

Efficiency of Set Optimization with Weighted Criteria

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1 Introduction

In this paper, we consider efficiency of set-valued optimization problems with weighted criteria. Let $(E, \leq)$ be an ordered topological vector space, C the ordering cone in $(E, \leq)$, and assume that C is a closed set. Also $C^+ = \{x^* \in E^* \mid \langle x^*, x \rangle \geq 0, \forall x \in C\}$ and we choose a *weight set* W , a subset of C^+ . Let $\mathcal{A}$ be the family of all nonempty compact convex sets in E , and $\mathcal{B}$ a nonempty subfamily of $\mathcal{A}$. Our purpose is to consider about minimal elements of $\mathcal{B}$ with weighted criteria.

In this paper, we introduce some concepts concerned with set-limit and cone-completeness, to characterize existence of such minimal elements. Also we consider completeness of some metric space including the whole space $\mathcal{A}$.

Definition 1.1 $\emptyset \neq A, B \subset E$,

$$A \leq_W^l B \stackrel{\text{def}}{\iff} \overline{\langle z^*, A + C \rangle} \supset \langle z^*, B \rangle, \forall z^* \in W$$

$$A \leq_W^u B \stackrel{\text{def}}{\iff} \langle z^*, A \rangle \subset \overline{\langle z^*, B - C \rangle}, \forall z^* \in W$$

Definition 1.2 (Minimal for a Family with Weight)

B_0 is (l, W) -minimal in $\mathcal{B}$ if $B_0 \in \mathcal{B}$ and condition $B \leq_W^l B_0$ implies $B_0 \leq_W^l B$.

B_0 is (u, W) -minimal in $\mathcal{B}$ if $B_0 \in \mathcal{B}$ and condition $B \leq_W^u B_0$ implies $B_0 \leq_W^u B$.

Similarly we can define (l, W) -maximal and (u, W) -maximal. In this paper we treat only the (l, W) -minimal notion.

2 Characterization of Efficiency

Definition 2.1 ((l, W) -Decreasing, (l, W) -Complete, (l, W) -Section)

A net of sets $\{A_\lambda\}$ in $\mathcal{A}$ is said to be (l, W) -decreasing if

$$\lambda < \lambda' \implies A_{\lambda'} \leq_W^l A_\lambda$$

A subfamily $\mathcal{D} \subset \mathcal{A}$ is said to be (l, W) -complete if there is no (l, W) -decreasing net $\{D_\lambda\}$ in $\mathcal{D}$ such that

$$\mathcal{D} \subset \{A \in \mathcal{A} \mid \exists \lambda \text{ such that } A \not\leq_W^l D_\lambda\}$$

Let $A \in \mathcal{A}$ and $\mathcal{D} \subset \mathcal{A}$. Then the family

$$\mathcal{D}(A) = \{D \in \mathcal{D} \mid D \leq_W^l A\}$$

is called an (l, W) -section in $\mathcal{D}$

Theorem 2.1 (Existence of (l, W) -minimal sets)

$\mathcal{B}$ has an (l, W) -minimal set if and only if $\mathcal{B}$ has a nonempty (l, W) -complete section

Definition 2.2 (W -limit, W -set limit)

Let $\{a_\lambda\}_\Lambda$ be a net of E , $x \in E$, then

$$\lim_W a_\lambda \ni x \stackrel{\text{def}}{\iff} \forall y^* \in W, \langle y^*, a_\lambda \rangle \rightarrow \langle y^*, x \rangle.$$

the set $\lim_W a_\lambda$ is called W -limit of $\{a_\lambda\}$ Also let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a net of $\mathcal{A}$, $x \in E$, then

$$\liminf_W A_\lambda \ni x \stackrel{\text{def}}{\iff} \exists \{a_\lambda\} \text{ such that } a_\lambda \in A_\lambda, \forall \lambda \in \Lambda \text{ and } \lim_W a_\lambda \ni x$$

$$\begin{aligned} \limsup_W A_\lambda \ni x &\stackrel{\text{def}}{\iff} \exists \{a_{\lambda'}\} \subset \{a_\lambda\}: \text{ a subnet such that } a_{\lambda'} \in A_{\lambda'}, \forall \lambda' \in \Lambda \\ \text{and } \lim_W a_{\lambda'} &\ni x \end{aligned}$$

these are called W -lower and W -upper limits, resp.

Definition 2.3 ((l, W) and (u, W) -Set limits)

$$\liminf_W^l A_\lambda = \liminf_W (A_\lambda + C)$$

$$\liminf_W^u A_\lambda = \liminf_W (A_\lambda - C)$$

$$\limsup_W^l A_\lambda = \limsup_W (A_\lambda + C)$$

$$\limsup_W^u A_\lambda = \limsup_W (A_\lambda - C)$$

Proposition 2.1 If $\mathcal{A}_\lambda$ is (l, W) -decreasing then

$$A \leq_W^l A_\lambda \iff A \leq_W^l \liminf_{\lambda \in \Lambda}^l A_\lambda$$

Theorem 2.2 The following are equivalent:

- $\mathcal{B}$ has an (l, W) -minimal set
- $\mathcal{B}$ has a nonempty (l, W) -complete section
- There exists $A_0 \in \mathcal{A}$ such that $\mathcal{B}(A_0) = \{B \in \mathcal{B} \mid B \leq_W^l A_0\}$ is (l, W) -complete
- For any (l, W) -decreasing net $\{B_\lambda\}$ in $\mathcal{B}$, there exists $A_0 \in \mathcal{A}$ such that $A_0 \leq_W^l \liminf_{\lambda \in \Lambda}^l B_\lambda$

Corollary 2.1 Let F be a set-valued map from a subset X of a topological space into E . If X is compact and

$$\begin{aligned} x_\lambda \rightarrow x_0, \{F(x_\lambda)\} : (l, W)\text{-decreasing} \\ \implies F(x_0) \leq_W^l \liminf_{\lambda \in \Lambda}^l F(x_\lambda) \end{aligned}$$

then there is an (l, W) -minimal set in $\{F(x) \mid x \in X\}$.

