a^2p^2q + 4a^2p^3 + 2paq^2)y^2 +$$

$$+ (8a^2p^2q + 8pa^2q^2 - 4a^2q^3 - 8p^2aq + 4q^3a)y + 3pa^2q^2 - pq^2 - 2paq^2)/(4a(py + q)^2) = 0. \quad (20)$$

and

$$y(3a^2y^4q^2p + (8a^2p^3 + 4a^2q^3)y^3 + (-2pa^2q^2 - 4p^3a + 16a^2p^2q + 4a^2p^3 + 2paq^2)y^2 + (8a^2p^2q + 8pa^2q^2 - 4a^2q^3 - 8p^2aq + 4q^3a)y + 3pa^2q^2 - pq^2 - 2paq^2)/(4a(py + q)^2) = 0.$$

It is sufficient to find only positive solutions of (20).

Denoting $\lambda = p/q$ we rewrite (20) in the form

$$3a^2\lambda y^4 + 4a^2(1 + 2\lambda^3)y^3 + 2a\lambda(1 - a + 8a\lambda - 2\lambda^2 + 2a\lambda^2)y^2 + 4a(1 - a + 2a\lambda - 2\lambda^2 + 2a\lambda^2)y - (1 - a)(1 + 3a)\lambda = 0.$$

Denote $\nu(y)$ polynomials at the left side of the equation. Then $\nu(0) = -(1-a)(1+3a)\lambda < 0$, and because the coefficient in higher degree of y is positive then there exists at least one positive root of the equation. Let us show that it is unique. It follows from the fact that the points where $\nu''(y) = 0$ are negative.

$$\nu''(y) = 36a^2\lambda y^2 + 24a^2(1 + 2\lambda^3)y + 4a\lambda((1 - a)(1 - 2\lambda^2) + 8a\lambda).$$

If this parabola has no roots then $\nu(y)$ is concave and the positive root is unique. Let there are two roots

$$y_{1,2} = \frac{-a(1 + 2\lambda^3) \pm \sqrt{a(4a\lambda^6 - 2a\lambda^4 + 2\lambda^4 - 4a\lambda^3 + a\lambda^2 - \lambda^2 + a)}}{3a\lambda}.$$

The root y_1 is negative. Coefficient in higher degree of y of $\nu''(y)$ is positive, hence, the largest root y_2 is negative, iff the coefficient in lower degree of $\nu(y)$ is positive. It is equal to $\xi(a, \lambda) = (1 - a)(1 - 2\lambda^2) + 8a\lambda$. We have: $\xi(a, 0) = 1 - a > 0$, the function $\xi(a, \lambda)$ is convex in λ , $\xi(a, 1) = 9a - 1$. If $a > \frac{1}{9}$, then $\xi(a, \lambda) > 0$, coefficient in lower degree in $\nu''(y)$ is positive, y_2 is negative, hence, the positive root of the equation is inique.

The solution takes place, iff $\beta_2 \geq 0$ or $\frac{ay^2+b}{2a(1+\lambda y)} \leq 1$. This enequality determines the lower border $a_3(p)$ of the region D_3 on the plane (p, a) . Notice, that in D_3 the inequality $a < \frac{1}{9}$ is satisfied automatically.

Theorem 5. For $(p, a) \in D_3$ the equilibrium is (F_1^*, F_2^*) of the form (7-8) with parameters detemined by (19-20). The value of the game: $v = q(2a - \beta_2) + p\alpha_3 - p(2b - 1 + \beta_2)\frac{\sqrt{\alpha_3-b}}{\sqrt{a}}$.

For fixed p , if a decreases from $a_2(p)$ to $a_3(p)$, then β decreases to zero. Finally, consider the case $a < a_3(p)$.

4.4 Equilibrium for $(p, a) \in D_4$

For $\alpha_1 = \alpha_2 = 2a$, $\alpha_4 = 1$, $\beta_1 = \beta_2 = 0$, $\beta_3 = 2b - 1$ the optimal strategies are

$$F_1^*(s_1) = \begin{cases} 0, & \text{if } s_1 \leq \alpha_3, \\ \frac{1}{q} \left(1 - \frac{\sqrt{\alpha_3-b}}{\sqrt{s_1-b}}\right), & \text{if } \alpha_3 < s_1 \leq 1, \\ 1, & \text{if } 1 < s_1, \end{cases} \quad (21)$$

$$F_2^*(s_2) = \begin{cases} 0, & \text{if } s_2 \leq 0, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}, & \text{if } 0 < s_2 \leq 2b - 1, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - s_2}}, & \text{if } 2b - 1 < s_2 \leq 2b - \alpha_3, \\ 1, & \text{if } 2b - \alpha_3 < s_2. \end{cases} \quad (22)$$

