

THE ABELIANIZATION OF THE CONGRUENCE IA-AUTOMORPHISM GROUP OF A FREE GROUP

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Dedicated to Professor Kojun Abe on the occasion of his 60th birthday

ABSTRACT. In this paper we consider the abelianizations of some normal subgroups of the automorphism group of a finitely generated free group. Let F_n be a free group of rank n . For $d \geq 2$, we consider a group consisting the automorphisms of F_n which act trivially on the first homology group of F_n with $\mathbf{Z}/d\mathbf{Z}$ -coefficients. We call it the congruence IA-automorphism group of level d and denote it by $IA_{n,d}$. Let $IO_{n,d}$ be the quotient group of the congruence IA-automorphism group of level d by the inner automorphism group of a free group. In this paper we determine the abelianization of $IA_{n,d}$ and $IO_{n,d}$ for $n \geq 2$ and $d \geq 2$. Furthermore, for $n = 2$ and odd prime p , we compute the integral homology groups of $IA_{2,p}$ for any dimension.

1. INTRODUCTION

Let F_n be a free group of rank n , and $\text{Aut } F_n$ the automorphism group of the free group F_n . We denote the abelianization of F_n by H . The abelianization homomorphism $F_n \rightarrow H$ induces a surjective homomorphism $\rho : \text{Aut } F_n \rightarrow GL(n, \mathbf{Z})$. In this paper we consider the abelianization of the preimages of the congruence subgroups of $GL(n, \mathbf{Z})$ by ρ . For $n \geq 2$ and $d \geq 2$, let $GL(n, d)$ be the general linear group over $\mathbf{Z}/d\mathbf{Z}$, and $\pi_d : GL(n, \mathbf{Z}) \rightarrow GL(n, d)$ the natural homomorphism induced by the mod reduction d . We call the kernel $\Gamma(n, d)$ of π_d the congruence subgroup of $GL(n, d)$ of level d . Classically, the congruence subgroups $\Gamma(n, d)$ have been studied by many authors, and there is a broad range remarkable results of them. In particular, it is well known that for $n \geq 3$ and odd prime integer p , Lee and Szczarba [10, Theorem 1.1] determined the structure of the abelianization of the congruence subgroup $\Gamma(n, p)$. More precisely, they showed that it is isomorphic to the Lie algebra $\mathfrak{sl}_n(\mathbf{F}_p)$ of trace-zero matrices over $\mathbf{F}_p$ as an $SL(n, \mathbf{F}_p)$ -module where $\mathbf{F}_p$ is the finite field of order p .

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In this paper we are also interested in the kernel IA_n of the natural map ρ . We call it the IA-automorphism group of F_n . Then we have an exact sequence

$$1 \rightarrow IA_n \rightarrow \text{Aut } F_n \xrightarrow{\rho} GL(n, \mathbf{Z}) \rightarrow 1.$$

This exact sequence plays important roles in the study of $\text{Aut } F_n$. Although Magnus [11] obtained a finitely many generating set of IA_n for $n \geq 3$, (See Subsection 2.1.) it is not known whether IA_n is finitely presented or not for $n \geq 4$. We remark that Krstić and McCool [9] showed that IA_3 is not finitely presentable.

For a group G , we denote the abelianization of G by G^{ab} . Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [8] independently showed that the abelianization IA_n^{ab} of IA_n is a free abelian group of rank $n^2(n-1)/2$. More precisely, it is isomorphic to $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ as a $GL(n, \mathbf{Z})$ -module where H^* is the dual group $\text{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ of H . We remark that the $GL(n, \mathbf{Z})$ -module structure of IA_n^{ab} is determined by using the first Johnson homomorphism of $\text{Aut } F_n$.

Here we consider subgroups of $\text{Aut } F_n$ which corresponds to the congruence subgroup of $GL(n, \mathbf{Z})$. Let $IA_{n,d}$ be the kernel of $\pi_d \circ \rho : \text{Aut } F_n \rightarrow GL(n, d)$, and call it the congruence IA-automorphism group of a free group F_n of level d . The first aim of this paper is to determine the structure of the abelianization of $IA_{n,d}$. Then our first result is

Theorem 1.1. *For $n \geq 2$ and $d \geq 2$,*

$$IA_{n,d}^{\text{ab}} \simeq (IA_n^{\text{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) \bigoplus \Gamma(n, d)^{\text{ab}}.$$

In Section 3, we prove this theorem using the “extended” Johnson homomorphism, introduced by Kawazumi [8], constructed from the $\mathbf{Z}/d\mathbf{Z}$ -valued Magnus expansion of $\text{Aut } F_n$. Considering the result of Lee and Szczarba [10] stated above, we see that for any odd prime p , the abelianization of IA_n is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n-1)(n^2+2n+2)}$ as an abelian group.

