

**Moduli Spaces of Holomorphic Mappings into Hyperbolically
Imbedded Complex Spaces and Locally Symmetric Spaces**

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Introduction

The aim of this talk is to describe the results of Noguchi [14]. Let X be a connected Zariski open subset of a compact reduced complex space $\bar{X}$ such that X is complete hyperbolic and hyperbolically imbedded into $\bar{X}$ (cf. [7, 16]). Let N be a Zariski open subset of a compact complex manifold $\bar{N}$ such that $\partial N = \bar{N} - N$ is a hypersurface with only normal crossings; in some case, we consider the case of $\partial N = \emptyset$. Here we study the structure of the moduli space $\text{Hol}(N, X)$ of all holomorphic mappings $f: N \rightarrow X$ of N into X . Especially interesting is the case where X is the quotient $\Gamma \backslash D$ of a

symmetric bounded domain D by a torsion-free arithmetic subgroup Γ of the identity component $\text{Aut}^0(D)$ of the holomorphic transformation group $\text{Aut}(D)$ of D . It is known that $\Gamma \backslash D$ is complete hyperbolic and hyperbolically imbedded into the Satake compactification $\overline{\Gamma \backslash D}$ of $\Gamma \backslash D$ (cf. [10, 2, 8, 9]). Besides the interesting results of [20, 21, 18], the present work is motivated by the results on the Parshin-Arakelov theorems for curves [17, 1] and Abelian varieties [5]. Cf. also [13]. Let $\pi: \bar{Y} \rightarrow \bar{N}$ be a fiber space over $\bar{N}$ which is smooth over N , such that the fibers $Y_x = \pi^{-1}(x)$ with $x \in N$ are curves with a given genus g or g -dimensional Abelian varieties with principal polarization. Then, roughly speaking, the fiber space naturally induces a holomorphic mapping $f: N \rightarrow \Gamma \backslash S_g$, where S_g denotes Siegel's generalized upper half space. Then the deformation of $\pi: \bar{Y} \rightarrow \bar{N}$ as fiber space over $\bar{N}$ with degeneration at most over ∂N and the total space of such fiber spaces correspond respectively to the deformation of the holomorphic mapping f and the moduli space $\text{Hol}(N, \Gamma \backslash S_g)$. Thus it is quite natural to deal with the case where N and $\Gamma \backslash D$ are non-compact. In the case where N is compact, there is an earlier work for a fiber space of Abelian varieties by [11].

§1. Holomorphic mappings into hyperbolically imbedded spaces

The natural topology of $\text{Hol}(N, X)$ which we endow with is the compact-open topology. We first prove an extension

and convergence theorem.

Theorem (1.1). Let X be a hyperbolic complex space and hyperbolically imbedded into $\bar{X}$. Let N be a complex manifold $\bar{N}$ minus a hypersurface with only normal crossings. If a sequence $\{f_\nu\}_{\nu=1}^\infty$ of $f_\nu \in \text{Hol}(N, X)$ converges to a holomorphic mapping $f: N \rightarrow \bar{X}$, then there are unique holomorphic extensions $\bar{f}_\nu: \bar{N} \rightarrow \bar{X}$ of f_ν and $\bar{f}: \bar{N} \rightarrow \bar{X}$ of f , and $\{\bar{f}_\nu\}$ converges uniformly on compact subsets of $\bar{N}$ to $\bar{f}$.

As for the extension theorem, this generalizes the result of [7], but the method of the proof is different. It will play a fundamental role in our arguments. In the proof of Theorem (1.1) we use the following lemma (cf. [14] for the details).

Lemma (1.2) (cf. [19]). Let $B(R)$ be the open ball of the m -dimensional complex vector space $\mathbb{C}^m$ with radius R and center 0 . Let S be an analytic subset of pure dimension k of $B(R)$ such that $0 \in S$. Then we have

$$\text{Vol}(S \cap B(r)) \geq \frac{\pi^k}{k!} r^{2k}$$

for $0 < r < R$. Moreover, if the equality holds for some $r > 0$, then S is a linear subspace of $\mathbb{C}^m$.

In what follows, we assume that $\bar{N}$ and $\bar{X}$ are compact, and that X is a Zariski open subset of $\bar{X}$, complete hyperbolic and hyperbolically imbedded into $\bar{X}$. Combining Theorem (1.1) with the Douady theory [3], we have

Theorem (1.3). Hol(N, X) carries a structure of a complex space with universal property, such that its underlying topology coincides with the compact-open topology, and

$$\Phi: (f, x) \in \text{Hol}(N, X) \times N \rightarrow f(x) \in X$$

is a holomorphic mapping, which is proper for every fixed $x \in N$. Moreover, $\text{Hol}(N, X)$ is a Zariski open subset of a compact complex space.

