

A relative bigness inequality and equidistribution theorem over function fields

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Plan of the Presentation

- 1 Review of bigness inequalities
- 2 Adelic line bundles and heights
- 3 Equidistribution theorem

Bigness criteria

- (Siu's inequality) Let X be a projective variety of dimension d , and L and M be nef line bundles. Then

$$\mathrm{vol}(L \otimes M^\vee) \geq c_1(L)^d - d \cdot c_1(L)^{d-1} c_1(M),$$

where $\mathrm{vol}(L) = \limsup_{n \rightarrow +\infty} \frac{h^0(X, L^{\otimes n})}{n^d/d!}$.

- (Arithmetic case by Yuan) Let $\mathcal{X} \rightarrow \mathrm{Spec} \mathcal{O}_K$ be an arithmetic variety of dimension d , where K is a number field. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be arithmetic ample line bundles, $\overline{\mathcal{E}}$ be a C^∞ -hermitian line bundle. Then

$$\chi_{\mathrm{sup}}(\overline{\mathcal{E}} \otimes (\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^\vee)^{\otimes n}) \geq \frac{\widehat{c}_1(\overline{\mathcal{L}})^d - d \cdot \widehat{c}_1(\overline{\mathcal{L}})^{d-1} \widehat{c}_1(\overline{\mathcal{M}})}{d!} n^d + o(n^d)$$

- How is the relative case?

Relative bigness inequality

Let k be a field of characteristic 0. Let Y be a projective and normal k -variety of dimension $e > 0$. Let $\pi : \mathcal{X} \rightarrow Y$ be a surjective morphism of projective varieties where $\mathcal{X}$ is of dimension $d + e$ ($d > 0$).

- A **polarization** of Y is a collection $\mathcal{H} := \{H_1, \dots, H_{e-1}\}$ where H_i are ample line bundles over Y .
- The **degree** of a coherent sheaf $\mathcal{F}$ with respect to $\mathcal{H}$ is defined as

$$\deg_{\mathcal{H}}(\mathcal{F}) := c_1(H_1) \cdots c_1(H_{e-1})c_1(\mathcal{F}).$$

- For $\alpha \in \mathrm{CH}_{e-1}(\mathcal{X})$, denote that $\alpha \cdot \pi^*\mathcal{H} := c_1(\pi^*H_1) \cdots c_1(\pi^*H_{e-1})\alpha$.

Theorem

Let $\mathcal{E}, \mathcal{L}, \mathcal{M}$ be line bundles over $\mathcal{X}$ such that $\mathcal{L}$ and $\mathcal{M}$ are nef. Then as $n \rightarrow +\infty$,

$$\deg_{\mathcal{H}}(\pi_*(\mathcal{E} \otimes (\mathcal{L} \otimes \mathcal{M}^\vee)^{\otimes n})) \geq \frac{c_1(\mathcal{L})^{d+1} \cdot \pi^*(\mathcal{H}) - (d+1)c_1(\mathcal{L})^d c_1(\mathcal{M}) \cdot \pi^*(\mathcal{H})}{(d+1)!} n^{d+1} + o(n^{d+1}).$$

Adelic line bundles

We fix the base variety Y (projective and normal) and polarization $\{H_1, \dots, H_{e-1}\}$.

- Let K be the function field $k(Y)$.
- Denote by $Y^{(1)} = \{\omega \in Y \mid \text{codim}_Y(\omega) = 1\}$. Then each $\omega \in Y^{(1)}$ gives an absolute value $|\cdot|_\omega : K \rightarrow \mathbb{R}_{\geq 0}$.
- Let K_ω be the completion of K with respect to $|\cdot|_\omega$.

Let X be a projective K -variety and L be a line bundle over X . We denote by X_ω^{an} the analytification of $X_\omega := X \times \text{Spec} K_\omega$ in the sense of Berkovich space. Let $L_\omega := L \otimes \mathcal{O}_{X_\omega}$.

- Let $\varphi = \{|\cdot|_{\varphi, \omega}\}_{\omega \in Y^{(1)}}$ be a family of continuous metrics, that is, each $|\cdot|_{\varphi, \omega}$ is a continuous metric on the analytification of L_ω .
- We say $\overline{L} = (L, \varphi)$ is an **adelic line bundle** over X if it satisfies certain measurability and dominance conditions.

Model metrics

Let $(\pi : \mathcal{X} \rightarrow Y, \mathcal{L})$ be a model of $(X, L^{\otimes r})$, that is, $\pi : \mathcal{X} \rightarrow Y$ is a surjective morphism of projective varieties, such that $\mathcal{X}_K \simeq X$ and $\mathcal{L}_K \simeq L^{\otimes r}$.

- For $\omega \in Y^{(1)}$, let K_ω° be the valuation ring of K_ω . We set

$$\begin{cases} \mathcal{X}_{K_\omega^\circ} := \mathcal{X} \times_Y K_\omega^\circ, \\ \mathcal{L}_\omega := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}_{K_\omega^\circ}}. \end{cases}$$

Then $(\mathcal{X}_{K_\omega^\circ}, \mathcal{L}_\omega)$ induces a model metric $|\cdot|_{\mathcal{L}, \omega}^{1/r}$ on L_ω .

