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INJECTIVITY OF GLOBAL MAPS OF CELLULAR AUTOMATA

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ABSTRACT. The purpose is to show briefly and visually that cellular automata $\mathcal{A}$'s of finite type with a quiescent state q are injective if and only if either $\mathcal{A}$ contains two mutually erasable configurations c_1, c_2 in Moore [2] or two not distinguished configurations d_1, d_2 in Myhill [3].

[1]

1. PRELIMINARIES

$$\circ\ \mathcal{A} = \{\mathbb{Z}^2, S, N, f\} \quad : \quad \text{a cellular automaton}$$

where

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\} \ \text{ : the rational integers,}$$

$$S = \{s_1, s_2, \cdots, s_t\} \ \text{ : the state set ,}$$

$$N(i) = \begin{Bmatrix} i+(-1,1) & i+(0,1) & i+(1,1) \\ i+(-1,0) & i+(0,0) & i+(1,0) \\ i+(-1,-1) & i+(0,-1) & i+(1,-1) \end{Bmatrix} \subseteq \mathbb{Z}^2$$

: neighbourhood of i for $i = (i_1, i_2) \in \mathbb{Z}^2$, ,

so $|N(i)| = 9$,

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$$f = \overbrace{S \times S \times \cdots \times S}^{9} \longrightarrow S \quad : \quad \text{local map} \ ,$$

$\circ\ C = S^{\mathbb{Z}^2}$: configuration set,

$$C \ni c = (\cdots, c(i), \cdots) \ : \ \mathbb{Z}^2 \to S, \ \text{ a map}$$

$\circ\ F : C \to C$: global map

$$c = (\cdots, c(i), \cdots) \mapsto c' = (\cdots, c'(i), \cdots)$$

where $c'(i) = f(c(N(i)))$ with $c(N(i)) = c|_{N(i)}$

$\circ\ C_n = \{c|_{\Delta_n} \mid c \in C\}$

where Δ_n is an $n \times n$ - square subset of $\mathbb{Z}^2$

(Note : Since F is homogeneous, the choice of Δ_n in $\mathbb{Z}^2$ is not essential.)

$\circ\ C_{n+2} = \{c|_{\Delta_{n+2}} \mid c \in C\}$

where Δ_{n+2} is the extension of Δ_n one cell on four sides.

$\circ\ C_{n+2} \setminus C_n = \{c|_{\Delta_{n+2} \setminus \Delta_n} \mid c \in C\}$

: the set called edges or frames

$\circ\ E = c \setminus c_n$: environment of c_n ,

where $c \in C$ and $c_n \in C_n$ and we write

$$c = c_n \vee E$$

$\circ$ For $c = c_n \vee e$ with $c \in C_{n+2}$, $c_n \in C_n$ and $e \in C_{n+2} \setminus C_n$, we have

$$c(N(i)) \subseteq c \ \text{ for } \ i \in \ \Delta_n$$

This allows us to define a function

$$F_{n,e} : C_n \longrightarrow C_n$$

$$c_n = (\cdots, c_n(i), \cdots) \longmapsto c_n' = (\cdots, c_n'(i), \cdots)$$

with

$$c_n'(i) = f(c(N(i))) \ \text{ and } \ c = c_n \vee e$$

$$\circ\ q \in S \text{ with } f(\underbrace{q, q, \cdots, q}_{9}) = q : \ \text{ quiescent state.}$$

$\circ\ c \in C$ with $|\{i \in \mathbb{Z}^2 \mid c(i) \neq q\}| < \infty$

: a configuration of finite type,

i.e., c : finite type $\Leftrightarrow c = c_n \vee E$

for some c_n in C_n and E an environment of which states are all q.

♣ Assumption.

(1) ${}^\exists q \in S$.

(2) ${}^\forall c \in C \Rightarrow$ c : finite type.

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2. Statement of the Theorem

Definition.

$d_1, d_2 \in C_{n-4}$ **with** $d_1 \neq d_2$ **: n-mutually erasable**

$\Leftrightarrow {}^\exists d' \in C_{n-4},\ {}^\exists g, g' \in C_{n-2} \setminus C_{n-4}$ and ${}^\exists h \in C_n \setminus C_{n-2}$

such that

$$\begin{tikzcd}[row sep=small] C_{n-2} \ni d_1 \vee g \arrow[dr, "F_{n-2,h}"] & \\ & d' \vee g' \in C_{n-2}. \\ C_{n-2} \ni d_2 \vee g \arrow[ur, "F_{n-2,h}"'] & \end{tikzcd}$$

Remark. (a) Let d_1, d_2 be n-mutually erasable. Then, for any l in $C_{n+2} \setminus C_n$ there exists h' in $C_n \setminus C_{n-2}$ such that

$$\begin{tikzcd}
C_n \ni c_1 = d_1 \vee g \vee h \arrow[dr, "F_{n,l}"] & \\
& d' \vee g' \vee h' \in C_n. \\
C_n \ni c_2 = d_2 \vee g \vee h \arrow[ur, "F_{n,l}"'] &
\end{tikzcd}$$

(b) Note that $d_1' = d_1 \vee g$ and $d_2' = d_2 \vee g$ are also (n+2)-mutually erasable by taking h, l for g, h and thus this procedure can be continued until to get their extensions $\tilde{c}_1, \tilde{c}_2$ in C.