Sketch of the proof. Let $\text{Hol}(\bar{N}, \bar{X})$ be the space of all holomorphic mappings from $\bar{N}$ into $\bar{X}$ with compact-open topology. Then, by Theorem (1.1) the mapping

$$f \in \text{Hol}(N, X) \rightarrow \bar{f} \in \text{Hol}(\bar{N}, \bar{X})$$

is an into-homeomorphism. Hence we identify the topological space $\text{Hol}(N, X)$ with its image in $\text{Hol}(\bar{N}, \bar{X})$. By making use of the distance decreasing property of hyperbolic distance for holomorphic mappings, we see that $\text{Hol}(N, X)$ is relatively compact. The complete hyperbolicity of X implies that $\text{Hol}(N, X)$ is open and closed in $\text{Hol}(\bar{N}, \bar{X})$ and then Theorem (1.1) yields that the topological closure of $\text{Hol}(N, X)$ in $\text{Hol}(\bar{N}, \bar{X})$ is a compact complex subspace which contains $\text{Hol}(N, X)$ as a Zariski open subset. The complete hyperbolicity of X also implies that $\Phi(\cdot, x): \text{Hol}(N, X) \rightarrow X$ is proper for every fixed $x \in N$. Q.E.D.

In general, let Y_1 and Y_2 be two complex spaces. For a holomorphic mapping $f: Y_1 \rightarrow Y_2$, we set

$$\text{rank } f = \sup \left\{ \dim_t Y_1 - \dim_t f^{-1}(f(t)); t \in Y_1 \right\}.$$

The following proposition follows from Lemma (1.2). It reveals a special nature of the complex analyticity of holomorphic mappings but is less known.

Proposition (1.4). Assume that Y_1 and Y_2 are compact. Let $\{f_\nu\}_{\nu=1}^\infty$ be a sequence of points of $\text{Hol}(Y_1, Y_2)$ converging to $f \in \text{Hol}(Y_1, Y_2)$. If $\text{rank } f_\nu = k$, then $\text{rank } f = k$.

We set

$$\text{Hol}(k; N, X) = \{f \in \text{Hol}(N, X); \text{rank } f = k\}.$$

Corollary (1.5). $\text{Hol}(k; N, X)$ is open and closed in $\text{Hol}(N, X)$.

§2. The moduli $\text{Hol}(N, \Gamma \backslash D)$

In this section we deal with the case where X is the quotient $\Gamma \backslash D$ of a symmetric bounded domain D by a torsion-free discrete subgroup Γ of $\text{Aut}(D)$. We assume that Γ is uniform or an arithmetic subgroup of $\text{Aut}^0(D)$. In the case where Γ is uniform and $N = \bar{N}$, the results of this section were already obtained in [20, 21, 18]. We are mainly interested in the case where $\Gamma \backslash D$ and N are non-compact, while our arguments work in the compact case. Let $l(D)$ (resp. $l(\Gamma)$) denote the maximum dimension of proper boundary components of D (resp. Γ -rational boundary components). Let

$\text{Hol}(k; N, \Gamma \backslash D)$ denote the set of all holomorphic mappings $f: N \rightarrow \Gamma \backslash D$ with $\text{rank } f = k$. Applying the results of the previous section, we have

Theorem (2.1). i) $\text{Hol}(N, \Gamma \backslash D)$ carries a structure of a complex space compatible with compact-open topology, such that the evaluation mapping

$$\Phi: (f, x) \in \text{Hol}(N, \Gamma \backslash D) \times N \rightarrow f(x) \in \Gamma \backslash D$$

is holomorphic. Moreover, $\text{Hol}(N, \Gamma \backslash D)$ is a Zariski open subset of the compact complex space $\overline{\text{Hol}(N, \Gamma \backslash D)}^{1)}$, and satisfies the universality property; i.e., for a complex space T and a holomorphic mapping $\psi: T \times N \rightarrow \Gamma \backslash D$, the natural mapping

$$t \in T \rightarrow \psi(t, \cdot) \in \text{Hol}(N, \Gamma \backslash D)$$

is holomorphic.

ii) Every connected component of $\text{Hol}(N, \Gamma \backslash D)$ is complete hyperbolic and the holomorphic mappings

$$\Phi_x: f \in \text{Hol}(N, \Gamma \backslash D) \rightarrow f(x) \in \Gamma \backslash D$$

are proper for all $x \in N$.

1) $\overline{\text{Hol}(N, \Gamma \backslash D)}$ is the closure of $\text{Hol}(N, \Gamma \backslash D)$ in $\text{Hol}(\bar{N}, \Gamma \backslash D)$.

- iii) $\text{Hol}(k; N, \Gamma \setminus D)$ are open and closed in $\text{Hol}(N, \Gamma \setminus D)$.
- iv) $\text{Hol}(k; N, \Gamma \setminus D)$ are compact for $k > \ell(\Gamma)$.
- v) $\text{Hol}(k; N, \Gamma \setminus D)$ are finite for $k > \ell(D)$.