- The pair $(L, \{|\cdot|_{\mathcal{L}, \omega}^{1/r}\}_{\omega \in Y^{(1)}})$ is an adelic line bundle.
- We say $(\mathcal{X}, \mathcal{L})$ is a relatively nef model if $\mathcal{L}$ is relatively nef.
- We say an adelic line bundle $\bar{L}$ is **semipositive** if there exists a sequence $\{\bar{L}_n = (L, \{|\cdot|_{\phi_n, \omega}\})\}$ of adelic line bundles induced by relatively nef models such that $\bar{L}_n$ uniformly converges to $\bar{L}$.
- We say an adelic line bundle $\bar{L}$ is **integrable** if $\bar{L} = \bar{L}_1 \otimes \bar{L}_2^\vee$ for semipositive line bundles $\bar{L}_1$ and $\bar{L}_2$.

Arakelov intersection theory over function fields

Definition

Let $\overline{L}_0, \dots, \overline{L}_d$ be integrable line bundles. We define the Arakelov intersection number

$$\widehat{c}_1(\overline{L}_0) \cdots \widehat{c}_1(\overline{L}_d) := \text{limit of geometric intersection number of models.}$$

Consider a purely d' -dimensional cycle $\alpha = \sum n_i Z_i$. We define that

$$\widehat{c}_1(\overline{L}_0) \cdots \widehat{c}_1(\overline{L}_{d'}) \cdot \alpha := \sum n_i \widehat{c}_1(\overline{L}_0|_{Z_i}) \cdots \widehat{c}_1(\overline{L}_{d'}|_{Z_i})$$

Definition

Let $\overline{L}$ be a semipositive line bundles, then we define the *height* of α by

$$h_{\overline{L}}(\alpha) = \frac{\widehat{c}_1(\overline{L})^{d'+1} \cdot [\alpha]}{c_1(L)^{d'} \cdot \alpha(d'+1)}.$$

Definition

Let $\bar{L}$ be a semipositive line bundle. Let Z be a d' -dimensional closed subvariety of X . By the **Chambert-Loir measure** $\mu_{\bar{L}, Z, \omega}$, we mean the unique probability measure $\frac{\hat{c}_1(\bar{L}|_Z)^{d'}}{\deg_{\mathcal{H}}(\omega) c_1(L)^{d'} \cdot Z}$ on X_{ω}^{an} with support on Z_{ω}^{an} with respect to $\bar{L}$.

- If $Z = X$, we denote the Chambert-Loir measure by $\mu_{\bar{L}, \omega}$.
- If Let $\rho : X' \rightarrow X$ be a generically finite surjective morphism, then $\mu_{\bar{L}, \omega} = \rho_* \mu_{\rho^* \bar{L}, \omega}$ due to the projection formula.

Remark

The Chambert-Loir measure can be also defined locally. Let $\omega \in Y^{(1)}$. Assume that the ω -adic metric of $\bar{L}$ can be given by a normal K_{ω}° -model $(\mathcal{X}, \mathcal{L})$, then $\hat{c}_1(\bar{L})_{\omega}^d = \sum_i m_i \deg_{\mathcal{L}}(W_i) \delta_{\eta_i}$, where W_i are irreducible components of the special fiber with multiplicities m_i and generic points η_i , and δ_{η_i} is the dirac-measure.

Equidistribution theorem over function fields

Assume that Y is a projective regular curve over k . Let X be a projective variety over the function field K of Y .

- Let I be an infinite directed set, and $\{Z_\iota\}_{\iota \in I}$ be a family of closed subvarieties $Z_\iota \subset X$ indexed by I .
- We say $\{Z_\iota\}$ is **generic** if for any proper closed subset $Z \subset X$, there exists an ι_0 , such that $Z_\iota \not\subset Z$ for any $\iota > \iota_0$. We say $\{Z_\iota\}$ is **small** if $\lim_{\iota \in I} h_{\overline{L}}(Z_\iota) = h_{\overline{L}}(X)$.

Theorem (W. Gubler, X. Faber)

Let $\overline{L}$ be an adelic line bundle such that L is ample, $\overline{L}$ is a limit of nef line bundles and $h_{\overline{L}}(X) = 0$. Let $\{Z_\iota\}_{\iota \in I}$ be a generic and small net of subvarieties of X . Then for any $\omega \in Y^{(1)}$, $\{Z_\iota\}$ is equidistributed in the sense that

$$\int_{X_\omega^{\text{an}}} f \mu_{\overline{L}, \omega} = \lim_{\iota \in I} \int_{X_\omega^{\text{an}}} f \mu_{\overline{L}, Z_\iota, \omega}$$

holds for any continuous function $f : X_\omega^{\text{an}} \rightarrow \mathbb{R}$.

Thank you for your attention.