Next we consider the outer automorphism group of a free group and the images of IA_n and $IA_{n,d}$ by a natural projection. An automorphism ι of F_n is called an inner automorphism of F_n if there exists some element $y \in F_n$ such that $x' = y^{-1}xy$ for any $x \in F_n$. Then the group $\text{Inn } F_n$ of inner automorphisms of F_n is a normal subgroup of $\text{Aut } F_n$. Let $\text{Out } F_n$ be the quotient group $\text{Aut } F_n / \text{Inn } F_n$. The groups $\text{Inn } F_n$ and $\text{Out } F_n$ are called the inner automorphism group and the outer automorphism group of the free group respectively. We define a group IO_n to be the

quotient group of IA_n by $\text{Inn } F_n$. The group IO_n is the kernel of the natural map $\bar{\rho} : \text{Out } F_n \rightarrow GL(n, \mathbb{Z})$ induced by ρ . It is also known that the abelianization IO_n^{ab} of IO_n is given by $IO_n^{\text{ab}} \simeq (H^* \otimes_{\mathbb{Z}} \Lambda^2 H)/H$. (See [8, Theorem 6.2].) For $n \geq 2$ and $d \geq 2$, we define $IO_{n,d}$ to be the quotient group of $IA_{n,d}$ by $\text{Inn } F_n$. The group $IO_{n,d}$ is the kernel of $\pi_d \circ \bar{\rho}$. The second aim of this paper is to determine the structure of the abelianization $IO_{n,d}^{\text{ab}}$. The result is

Theorem 1.2. *For $n \geq 2$ and $d \geq 2$,*

$$IO_{n,d}^{\text{ab}} \simeq (IO_n^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/d\mathbb{Z}) \bigoplus \Gamma(n, d)^{\text{ab}}.$$

Finally, in Section 5, we compute the integral homology groups of $IA_{2,p}$ for an odd prime p . In general, to compute the integral homology groups of $IA_{n,d}$ is quite difficult as well as that of IA_n . In the case where $n = 2$ and $d = p$, we can compute those as follows:

Theorem 1.3. *For any prime p ,*

$$H_q(IA_{2,p}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}^{\oplus \alpha(p)} \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus 2} & \text{if } q = 1, \\ \mathbb{Z}^{\oplus (2\alpha(p)-2)} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3 \end{cases}$$

where $\alpha(p) = 1 + \frac{(p-1)p(p+1)}{12}$ is the rank of $\Gamma(2, p)$ as a free group.

We remark that $IO_{2,p}$ is isomorphic to the congruence subgroup $\Gamma(2, p)$ since $IA_2 = \text{Inn } F_2$ due to Nielsen [12], and hence, it is a free group of rank $\alpha(p)$.

2. PRELIMINARIES

In this section we review the IA-automorphism group of a free group and the first Johnson homomorphism of the automorphism group of a free group. Throughout this paper we use the following notation and conventions.

- The group $\text{Aut } F_n$ acts on F_n from the right.
- For any $\sigma \in \text{Aut } F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ .
- For elements x and y of a group, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.1. The IA-automorphism group.

In this subsection, we prepare generators of IA_n , and some basic exact sequences which is required to prove our main theorems.

Let F_n be a free group on $\{x_1, \dots, x_n\}$. Magnus [11] showed that IA_n is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto x_j^{-1} x_i x_j, \\ x_t & \mapsto x_t, \end{cases} \quad (t \neq i)$$

for distinct $i, j \in \{1, 2, \dots, n\}$ and

$$K_{klm} : \begin{cases} x_k & \mapsto x_k x_l x_m x_l^{-1} x_m^{-1}, \\ x_t & \mapsto x_t, \end{cases} \quad (t \neq k)$$

for distinct $k, l, m \in \{1, 2, \dots, n\}$ such that $l < m$. Since IO_n is the quotient group $IA_n/\text{Inn } F_n$, IO_n is also generated by (the coset classes of) automorphisms K_{ij} and K_{ijk} .