In the last of this section, we study in details the structure of $\text{Hol}(N, \Gamma \setminus D)$, assuming that $\bar{N}$ is Kähler and ∂N is a hypersurface with only simple normal crossings. We use the following result on harmonic mappings by [18]:

(2.2) Let $F: N \rightarrow \Gamma \setminus D$ and $G: N \rightarrow \Gamma \setminus D$ be free homotopic harmonic mappings with finite energy. Then there is a harmonic mapping $\Psi: \mathbb{R} \times N \rightarrow \Gamma \setminus D$ with respect to the product metric $dt \oplus h$ on $\mathbb{R} \times N$ such that

- i) $\Psi(0, x) = F(x)$, $\Psi(1, x) = G(x)$ and Ψ provides a free homotopy between F and G , equivalent to the given one;
- ii) for every $x \in N$, the curve $\gamma_x: t \in \mathbb{R} \rightarrow \Psi(t, x) \in \Gamma \setminus D$ is a parametrization of a geodesic with constant speed, independent of x , and $e(\Psi(t, \cdot))(x)$ is constant in t .

Lemma (2.3). Let F and G be as in (2.2). If F is holomorphic, then so is G .

Remark. 1) In case N is compact, this is a theorem due to Lichnerowicz (cf. Theorem (8.6) of [4]).

2) Since (2.2) actually holds for harmonic mappings from a complete Riemannian manifold with finite volume into a complete Riemannian manifold with non-positive sectional curvatures, Lemma (2.3) is also true for harmonic mappings F and G from a complete Kähler manifold with finite volume into a complete Kähler manifold with non-positive sectional curvatures, provided that F and G have finite energies.

The main result is the following:

Theorem (2.4). i) $\text{Hol}(N, \Gamma \backslash D)$ is smooth and quasi-projective.

ii) For a connected component Z of $\text{Hol}(N, \Gamma \backslash D)$ and a point $x \in N$, the evaluation mapping at x

$$\phi_x: f \in Z \rightarrow f(x) \in \Gamma \backslash D$$

is a proper holomorphic immersion onto a totally geodesic complex submanifold, so that Z is a free quotient of a symmetric bounded domain.

iii) For a connected component Z of $\text{Hol}(N, \Gamma \backslash D)$, there is a normal complex projective variety $\tilde{Z}$ such that Z is hyperbolically imbedded into $\tilde{Z}$ and ϕ_x holomorphically extends to $\bar{\phi}_x: \tilde{Z} \rightarrow \overline{\Gamma \backslash D}$.

iv) $\dim \text{Hol}(k; N, \Gamma \backslash D) \leq \ell(D)$ for $k > 0$.

- v) For $f \in \text{Hol}(N, \Gamma \backslash D)$ with $\bar{f}^{-1}(\partial \Gamma \backslash D) \neq \emptyset$,
 $\dim_f \text{Hol}(N, \Gamma \backslash D) \leq \ell(\Gamma)$.

As a corollary, we have the following.

Corollary (2.5) (Rigidity). Let $f: N \rightarrow \Gamma \backslash D$ be a holomorphic mapping. Then f is a unique holomorphic mapping among the free homotopy class of f , if f satisfies one of the following conditions:

- a) The image of f is not contained in a totally geodesic complex proper submanifold of $\Gamma \backslash D$;
- b) $\text{rank } f > \ell(D)$;
- c) $\bar{f}^{-1}(\partial \Gamma \backslash D) \neq \emptyset$ and $\text{rank } f > \ell(\Gamma)$.

In general, a holomorphic mapping $f \in \text{Hol}(N, \Gamma \backslash D)$ admits a deformation (cf. [5]). But in the special case where D is the n -th product H^n of the upper half plane $H \subset \mathbb{C}$, we see that any $f \in \text{Hol}(N, \Gamma \backslash H^n)$ is rigid. That is, by making use of the rigidity Theorem 6 of [6], we have

Theorem (2.6). Let $\Gamma \subset (\text{PSL}(2, \mathbb{R}))^n$ be an irreducible torsion-free discrete subgroup with $\text{Vol}(\Gamma \backslash H^n) < \infty$. Then

- i) if $f: N \rightarrow \Gamma \backslash H^n$ is a non-constant holomorphic mapping, f is a unique holomorphic mapping among the free homotopy class of f ,

so that

ii) there are only finitely many non-constant holomorphic mappings from N into $\Gamma \backslash H^n$.

Remark. 1) It must be noted that if $\Gamma \backslash H^n$ is not compact, then Γ is arithmetic ([12]). Therefore Γ satisfies our requirement for discrete subgroups.

2) In the case of $\dim N = 1$, i) was proved in [6].

3) By the same arguments as in [15], we see that the Kähler assumption for $\bar{N}$ is not necessary in ii). The proof is reduced to the present case.

4) For a compact quotient $\Gamma \backslash H^2$ and a compact complex manifold N , ii) was proved in [15]. For an algebraic curve N and compact $\Gamma \backslash H^n$, it was proved in [6].

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