Then, for $s_1 \in (\alpha_3; 1]$

$$H_2 = H(s_1, F_2^*(s_2)) = p \left\{ 0 \cdot F_2^*(0) + \int_{2b-1}^{2b-\alpha_3} s_2 dF_2^*(s_2) \right\} + \\ + q \left\{ s_1 F_2^*(2b - s_1) + \int_{2b-s_1}^{2b-\alpha_3} s_2 dF_2^*(s_2) \right\} = \alpha_3 - \frac{p\sqrt{\alpha_3 - b}}{\sqrt{a}}.$$

If $s_2 = 0$, then

$$H_3 = H(F_1^*(s_1), s_2) = q \left\{ \int_{\alpha_3}^1 s_1 dF_1^*(s_1) + 1 \cdot (1 - F_1^*(1)) \right\} = 2\sqrt{a}\sqrt{\alpha_3 - b} + 2b - \alpha_3 - p.$$

If $s_2 \in (2b - 1; 2b - \alpha_3]$, then

$$H_4 = H(F_1^*(s_1), s_2) = ps_2 + q \left\{ \int_{\alpha_3}^{2b-s_2} s_1 dF_1^*(s_1) + s_2(1 - F_1^*(2b - s_2)) \right\} = 2b - \alpha_3.$$

$F_1^*(s_1), F_2^*(s_2)$ be optimal iff

$$\begin{cases} 2b - \alpha_3 = \alpha_3 - \frac{p\sqrt{\alpha_3 - b}}{\sqrt{a}}, \\ 2b - \alpha_3 = 2\sqrt{a}\sqrt{\alpha_3 - b} + 2b - \alpha_3 - p. \end{cases}$$

Solution of this system: $\alpha_3 = b + \frac{p^2}{4a}$.

This form for H_2-H_4 takes place, iff $\alpha_3 \leq 1$ or, equivalently, $a > p/2$. That determines the region D_4 on the plane (p, a) .

Theorem 6. For $(p, a) \in D_4$ the equilibrium is (F_1^*, F_2^*) of the form (21-22). The value of the game: $v = b - \frac{p^2}{4a}$.

The case $a < p/2$ was analysed in section 2.

5 Solution for $p > 0.5$

At the begining we assumed $p \leq 0.5$. In case $p > 0.5$ the solution follows from the following theorem.

Theorem 7. Let for some fixed values of a and p we found the optimal strategies $F_1^*(s_1, p, a)$ and $F_2^*(s_2, p, a)$ in the game with

$$P\{\alpha = a\} = p, \quad P\{\alpha = b\} = q, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \leq q.$$

Then the optimal strategies in the game for the same values a, p and for

$$P\{\alpha = a\} = q, \quad P\{\alpha = b\} = p, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \leq q,$$

are

$$G_1^*(s_1, q, a) = 1 - F_2^*(1 - s_1, p, a), \quad G_2^*(s_2, q, a) = 1 - F_1^*(1 - s_2, p, a).$$

Proof. We have

$$G_1^*(s_1, q, a) = \begin{cases} 0, & \text{if } s_1 \leq 1 - 2b + \alpha_3, \\ 1 - \frac{\sqrt{\alpha_3 - b}}{\sqrt{s_1 - a}}, & \text{if } 1 - 2b + \alpha_3 < s_1 \leq 1 - \beta_3, \\ 1 - \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}, & \text{if } 1 - \beta_3 < s_1 \leq 1 - \beta_2, \\ 1 + \frac{q}{p} - \left(\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + \frac{q}{p} \right) \frac{\sqrt{a - \beta_2}}{\sqrt{s_1 - b}}, & \text{if } 1 - \beta_2 < s_1 \leq 1 - 2a + \alpha_2, \\ 1, & \text{if } 1 - 2a + \alpha_2 < s_1, \end{cases}$$

$$G_2^*(s_2, q, a) = \begin{cases} 0, & \text{if } s_2 \leq 1 - 2b + \beta_3, \\ \left(\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + \frac{p}{q} \right) \frac{\sqrt{\alpha_3 - b}}{\sqrt{a - s_2}} - \frac{p}{q}, & \text{if } 1 - 2b + \beta_3 < s_2 \leq 1 - \alpha_3, \\ \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}, & \text{if } 1 - \alpha_3 < s_2 \leq 1 - \alpha_2, \\ \frac{\sqrt{a - \beta_2}}{\sqrt{b - s_2}}, & \text{if } 1 - \alpha_2 < s_2 \leq 1 - 2a + \beta_2, \\ 1, & \text{if } 1 - 2a + \beta_2 < s_2. \end{cases}$$