Next we give some basic exact sequences. Since the natural maps ρ and $\bar{\rho}$ are surjective, for any $n \geq 2$ and $d \geq 2$, we have exact sequences

$$(1) \quad 1 \rightarrow IA_n \rightarrow IA_{n,d} \xrightarrow{\rho} \Gamma(n, d) \rightarrow 1$$

and

$$(2) \quad 1 \rightarrow IO_n \rightarrow IO_{n,d} \rightarrow \Gamma(n, d) \rightarrow 1$$

respectively. Furthermore, by definition, we have

$$(3) \quad 1 \rightarrow \text{Inn } F_n \rightarrow IA_{n,d} \rightarrow IO_{n,d} \rightarrow 1.$$

These exact sequences are used in later sections.

2.2. The first Johnson homomorphism.

In this subsection, we review the first Johnson homomorphism of the automorphism group of a free group. For each $k \geq 1$, let $\Gamma_n(k)$ be the k -th subgroup of the lower central series of F_n defined by

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 1.$$

We denote the graded quotients $\Gamma_n(k)/\Gamma_n(k+1)$ by $\mathcal{L}_n(k)$. Set $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$. Then it is well known that $\mathcal{L}_n$ naturally has a structure of a graded Lie algebra over $\mathbb{Z}$ induced from the commutator bracket on F_n and it is naturally isomorphic to a graded free Lie algebra over H . In particular, we have $\mathcal{L}_n(1) = H$ and $\mathcal{L}_n(2) = \Lambda^2 H$. (For details, see [14].)

In this paper, for any $x \in \Gamma_n(k)$, we also denote by x the coset class of x in $\mathcal{L}_n(k)$.

For each $k \geq 1$, let

$$\tau' : IA_n \rightarrow \text{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(2)) = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

be the homomorphism defined by

$$\sigma \mapsto (x \mapsto x^{-1}x^\sigma).$$

for $\sigma \in IA_n$ and $x \in H$. The map τ' naturally induces a homomorphism

$$\tau : IA_n^{\text{ab}} \rightarrow H^* \otimes_{\mathbf{Z}} \Lambda^2 H.$$

These homomorphisms τ' and τ are called the first Johnson homomorphisms of $\text{Aut } F_n$. The map τ is a $GL(n, \mathbf{Z})$ -equivariant isomorphism. In particular, IA_n^{ab} is a free abelian group of rank $\frac{1}{2}n^2(n-1)$. (See [8, Theorem 6.1].)

Next, we consider IO_n^{ab} . For any $y \in F_n$, we denote by ι_y the inner automorphism of F_n such that $x^\iota_y = y^{-1}xy$ for any $x \in F_n$. Considering a natural isomorphism $\text{Inn } F_n \rightarrow F_n; \iota \mapsto y$, we often identify $\text{Inn } F_n$ with F_n . Then we have $(\text{Inn } F_n)^{\text{ab}} = H$. It is also known that the induced homomorphism $(\text{Inn } F_n)^{\text{ab}} = H \rightarrow H^* \otimes_{\mathbf{Z}} \Lambda^2 H = IA_n^{\text{ab}}$ from the inclusion map $\text{Inn } F_n \hookrightarrow IA_n$ is injective, and whose image, which we identify with H by this map, is a direct summand of $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ as a $\mathbf{Z}$ -module. Hence we see IO_n^{ab} is isomorphic to a free abelian group $(H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H$ of rank $\frac{1}{2}n(n+1)(n-2)$. For details, see [8, Theorem 6.2].

Now, the first Johnson homomorphism τ' induces a homomorphism

$$\tau'_d : IA_n \rightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

Finally, we recall that τ'_d is extended to a homomorphism from $IA_{n,d}$ to $(H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$. For any $\mathbf{Z}/d\mathbf{Z}$ -valued Magnus expansion θ , Kawazumi [8] constructed a crossed homomorphism

$$\tau^\theta : \text{Aut } F_n \rightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$$

and showed that if we restrict it to $IA_{n,d}$, then the map τ^θ is a homomorphism. Furthermore he also show that $\tau^\theta \equiv \tau'_d$ on IA_n . Especially the restriction $\tau^\theta|_{IA_n}$ is independent of the choice of the Magnus expansion θ . For details, see [8, Theorem 3.1].