3 Completeness

In this section, we consider about completeness of metric space $(\mathcal{A}/\equiv_W^l, d)$. At first we define a quotient space $\mathcal{A}/\equiv_W^l$ as follows:

$$\mathcal{A}/\equiv_W^l = \{[A] \mid A \in \mathcal{A}\},$$

where $[A] = \{B \in \mathcal{A} \mid A \equiv_W^l B\}$ for each $A \in \mathcal{A}$. In this space, we define an order relation. For $[A], [B] \in \mathcal{A}/\equiv_W^l$,

$$[A] \leq_W^l [B] \stackrel{\text{def}}{\iff} A \leq_W^l B$$

Then $\leq_W^l$ is an order relation on $\mathcal{A}/\equiv_W^l$. Next, we define a metric on the space. For $[A], [B] \in \mathcal{A}/\equiv_W^l$,

$$d([A], [B]) = \sup_{y^* \in W} |\min \langle y^*, A \rangle - \min \langle y^*, B \rangle|$$

Then d is a metric on $\mathcal{A}/\equiv_W^l$.

Now we have a question. Is d complete?

Counterexample 3.1 $E = \mathbf{R}^2$, $C = \mathbf{R}_+^2$, $W = [(1, 0), (0, 1)]$, $A_n = \{(x_1, x_2) \in E \mid 0 \leq x_1, x_2 \leq n, 1 \leq x_1 x_2\}$. Then $\{[A_n]\}$ is a Cauchy sequence in $\mathcal{A}/\equiv_W^l$, but $\{[A_n]\}$ does not converges to any elements of $\mathcal{A}/\equiv_W^l$. (For example, $A_0 = \{(x_1, x_2) \in E \mid 0 \leq x_1, x_2, 1 \leq x_1 x_2\}$, $d(A_n, A_0) \rightarrow 0$ as $n \rightarrow \infty$)

How conditions assure the completeness? Concerning the question, we have the following two theorems.

Theorem 3.1 $\{[A_n]\}$ is a Cauchy sequence in $\mathcal{A}/\equiv_W^l$, and there exists a compact subset K of E such that $A_n \subset K$ for each n .

Proof. Let $\mu_{A_n} : W \rightarrow \mathbf{R}$ defined by

$$\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \quad y^* \in W$$

then there exists a continuous function $\mu_0 : W \rightarrow \mathbf{R}$ such that μ_{A_n} converges to μ_0 uniformly on W . For $y^* \in W$, there exists $a_{y^*} \in K$ such that $\mu_0(y^*) = \langle y^*, a_{y^*} \rangle$. Let $A_0 := \{a_{y^*} \mid y^* \in W\}$, then $\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \text{co} A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\text{co}} A_0} \langle y^*, a \rangle$. Also we have $\overline{\text{co}} A_0 \in \mathcal{A}$, and then we conclude the proof. $\square$

Theorem 3.2 $\{[A_n]\}$ is a Cauchy sequence in $\mathcal{A}/\equiv_W^l$, and there exists a compact subset K of E and a sequence $\{x_n\} \subset E$ such that $x_n + A_n \subset K$ for each n . Assume that $C^+ - C^+ = E^*$ and E is reflexive, then $\{[A_n]\}$ converges some element of $\mathcal{A}$.

Proof. Let $\mu_{A_n} : W \rightarrow \mathbf{R}$ defined by

$$\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \quad y^* \in W$$

then there exists a continuous function $\mu_0 : W \rightarrow \mathbf{R}$ such that μ_{A_n} converges to μ_0 uniformly on W . From condition $x_n + A_n \subset K$, there exists M such that $|\langle y^*, x_n \rangle| \leq M$ for each $y^* \in W$ and n , and by assumption $C^+ - C^+ = E^*$, we have $|\langle y^*, x_n \rangle| \leq M$ for each $y^* \in E^*$ and n . Using uniform boundedness theorem, we have $\|x_n\| \leq M$ for each n . Then we can choose a subsequence $\{x_{n'}\}$ and $x_0 \in E$ such that $\{x_{n'}\}$ converges to x_0 weakly.

For $y^* \in W$, there exists $a_{y^*} \in K$ such that $\langle y^*, x_0 \rangle + \mu_0(y^*) = \langle y^*, a_{y^*} \rangle$. Let $A_0 := \{a_{y^*} - x_0 \mid y^* \in W\}$, then $\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \text{co} A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\text{co}} A_0} \langle y^*, a \rangle$ for each $y^* \in W$. Also we have $\overline{\text{co}} A_0 \in \mathcal{A}$, then we complete the proof. $\square$

References

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