Definition.

$$d_1, d_2 \in C_{n-4} \textbf{ with } d_1 \neq d_2 \textbf{ : } n\textbf{-not distinguished}$$

$$\Leftrightarrow {}^{\exists}c' \in C \quad \text{and } {}^{\exists}E \in C \setminus C_{n-4}$$

such that

$$\begin{tikzcd}
C \ni c_1 = d_1 \vee E \arrow[dr, "F"] & \\
& c' \in C. \\
C \ni c_2 = d_2 \vee E \arrow[ur, "F"'] &
\end{tikzcd}$$

Now we state our theorem.

Theorem. The following are equivalent:

(I) $F \neq$ injective.

($\mathrm{I_n}$) ${}^{\exists}\mathrm{d_1, d_2} \in \mathrm{C_{n-4}}$: n- mutually erasable for some n in $\mathbb{N}$.

($\mathrm{I'_n}$) ${}^{\exists}\mathrm{d_1, d_2} \in \mathrm{C_{n-4}}$: n- not distinguished for some n in $\mathbb{N}$.

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3. Proof for the Theorem

(a)

(I) : $F \neq$ injective $\Rightarrow {}^{\exists}c_1, c_2 \in C$ such that

(i) $c_1 \neq c_2$

(ii) $F(c_1) = F(c_2)$

$\Rightarrow$ Since

c_1, c_2 : finite type,

we have

${}^{\exists}d_1, d_2 \in C_{n-4}$ and

${}^{\exists}E \in C \setminus C_{n-4}$ of which states are all q

such that

$c_i = d_i \vee E$ for $i = 1, 2$,

where

(i) $d_1 \neq d_2$ by (i),

(ii) $F(c_1) = F(c_2)$ by (ii)

$\Rightarrow$ (I'_n)

(b)

(I'_n) : ${}^{\exists}d_1, d_2 \in C_{n-4}$ with $d_1 \neq d_2$ and ${}^{\exists}E \in C \setminus C_{n-4}$

such that

$$F(d_1 \vee E) = F(d_2 \vee E)$$

$\Rightarrow$ expressing E as

$$E = g \vee h \vee E', \quad \text{where}$$

g in $C_{n-2} \setminus C_{n-4}$, h in $C_n \setminus C_{n-2}$ and

E' an environment of C_n,

we have d' in C_{n-4} and g' in $C_{n-2} \setminus C_{n-4}$ such that

$$\begin{tikzcd}
d_1 \vee g \arrow[dr, "F_{n-2,h}"] & \\
& d' \vee g', \\
d_2 \vee g \arrow[ur, "F_{n-2,h}"'] &
\end{tikzcd}$$

$\Rightarrow$ $(\mathrm{I_n})$.

(c)

(I_n) : ${}^\exists d_1, d_2, d'$ in C_{n-4} with $d_1 \neq d_2$, ${}^\exists g, g'$ in $C_{n-2} \setminus C_{n-4}$,
${}^\exists h$ in $C_n \setminus C_{n-2}$
such that

$$\begin{tikzcd}
d_1 \vee g \arrow[dr, "F_{n-2,h}"] & \\
& d' \vee g' \\
d_2 \vee g \arrow[ur, "F_{n-2,h}"'] &
\end{tikzcd}$$

$\Rightarrow$ by (b) of Remark

${}^\exists g_1$ in $C_n \setminus C_{n-2}$, ${}^\exists g_2$ in $C_{n+2} \setminus C_n$, $\cdots$ and
${}^\exists c \in C$
such that

$$\begin{tikzcd}
C \ni c_1 = d_1 \vee g \vee g_1 \vee g_2 \vee \cdots \arrow[dr, "F"] & \\
& c', \\
C \ni c_2 = d_2 \vee g \vee g_1 \vee g_2 \vee \cdots \arrow[ur, "F"'] &
\end{tikzcd}$$

$\Rightarrow c_1 \neq c_2$ by $d_1 \neq d_2$, and $F(c_1) = F(c_2)$

$\Rightarrow$ (I).

References

[1] B. Durand, Global Properties of Cellular automata, [E. Goles, S. Martinez, Eds. , Cellular automata Complex Systems], Kluwer Academic Pblishers, (1999) 1-22.

[2] K. Jarkko, [Part 1: Spcial issue: Current research trends in cellular automata theory], Nat. Comput. 16, no. 3 (2017) 365-366.

[3] E. F. Moore, Machine Models of Self-Reproduction, Proc. Symp. Apl. Math. 14 (1963) 17-34.

[4] J. Myhill, The Convers to Moore's Garden-of-Eden Theorem, Proc. Am. Math. Soc. 14 (1963) 685-686.

[5] H. Nishio, T. Saito, Information of Cellular Automata I - An Algebraic Sutady -, Fundamenta Informaticae 58 (2003) 399-420.

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