3. THE ABELIANIZATION OF $IA_{n,d}$.

In this section we give a proof of Theorem 1.1. First, we see that since the first Johnson homomorphism τ is a $GL(n, \mathbf{Z})$ -equivariant isomorphism, τ induces a surjective homomorphism

$$\tilde{\tau} : H_0(\Gamma(n, d), IA_n^{\text{ab}}) \rightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

To show $\tilde{\tau}$ is an isomorphism, we use

Lemma 3.1. *For $n \geq 2$ and $d \geq 2$, we have*

$$d[K_{ij}] = 0 \quad \text{and} \quad d[K_{klm}] = 0$$

in $H_0(\Gamma(n, d), IA_n^{\text{ab}})$.

Then considering the homological five term exact sequence

$$\begin{aligned} H_2(IA_n, \mathbf{Z}) \rightarrow H_2(IA_{n,d}, \mathbf{Z}) \rightarrow H_0(\Gamma(n, d), IA_n^{\text{ab}}) \\ \xrightarrow{\eta} IA_{n,d}^{\text{ab}} \rightarrow \Gamma(n, d)^{\text{ab}} \rightarrow 0. \end{aligned}$$

of (1), and a homomorphism

$$\tau^\theta : IA_{n,d} \rightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$$

induced from a Magnus expansion θ , we see η is injective. Hence we have a split exact sequence

$$0 \rightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \xrightarrow{\eta} IA_n^{\text{ab}} \rightarrow \Gamma(n, d)^{\text{ab}} \rightarrow 0.$$

This completes the proof of Theorem 1.1.

Here we consider the case where d equals to an odd prime integer p . For $n \geq 3$, Lee and Szczarba [10] showed that the abelianization $\Gamma(n, p)^{\text{ab}}$ of the congruence subgroup $\Gamma(n, p)$ is a $\mathbf{Z}/p\mathbf{Z}$ -vector space of dimension $n^2 - 1$. For $n = 2$, Frasch [5] showed that the congruence subgroup $\Gamma(2, p)$ is a free group of rank $\alpha(p) := 1 + \frac{(p-1)p(p+1)}{12}$. Furthermore Nielsen [12] showed that $IA_2 = \text{Inn } F_2$. Hence we have

Corollary 3.1.

$$IA_{n,p}^{\text{ab}} = \begin{cases} \mathbf{Z}^{\oplus \alpha(p)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if } n = 2, \\ (\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n-1)(n^2+2n+2)} & \text{if } n \geq 3. \end{cases}$$

4. THE ABELIANIZATION OF $IO_{n,d}$.

In this section we give a proof of Theorem 1.2. Considering the homological five term exact sequence of (3), we have

$$H_2(IA_{n,d}, \mathbf{Z}) \rightarrow H_2(IO_{n,d}, \mathbf{Z}) \rightarrow H_0(IO_{n,d}, (\text{Inn } F_n)^{\text{ab}}) \xrightarrow{\delta} IA_{n,d}^{\text{ab}} \rightarrow IO_{n,d}^{\text{ab}} \rightarrow 0.$$

Since H and $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ is free abelian groups, the injective homomorphism

$$H = (\text{Inn } F_n)^{\text{ab}} \hookrightarrow IA_n^{\text{ab}} = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

induces an injective homomorphism

$$\psi_d : H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \hookrightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

For each i , $1 \leq i \leq n$, set $\iota_i := \iota_{x_i}$. Then, $\text{Inn } F_n$ is a free group on $\{\iota_1, \dots, \iota_n\}$.

Lemma 4.1. *For $n \geq 2$ and $d \geq 2$,*

$$d[\iota_i] = 0, \quad 1 \leq i \leq n$$

in $H_0(IO_{n,d}, (\text{Inn } F_n)^{\text{ab}})$.

Hence there exists an isomorphism

$$\bar{\xi} : H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \rightarrow H_0(IO_{n,d}, (\text{Inn } F_n)^{\text{ab}})$$

such that $\psi_d = \delta \circ \bar{\xi}$, and we have a short exact sequence

$$0 \rightarrow H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \xrightarrow{\delta} IA_{n,d}^{\text{ab}} \rightarrow IO_{n,d}^{\text{ab}} \rightarrow 0,$$

and hence

$$\begin{aligned} & ((H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) / (H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) \\ & \simeq ((H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}, \\ & \simeq IO_n^{\text{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}. \end{aligned}$$

This completes the proof of Theorem 1.2.