These functions will represent the optimal strategies, iff

$$H(s_1, G_2^*(s_2, q, a)) = \text{const for } s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3] \cup (1 - \beta_2; 1 - 2a + \alpha_2],$$

$$H(G_1^*(s_1, q, a), s_2) = \text{const for } s_2 \in (1 - 2b + \beta_3; 1 - \alpha_3] \cup (1 - \alpha_2; 1 - 2a + \beta_2].$$

Denote $G_{1,12}^*(s_1)$ and $G_{1,34}^*(s_1)$ as the form of function $G_1^*(s_1, q, a)$ at the intervals $(1 - 2b + \alpha_3; 1 - \beta_3]$ and $(1 - \beta_2; 1 - 2a + \alpha_2]$ and $G_{2,12}^*(s_1)$, $G_{2,34}^*(s_1)$ for the $G_2^*(s_1, q, a)$ at the intervals $(1 - 2b + \beta_3; 1 - \alpha_3]$, $(1 - \alpha_2; 1 - 2a + \beta_2]$, respectively.

We obtain for $s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3]$

$$H'_1 = H(s_1, G_2^*(s_1, q, a)) = q \left\{ s_1 G_{2,12}^*(2a - s_1) + \int_{2a - s_1}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int_{1 - \alpha_2}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} +$$

$$+ p s_1 = q \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} ((\alpha_3 - 2b) - (\alpha_2 - 2a)) + p(\alpha_3 + 2a - 1) + q(1 - \beta_2).$$

If $s_1 \in (1 - \beta_2; 1 - 2a + \alpha_2]$, then

$$H'_2 = H(s_1, G_2^*(s_1, q, a)) = q \left\{ 0 \cdot G_2^*(0, q, a) + \int_{1 - 2b + \beta_3}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int_{1 - \alpha_2}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} +$$

$$+ p \left\{ s_1 G_{2,34}^*(2b - s_1) + \int_{2b - s_1}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} =$$

$$\begin{aligned}
&= q \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} ((\alpha_3-2b) - (\alpha_2-2a)) + p(\alpha_3+2a-1) + q(1-\beta_2) - \\
&\quad - q(1-\beta_3) \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} \cdot \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} + p(1-\beta_2) - p(1-\beta_3) \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}.
\end{aligned}$$

If $s_2 \in (1-2b+\beta_3; 1-\alpha_3]$, then

$$\begin{aligned}
H'_3 &= H(G_1^*(s_1, q, a), s_2) = q \left\{ \int_{1-2b+\alpha_3}^{2a-s_2} s_1 dG_{1,12}^*(s_1) + s_2(1 - G_{1,12}^*(2a-s_2)) \right\} + \\
&\quad + p \left\{ \int_{1-2b+\alpha_3}^{1-\beta_3} s_1 dG_{1,12}^*(s_1) + \int_{1-\beta_2}^{1-2a+\alpha_2} s_1 dG_{1,34}^*(s_1) + 1 \cdot (1 - G_1^*(1)) \right\} = \\
&= p \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} ((2b-\beta_3) - (2a-\beta_2)) + p(1-\alpha_3) - q(1-2b-\beta_2) - \\
&\quad - p(1-\alpha_2) \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} + q(1-\alpha_3) - q(1-\alpha_2) \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} + p\eta.
\end{aligned}$$

If $s_2 \in (1-\alpha_2; 1-2a+\beta_2]$, then

$$\begin{aligned}
H'_4 &= H(G_1^*(s_1, q, a), s_2) = qs_2 + p \left\{ \int_{1-2b+\alpha_3}^{1-\beta_3} s_1 dG_{1,12}^*(s_1) + \right. \\
&\quad \left. + \int_{1-\beta_2}^{2b-s_2} s_1 dG_{1,34}^*(s_1) + s_2(1 - G_{1,34}^*(2b-s_2)) \right\} = \\
&= p \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} ((2b-\beta_3) - (2a-\beta_2)) + p(1-\alpha_3) - q(1-2b-\beta_2),
\end{aligned}$$

where $\eta = \begin{cases} 0, & \text{if } G_1^*(1) = 1, \\ -\frac{q}{p} + \left(\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} + \frac{q}{p} \right) \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}, & \text{if } G_1^*(1) < 1. \end{cases}$

We have

$$\begin{aligned}
\psi_1 &= H'_2 - H'_1 = -q(1-\beta_3) \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} \cdot \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} + p(1-\beta_2) - p(1-\beta_3) \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}, \\
\psi_2 &= H'_3 - H'_4 = -p(1-\alpha_2) \frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}} \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} + q(1-\alpha_3) - q(1-\alpha_2) \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}} + p\eta.
\end{aligned}$$

There are only four possible forms for the functions $F_1^*(s_1, p, a)$ and $F_2^*(s_2, p, a)$. With $\chi_1 = \chi_2 = 0$, it gives:

1. For $\alpha_2 = \alpha_3 = A$, $\beta_2 = \beta_3 = B$ take place $\frac{\chi_1}{A} = \frac{\psi_2}{1-A}$ and $\frac{\chi_2}{B} = \frac{\psi_2}{1-B}$, consequently, $\psi_1 = \psi_2 = 0$.
2. For $\alpha_2 = \alpha_3 = A$, $\beta_3 = 2b-1$ take place $\frac{\chi_1}{A} = \frac{\psi_2}{1-A}$ and $\chi_2 = -\psi_1$, consequently, $\psi_1 = \psi_2 = 0$.

. For $\alpha_2 = 2a$, $\beta_3 = 1 - 2a$ take place $\chi_1 = -\psi_2$ and $\chi_2 = -\psi_1$, consequently, $\psi_1 = \psi_2 = 0$.

. For $\alpha_1 = \alpha_2 = 2a$, $\alpha_4 = 1$, $\beta_1 = \beta_2 = 0$, $\beta_3 = 2b - 1$, the form of $G_1^*(s_1)$, $G_2^*(s_2)$ is:

$$G_1^*(s_1, q, a) = \begin{cases} 0, & \text{if } s_1 \leq a + \frac{p^2}{4a}, \\ 1 - \frac{p}{2\sqrt{a}\sqrt{s_1-a}}, & \text{if } a + \frac{p^2}{4a} < s_1 \leq 2a, \\ 1 - \frac{p}{2a}, & \text{if } 2a < s_1 \leq 1, \\ 1, & \text{if } 1 < s_1, \end{cases}$$

$$G_2^*(s_2, q, a) = \begin{cases} 0, & \text{if } s_2 \leq 0, \\ 1 - \frac{1}{q} \left(1 - \frac{p}{2\sqrt{a}\sqrt{a-s_2}} \right), & \text{if } 0 < s_2 \leq a - \frac{p^2}{4a}, \\ 1, & \text{if } a - \frac{p^2}{4a} < s_2. \end{cases}$$

Then for $s_2 \in \left(0; a - \frac{p^2}{4a}\right]$

$$\begin{aligned} H(G_1^*(s_1, q, a), s_2) &= q \left\{ \int_{a+\frac{p^2}{4a}}^{2a-s_2} s_1 dG_1^*(s_1, q, a) + s_2(1 - G_1^*(2a - s_2, q, a)) \right\} + \\ &+ p \left\{ \int_{a+\frac{p^2}{4a}}^{2a} s_1 dG_1^*(s_1, q, a) + 1 \cdot (1 - G_1^*(1, q, a)) \right\} = a + \frac{p^2}{4a}. \end{aligned}$$

For $s_1 \in \left(a + \frac{p^2}{4a}; 2a\right]$

$$H(s_1, G_2^*(s_2, q, a)) = q \left\{ s_1 G_2^*(2a - s_1, q, a) + \int_{2a-s_1}^{a-\frac{p^2}{4a}} s_2 dG_2^*(s_2, q, a) \right\} + ps_1 = a + \frac{p^2}{4a}.$$

Finally, for $s_1 = 1$

$$H(s_1, G_2^*(s_2, q, a)) = q \int_0^{a-\frac{p^2}{4a}} s_2 dG_2^*(s_2, q, a) + p = a + \frac{p^2}{4a}.$$

In all cases the payoff is constant, and with $H_1 + H'_4 = 1$, $H_4 + H'_1 = 1$ and $H_1 = H_4$, gives $H'_1 = H'_4$, and all $H'_i, i = 1, \dots, 4$ are equal. It proves the optimality $G_1^*(s_1, q, a)$ and (s_2, q, a) .

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