For $n \geq 2$ and an odd prime p , by an argument similar to that in Corollary 3.1, we obtain

Corollary 4.1.

$$IO_{n,p}^{\text{ab}} = \begin{cases} \mathbf{Z}^{\oplus \alpha(p)} & \text{if } n = 2, \\ (\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n+1)(n^2-2)} & \text{if } n \geq 3. \end{cases}$$

5. THE INTEGRAL HOMOLOGY GROUPS OF $IA_{2,p}$

In this section, we compute the integral homology groups of $IA_{2,p}$ for any odd prime p . Since the groups IA_2 and $\Gamma(2, p)$ are free groups stated above, considering the homological Lyndon-Hochschild-Serre spectral sequence of (1) for $n = 2$ and $d = p$, we see the homological dimension of $IA_{2,p}$ is 2. On the other hand, since the first homology group $H_1(IA_{2,p}, \mathbf{Z})$ is obtained in Section 3, it suffices to compute the second homology group $H_2(IA_{2,p}, \mathbf{Z})$. Our result is

Theorem 5.1. *For any odd prime p ,*

$$H_2(IA_{2,p}, \mathbf{Z}) = \mathbf{Z}^{\oplus (2\alpha(p)-2)}$$

where $\alpha(p) = 1 + \frac{(p-1)p(p+1)}{12}$.

To prove this theorem, first, we directly compute the second cohomology groups of $IA_{2,p}$. Then, using the universal coefficients theorem, we obtain the second homology group of $IA_{2,p}$.

Proposition 5.1. *For any odd prime integer p , we have*

$$H^2(IA_{2,p}, \mathbf{Z}) = \mathbf{Z}^{\oplus (2\alpha(p)-2)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2}.$$

Similarly, we obtain

Proposition 5.2. *For any odd prime integer p , we have*

$$H^2(IA_{2,p}, \mathbf{Z}/q\mathbf{Z}) \simeq \begin{cases} (\mathbf{Z}/q\mathbf{Z})^{\oplus (2\alpha(p)-2)} & \text{if } (q, p) = 1, \\ (\mathbf{Z}/q\mathbf{Z})^{\oplus (2\alpha(p)-2)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if } q = p^e. \end{cases}$$

Using Propositions 5.1 and 5.2, we obtain the second homology group $H_2(IA_{2,p}, \mathbf{Z})$ by the universal coefficients theorem.

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REFERENCES

- [1] D. Arlettaz; On the homology and cohomology of congruence subgroups, J. of Pure and Applied Algebra 44 (1987), 3-12.
- [2] F. Cohen and J. Pakianathan; On Automorphism Groups of Free Groups, and Their Nilpotent Quotients, preprint.
- [3] F. Cohen and J. Pakianathan; On subgroups of the automorphism group of a free group and associated graded Lie algebras, preprint.
- [4] B. Farb; Automorphisms of F_n which act trivially on homology, in preparation.
- [5] H. Frasci; Die Erzeugenden der Hauptkongruenzgruppen für Primzahlstufen, Math. Ann. 108 (1933), 230-252.
- [6] S. M. Gersten; A presentation for the special automorphism group of a free group, J. of Pure and Applied Algebra 33 (1984), 269-279.
- [7] A. Hatcher and K. Vogtmann; Rational homology of $\text{Aut}(F_n)$, Math. Res. Lett. 5 (1998), 759-780.
- [8] N. Kawazumi; Cohomological aspects of Magnus expansions, preprint, The University of Tokyo. UTMS 2005-18 (2005), <http://xxx.yukawa.kyoto-u.ac.jp/abs/math.GT/0505497>.
- [9] S. Krstić and J. McCool; The non-finite presentability in $IA(F_3)$ and $GL_2(\mathbb{Z}[t, t^{-1}])$, Invent. Math. 129 (1997), 595-606.
- [10] R. Lee and R. H. Szczarba; On the homology and cohomology of congruence subgroups, Invent. Math. 33 (1976), 15-53.
- [11] W. Magnus; Über n -dimensionale Gittertransformationen, Acta Math. 64 (1935), 353-367.
- [12] J. Nielsen; Die Isomorphismen der allgemeinen unendlichen Gruppe mit zwei Erzeugenden, Math. Ann. 78 (1918), 385-397.
- [13] J. Nielsen; Die Isomorphismengruppe der freien Gruppen, Math Ann., 91 (1924), 169-209.
- [14] C. Reutenauer; Free Lie Algebras, London Mathematical Society monographs, new series, no. 7, (Oxford University Press, 1993).

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